Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 52(1), 2025, Pages 257–261, DOI: 10.52846/ami.v52i1.2075
ISSN: 1223-6934

A New Sequence Related to the Constant e

Cristinel Mortici

Abstract. The aim of this work is to introduce a new sequence related to the constant e. Some properties of this sequence are given.

2020 *Mathematics Subject Classification.* Primary 26A06; Secondary 26A09.
Key words and phrases. constant e, sequences, inequalities, functions variation.

1. Introduction

In mathematics, certain constants stand out due to their fundamental importance and pervasive presence across various fields. Among these, the number $e = 2.71828...$ holds a distinguished position. Often referred to as Euler's number, after the Swiss mathematician Leonhard Euler (1707-1783), e is a transcendental number that arises naturally in the context of growth and decay processes, complex analysis, and calculus.

It was first defined in connection with the problem of compound interest, where it arises naturally as the limit of a particular sequence.

The first formula defining e can be expressed as:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

Ever since, other approximating sequences were introduced and we mention here the family

$$e_n(a) = \left(1 + \frac{1}{n}\right)^{n+a}, \quad n \geq 1,$$

where $a \in [0,1]$ is a real parameter. Note that $(e_n(a))_{n\geq 1}$ is increasing, or decreasing, as $a \in [0,1/2]$, or $a \in (1/2,1]$, respectively.

The number e can be written in terms of logarithmic mean:

$$L(x,y) = \frac{x-y}{\ln x - \ln y}, \quad x > y > 0,$$

namely:

$$e = \left(1 + \frac{1}{n}\right)^{L(n,n+1)} .$$

We define in this work a new sequence $(a_n)_{n\geq 1}$ by the formula:

$$\left(1 + \frac{a_n}{n}\right)^{a_n} = e, \quad n \geq 1.$$

Received December 24, 2024. Accepted March 16, 2025.

We prove that such a sequence does uniquely exist and we establish some of its properties. Note that if we replace a_n by $\sqrt{n}$, we obtain the sequence:

$$\left(1+\frac{\sqrt{n}}{n}\right)^{\sqrt{n}}=\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}},$$

that converges to e. This remark anticipates our result according to which the sequence $(a_n)_{n\geq 1}$ has the order $\sqrt{n}$, as n tends to infinity.

2. The results

Let us consider the relation:

$$\left(1+\frac{a_n}{n}\right)^{a_n}=e, \quad n\geq 1. \tag{1}$$

We show that for every integer $n\geq 1$, there exist $a_n>0$ such that (1) holds true. First we pass to a continuous variable, to prove the following:

Theorem 2.1. *There exists a function* $f:(0,\infty)\to(0,\infty)$ *such that for every* $x\in(0,\infty)$, *we have:*

$$\left(1+\frac{f(x)}{x}\right)^{f(x)}=e. \tag{2}$$

Proof. By taking the logarithm, we deduce:

$$f(x)\ln\left(1+\frac{f(x)}{x}\right)=1,$$

then

$$\ln\left(1+\frac{f(x)}{x}\right)=\frac{1}{f(x)}.$$

Further,

$$\frac{f(x)}{x}=e^{\frac{1}{f(x)}}-1,$$

or

$$\frac{f(x)}{e^{\frac{1}{f(x)}}-1}=x. \tag{3}$$

Now let us introduce the function $g:(0,\infty)\to(0,\infty)$, by the formula:

$$g(x)=\frac{x}{e^{\frac{1}{x}}-1}, \quad x\in(0,\infty). \tag{4}$$

We have:

$$g(x)=\frac{1}{h\left(\frac{1}{x}\right)}, \tag{5}$$

where $h(x)=x(e^x-1)$, $x\in(0,\infty)$. But $h'(x)=xe^x+(e^x-1)>0$, so h is strictly increasing. It follows by (5) that g is strictly increasing, too. Moreover, g is inversable, with an increasing inverse g^{-1}, since $\lim_{x\to 0_+}g(x)=0$ and $\lim_{x\to\infty}g(x)=\infty$.
Now, (3) can be rewritten as:

$$g(f(x))=x.$$

We deduce that:

$$f(x)=g^{-1}(x),$$

so the requested function f satisfying (2) is the function g^{-1}. The proof is completed. $\square$

As a direct consequence, we give the following:

Theorem 2.2. *There exists a strictly increasing sequence $(a_n)_{n\geq 1}$ such that the equality (1) holds true, for every $n \in \mathbb{N}^*$.*

Proof. By taking $x = n \in \mathbb{N}^*$ in (2), we obtain the sequence $a_n = f(n)$ satisfying (1). As g is strictly increasing, so it is its inverse f. Thus $f(n) < f(n+1)$, for every integer $n \geq 1$, that is $a_n < a_{n+1}$. $\square$

Next we concentrate to evaluate the size of the function f and consequently, of the sequence $(a_n)_{n\geq 1}$. To do this, we study the function g.

If we look carefully at (4), we see that g is closely related to the generating series of Bernoulli's numbers B_n, $n \in \mathbb{N}$. These numbers are defined by the relation:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi. \tag{6}$$

The first few Bernoulli's numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, ..., while $B_{2k+1} = 0$, for every integer $k \geq 1$. For further properties, please see [1, p. 804].

By replacing t by $1/x$ in (6), we get:

$$\frac{\frac{1}{x}}{e^{\frac{1}{x}} - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n! x^n},$$

then, by multiplying by x^2, we get

$$g(x) = \sum_{n=0}^{\infty} \frac{B_n}{n! x^{n-2}}.$$

The first few terms of this series are:

$$g(x) = x^2 - \frac{1}{2}x + \frac{1}{12} - \frac{1}{720x^2} + \frac{1}{30\,240x^4} - \frac{1}{1209\,600x^6} + \dots . \tag{7}$$

In general, by truncation of series (6), under- and upper-approximations for $t/(e^t - 1)$ are obtained. This fact and (7) motivates us to give the following:

Theorem 2.3. *The following inequalities hold true, for every $x > 1/2$:*

$$x^2 - \frac{1}{2}x < g(x) < x^2 - \frac{1}{2}x + \frac{1}{12}. \tag{8}$$

Proof. First, let us replace x by $1/x$:

$$\frac{1}{x^2} - \frac{1}{2x} < \frac{1}{x(e^x - 1)} < \frac{1}{x^2} - \frac{1}{2x} + \frac{1}{12}$$

(from now we have $0 < x < 2$). We equivalently have:

$$\frac{1}{x\left(\frac{1}{x^2} - \frac{1}{2x} + \frac{1}{12}\right)} < e^x - 1 < \frac{1}{x\left(\frac{1}{x^2} - \frac{1}{2x}\right)},$$

or

$$1 + \frac{1}{x\left(\frac{1}{x^2} - \frac{1}{2x} + \frac{1}{12}\right)} < e^x < 1 + \frac{1}{x\left(\frac{1}{x^2} - \frac{1}{2x}\right)}.$$

By taking the logarithm, we obtain:

$$\ln\left(1+\frac{1}{x\left(\frac{1}{x^2}-\frac{1}{2x}+\frac{1}{12}\right)}\right)<x<\ln\left(1+\frac{1}{x\left(\frac{1}{x^2}-\frac{1}{2x}\right)}\right).$$

Hence we have to prove that $u(x)>0$ and $v(x)<0$, $x\in(0,2)$, where:

$$u(x)=x-\ln\left(1+\frac{1}{x\left(\frac{1}{x^2}-\frac{1}{2x}+\frac{1}{12}\right)}\right)$$

and

$$v(x)=x-\ln\left(1+\frac{1}{x\left(\frac{1}{x^2}-\frac{1}{2x}\right)}\right).$$

We have:

$$u'(x)=\frac{x^4}{(x^2-6x+12)(x^2+6x+12)}>0$$

and

$$v'(x)=-\frac{x^2}{4-x^2}<0.$$

It follows that u is strictly increasing, while v is strictly decreasing on $(0,2)$. As $\lim_{x\to 0_+}u(x)=\lim_{x\to 0_+}v(x)=0$, it results that $u>0$ and $v<0$, on $(0,2)$. The proof is completed. □

Now, we use the monotonicity of g (and $g^{-1}=f$). By applying $g^{-1}=f$ in (8), we obtain:

$$f\left(x^2-\frac{1}{2}x\right)<x<f\left(x^2-\frac{1}{2}x+\frac{1}{12}\right). \tag{9}$$

By replacing x by $\frac{1}{4}+\sqrt{x+\frac{1}{16}}$ (the inverse function of $x\mapsto x^2-\frac{1}{2}x$) in the left-hand side inequality (9), we get:

$$f(x)<\frac{1}{4}+\sqrt{x+\frac{1}{16}}.$$

Analogously, by replacing x by $\frac{1}{4}+\sqrt{x-\frac{3}{144}}$ (the inverse function of $x\mapsto x^2-\frac{1}{2}x+\frac{1}{12}$) in the right-hand side inequality (9), we get:

$$f(x)>\frac{1}{4}+\sqrt{x-\frac{3}{144}}.$$

We obtained the following:

Theorem 2.4. *For every $x>1/2$, we have:*

$$\frac{1}{4}+\sqrt{x-\frac{3}{144}}<f(x)<\frac{1}{4}+\sqrt{x+\frac{1}{16}}.$$

Now, we can give the following estimate for the sequence $(a_n)_{n\geq 1}$:

Theorem 2.5. *There exists an unique sequence $(a_n)_{n\geq 1}\subset(0,\infty)$ such that for every integer $n\geq 1$, we have:*

$$\left(1+\frac{a_n}{n}\right)^{a_n}=e.$$

Moreover, $(a_n)_{n\geq 1}$ *is strictly increasing and*

$$\frac{1}{4}+\sqrt{n-\frac{3}{144}}<a_n<\frac{1}{4}+\sqrt{n+\frac{1}{16}}, \quad n\geq 1.$$

Finally, the sequence $(a_n)_{n\geq 1}$ is of the form:

$$a_n=\frac{1}{4}+\sqrt{n}+\theta_n,$$

where $(\theta_n)_{n\geq 1}$ converges to zero.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* National Bureau of Standards, Applied Mathematics Series, 55, Washington, 1964.

(Cristinel Mortici) Valahia University of Târgovişte, Aleea Sinaia 13, 130004 Târgovişte, Romania

National University of Science and Technology "Politehnica" Bucharest, Splaiul Independenţei 313, 060042 Bucharest, Romania

Academy of Romanian Scientists, 3 Ilfov, 050044 Bucharest, Romania
E-mail address: `cristinel.mortici@hotmail.com`