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Entropy solutions of a stationary problem associated to a nonlinear parabolic strongly degenerate problem in one space dimension

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Abstract. We study a nonlinear elliptic degenerate equation with the form: $b(u)-a(u,\varphi(u)_x)_x = f$ which is associated to the elliptic-parabolic-hyperbolic equation of the form: $b(u)_t-a(u,\varphi(u)_x)_x = f$. We prove in this work without Alt and Luckhaus structure condition(1983) existence and uniqueness of entropy solutions of the associated Dirichlet problem. We also define an operator associated to the evolution problem and prove some useful properties of this operator .

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1. Introduction and notations

We consider the Dirichlet problem

$$(SP)\quad \begin{cases} b(u)-a(u,\varphi(u)_x)_x = f & \text{in} \quad I \\ u=0 & \text{on } \Gamma=\partial I, \end{cases}$$

associated to the Cauchy-Dirichlet problem:

$$(CP)\quad \begin{cases} b(u)_t-a(u,\varphi(u)_x)_x = f & \text{in} \quad \mathbb{Q}=]0,T[\times I \\ b(u)(0,.)=v_0 & \text{on } I \\ u=0 & \text{on } \Gamma=]0,T[\times\partial I \end{cases}$$

with $T>0$ and I is an open bounded interval of $\mathbb{R}$,
where

$$\begin{cases} a:(z,\xi)\in\mathbb{R}\times\mathbb{R}\to\mathbb{R} \text{ is continuous, nondecreasing in } \xi\in\mathbb{R} \text{ with } a(0,0)=0; \\ b:\mathbb{R}\to\mathbb{R} \text{ is continuous, nondecreasing and } b \text{ surjective with } b(0)=0; \\ \varphi:\mathbb{R}\to\mathbb{R} \text{ is continuous, nondecreasing with } \varphi(0)=0; \end{cases}$$

and
$f\in L^\infty(\mathbb{Q}), v_0\in L^\infty(I)$.

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We define the operators H^+, H_0 and H_ϵ by:

$$H_0(r) = \begin{cases} 1 \ \text{ if } r > 0 \\ 0 \text{ otherwise} \end{cases} ; \ H^+(r) = \begin{cases} 1 \text{ if } \ r > 0 \\ [0,1] \ \text{ if } r = 0 \\ 0 \text{ if } \ r < 0, \end{cases}$$

and $H_\epsilon(r) = min(\frac{r^+}{\epsilon}, 1)$.

Let γ be a maximal monotone operator defined on $\mathbb{R}$. We recall the definition of the main section γ_0 of γ.

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s) \text{ if } \gamma(s) \neq \emptyset \\ +\infty \text{ if } [s, +\infty) \cap D(\gamma) = \emptyset \\ -\infty \text{ if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

$H(k) = a(k, 0)$ for $k \in \mathbb{R}$, $h = a(u, \varphi(u)_x)$.

Whenever u takes a value such that $b(u)$ is constant, (CP) degenerates into an elliptic problem of the form:

$$(CP_1) \begin{cases} -a(u, \varphi(u)_x)_x = f & \text{in } \mathbb{Q} =]0, T[\times I \\ b(u)(0, .) = v_0 & \text{on } I \\ u = 0 & \text{on } \Gamma =]0, T[\times I. \end{cases}$$

Take $b = id$, on each part where u takes a value such that $\varphi(u)$ is constant, (CP) degenerates to a scalar conservation law of the form:

$$(CP_2) \quad \begin{cases} u_t - a(u, 0)_x = f & \text{in } \mathbb{Q} =]0, T[\times I \\ u(0, .) = u_0 & \text{on } I \\ u = 0 & \text{on } \Gamma =]0, T[\times I. \end{cases}$$

It is clear that we include in (CP), some first order hyperbolic problems, for which (even under assumptions of regularity on data) there is no hope of getting classical global solutions.
Elliptic-parabolic-hyperbolic problems arise as a model in many applications, for example as a model of flow through porous media (cf. [Bj], [DT]).

There already exist a vast literature on problems of this type. Several authors have studied the degenerate parabolic equation of the type we consider here (see e.g [AGW],[AL], [AW], [BT], [BW], [Ca], [CW], [DT], [Of],[OT_1], [OT_2], ...). Some of these authors proved existence and uniqueness of weak solutions of the Dirichlet boundary value problems under various additionnal conditions. Among these results, the pioneering work of Alt and Luckhaus [AL] established existence and uniqueness of weak solutions under some energy condition provided $u_t \in L^1(Q)$ and under some structure condition on the vector field a:

$$b(r) = b(s) \Longrightarrow a(r,\xi) = a(s,\xi) \text{ for all } r,s \in \mathbb{R},\ \xi \in \mathbb{R},$$

which is equivalent to the condition:

$$a(r,\xi) = \widetilde{a}(b(r),\xi) \text{ for all } r \in \mathbb{R},\ \xi \in \mathbb{R},$$

for some continuous vector field $\widetilde{a} : R(b) \times \mathbb{R} \longrightarrow \mathbb{R}$ nondecreasing with respect to the second variable.

In this paper, we study the stationary problem (SP) associated to the Cauchy-Dirichlet problem (CP)(the study of (CP) which is based on the results of this paper will be made in a forthcoming work [Os]). The paper is divided into three sections: in the first section, we define the concepts of solutions used and study the uniqueness of the entropy solution. In the second section of this article, using the results of the first section, we show the existence of entropy solutions and finally in the third and last section, we define an operator in L^1 associated with the evolution problem (CP) and show that this operator is T-accretive, with dense domain and checks the image condition.

2. Uniqueness of entropy solutions

We consider the following stationary problem $(SP) = (SP)(b, a, \varphi, f)$ defined by:

$$\begin{cases} b(u) - a(u, \varphi(u)_x)_x = f & \text{in } I \\ \\ u = 0 & \text{on } \partial I. \end{cases}$$

We begin by introducing the following notions of weak and entropy solutions for this problem.

Definition 2.1. *Let $f \in L^\infty(I)$; a weak solution of (SP) is a measurable function u such that $b(u) \in L^1(I)$, $\varphi(u) \in W^{1,\infty}(I)$, $h \in L^2(I)$ and*

$$b(u) - a(u, \varphi(u)_x)_x = f \ \text{ in } D'(I),$$

or equivalent to

$$\int_I \{b(u)\xi + h\xi_x\}\, dx = \int_I f\xi dx \quad \text{for any } \xi \in H_0^1(I) \cap L^\infty(I).$$

Remark 2.1. *We easily check that if u is a weak solution of $(SP)(b, a, \varphi, f)$ then $-u$ is a weak solution of $(SP)(\widetilde{b}, \widetilde{a}, \widetilde{\varphi}, -f)$ where $\widetilde{b}(s) = -b(-s)$, $\widetilde{\varphi} = -\varphi(-s)$, and $\widetilde{a}(s,k) = -a(-s,-k)$.*

It is well known that there is no uniqueness of weak solutions in general. In order to get uniqueness, we may introduce entropy solutions following the notion of entropy solution of S.N. Kruzkhov for conservation laws (see e.g.[Ks], [ABK]).

Definition 2.2. *Let $f \in L^\infty(I)$; an entropy solution of (SP) is a weak solution u satisfying:*
(i) there exists $h \in C(I)$ such that $h = a(u, \varphi(u)_x)$ a.e on I
(ii) the following entropy inequality

$$(a) \int_I H_0(u-k)\left\{(H(k) - h)\xi_x + \xi(f - b(u))\right\} dx \geq 0$$

for any $(k,\xi) \in I\!R \times (H_0^1(I) \cap L^\infty(I))$ *such that* $\xi \geq 0$ *, and for any* $(k,\xi) \in I\!R \times (H^1(I) \cap L^\infty(I))$ *such that* $\xi \geq 0$ *and such that* $k \geq 0$*;*

$$(b) \int_I H_0(k-u)\left\{(H(k)-h)\xi_x + \xi(f-b(u))\right\} dx \leq 0$$

for any $(k,\xi) \in I\!R \times (H_0^1(I) \cap L^\infty(I))$ *such that* $\xi \geq 0$ *, and for any* $(k,\xi) \in I\!R \times (H^1(I) \cap L^\infty(I))$ *such that* $\xi \geq 0$ *and such that* $k \leq 0$.

Remark 2.2. *It is easy to see that if* u *is an entropy solution of* $(SP)(b,a,\varphi,f)$ *then* $(-u)$ *is an entropy solution of* $(SP)(\widetilde{b},\widetilde{a},\widetilde{\varphi},\widetilde{f})$ *where* $\widetilde{b}(r) = -b(-r)$*,* $\widetilde{a}(r,k) = -a(-r,-k)$*,* $\widetilde{\varphi}(r) = -\varphi(-r)$ *and* $\widetilde{f} = -f$*.*

For a φ, we define: $E = \{r \in Im(\varphi)/;(\varphi^{-1})_0 \text{ is discontinuous at } r\}$.
Since $(\varphi^{-1})_0$ is a monotone function, E is a countable subset of $\mathbb{R}$, then we have:

$$\varphi(u)_x = 0 \text{ a.e in } O = \{x \in I/;\varphi(u(x)) \in E\}.$$

If $\varphi(s) \notin E$ then we have that $H_0(u-s) = H_0(\varphi(u)-\varphi(s))$ a.e in $\mathbb{R}$.

Our main assumption is the coerciveness of a with respect to ξ, for k bounded; more precisely:

$$(H_1) \qquad \lim_{|\xi|\to\infty} \inf_{|k|<\mathbb{R}} \mid a(k,\xi) \mid = +\infty \ \forall\, \mathbb{R} > 0.$$

In this part (uniqueness), we make the following additional assumption

$$(H_2)\begin{cases} (a(r,\xi)-a(s,\eta)).(\xi-\eta) + M(r,s)(1+|\xi|^2+|\eta|^2)\,|\varphi(r)-\varphi(s)| \geq \\ \Gamma(\varphi(r),\varphi(s)).\xi + \widehat{\Gamma}(\varphi(r),\varphi(s)).\eta \end{cases}$$

for all r, s, ξ, $\eta \in \mathbb{R}$, where $M : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$, Γ, $\widehat{\Gamma} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous.

Remark 2.3. *(i) Assumption* (H_2) *implies* $\Gamma(\varphi(r),\varphi(r)) = \widehat{\Gamma}(\varphi(r),\varphi(r)) = 0$ *for all* $r \in I\!R$*. Indeed, choosing* $r = s$*,* $\eta = 0$*,* $\xi = t\nu$*,* $t > 0$*,* $\nu \in I\!R$ *in* (H_2)*, we get* $t\nu[a(r,t\nu) - a(r,0)] \geq \Gamma(\varphi(r),\varphi(r))t\nu$*. Dividing by* t *and passing to the limit with* $t \to 0$*, we find* $\Gamma(\varphi(r),\varphi(r))\nu \leq 0$ *for all* $\nu \in I\!R$*; hence* $\Gamma(\varphi(r),\varphi(r)) = 0$*. Using the same arguments we obtain the corresponding result for* $\widehat{\Gamma}$*.*
(ii) (H_2) *implies that* a *is monotone with respect to the second variable (see [CW]-Remark 2.2 for the proof).*

For the proof of uniqueness, we need the following result, which we present as lemma:

Lemma 2.1. *Let* $f \in L^\infty(I)$ *and* u *be a weak solution of* (SP)*, then:*

$$\begin{cases} \displaystyle\int_I H_0(u-k)\left\{\xi_x(H(k)-h) + \xi(f-b(u))\right\} dx = \\ \displaystyle\lim_{\epsilon\longrightarrow 0}\int_I (h-H(k))\,\varphi(u)_x H'_\epsilon(\varphi(u)-\varphi(k))\xi dx \end{cases} \tag{1}$$

for any $(k,\xi) \in I\!R \times (H_0^1(I) \cap L^\infty(I))$ *such that* $\varphi(k) \notin E$ *and* $\xi \geq 0$ *, and for any* $(k,\xi) \in I\!R \times (H^1(I) \cap L^\infty(I))$ *such that* $\varphi(k) \notin E$*,* $\xi \geq 0$ *and* $k \geq 0$*. Moreover,*

$$\begin{cases} \displaystyle\int_I H_0(k-u)\left\{\xi_x(H(k)-h)+\xi(f-b(u))\right\}dx = \\ \displaystyle -\lim_{\epsilon \longrightarrow 0}\int_I (h-H(k))\,\varphi(u)_x H'_\epsilon(\varphi(k)-\varphi(u))\xi dx \end{cases} \tag{2}$$

for any $(k,\xi) \in I\!R \times (H_0^1(I) \cap L^\infty(I))$ *such that* $\varphi(k) \notin E$ *and* $\xi \geq 0$ *, and for any* $(k,\xi) \in I\!R \times (H^1(I) \cap L^\infty(I))$ *such that* $\varphi(k) \notin E$*,* $\xi \geq 0$ *and* $k \leq 0$*.*

Proof. Let (k,ξ) be as in lemma 6 for (1). Then the function $H_\epsilon(\varphi(u)-\varphi(k))\xi \in H_0^1(I) \cap L^\infty(I)$ and since u is weak solution of (SP), we have

$$\int_I (b(u)-f)H_\epsilon(\varphi(u)-\varphi(k))\xi dx + \int_I h[H_\epsilon(\varphi(u)-\varphi(k))\xi]_x dx = 0;$$

Note that $\displaystyle\int_I H(k)[H_\epsilon(\varphi(u)-\varphi(k))\xi]_x dx = 0$ since $H(k)$ is constant and $(H_\epsilon(\varphi(u)-\varphi(k))\xi)|_{\partial I} = 0$.

Then, we have

$$\int_I (b(u)-f)H_\epsilon(\varphi(u)-\varphi(k))\xi dx + \int_I (h-H(k))[H_\epsilon(\varphi(u)-\varphi(k))\xi]_x dx = 0;$$

which is equivalent after passing to the limit when $\epsilon \to 0$ (since $\varphi(k) \notin E$) to:

$$\begin{cases} \displaystyle\int_I H_0(u-k)\left\{\xi_x(H(k)-h)+\xi(f-b(u))\right\}dx = \\ \displaystyle \lim_{\epsilon \longrightarrow 0}\int_I (h-H(k))\,\varphi(u)_x H'_\epsilon(\varphi(u)-\varphi(k))\xi dx. \end{cases}$$

By the same method of proof, we can prove the second equality.
We can now establish the so called Kato's inequality (cf.[AB]), for two entropy solutions of the stationary problem.

Theorem 2.1. *Let* $f_1 \in L^\infty(I)$*,* $f_2 \in L^\infty(I)$*. Let* u_1*,* u_2 *be entropy solutions with respect to* $(SP)(b,a,\varphi,f_1)$*,* $(SP)(b,a,\varphi,f_2)$ *respectively. Then:*

$$\begin{cases} \displaystyle\int_I H_0(u_1-u_2)\left\{(h_1-h_2)\xi_x+(b(u_1)-b(u_2))\xi\right\}dx \leq \\ \displaystyle\int_I H_0(u_1-u_2)(f_1-f_2)\xi dx \end{cases} \tag{3}$$

for all $\xi \in H_0^1(I) \cap L^\infty(I)$ *and* $\xi \geq 0$*.*

Proof: We consider two different variables x and y of $\mathbb{R}$ and define $\partial_x = \frac{\partial}{\partial x}$ and $\partial_y = \frac{\partial}{\partial y}$.
We denote $u_1 = u_1(x)$, $f_1 = f_1(x)$, and $u_2 = u_2(y)$, $f_2 = f_2(y)$.
Let ξ be a positive test function of $D(I \times I)$, then for all $x \in I$, $y \in I$:

$$\begin{cases} y \mapsto \xi(x,y) \in D^+(I) \text{ for all } x \in I \\ x \mapsto \xi(x,y) \in D^+(I) \text{ for all } y \in I. \end{cases} \tag{4}$$

Let:

$$O_1 = \{x \in I/; \varphi(u_1(x)) \in E\} \tag{5}$$

and

$$O_2 = \{y \in I/; \varphi(u_2(y)) \in E\}. \tag{6}$$

We deduce that

$$\begin{cases} \partial_x \varphi(u_1(x)) = 0 \text{ on } O_1 \\ \partial_y \varphi(u_2(y)) = 0 \text{ on } O_2. \end{cases} \tag{7}$$

Then

$$H_0(u_1 - u_2) = H_0(\varphi(u_1) - \varphi(u_2)) \text{ in } [I \times (I \setminus O_1)] \cup [(I \setminus O_2) \times I].$$

We replace in (1), u by u_1 and k by u_2 and integrate on $I \setminus O_2$; in (a), u by u_1 and k by u_2 and integrate on O_2. Then, one adds the two relations which gives:

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1 - u_2)\{(h_1 - a(u_2,0))\xi_x + (b(u_1) - f_1)\xi\}\,dxdy \le \\ \displaystyle -\lim_{\epsilon \longrightarrow 0} \int_{(I\setminus O_2)\times I} (h_1 - a(u_2,0))\,\varphi(u_1)_x H'_\epsilon(\varphi(u_1) - \varphi(u_2))\xi dxdy. \end{cases} \tag{8}$$

In the same way, we replace in (2), k by u_1 and u by u_2 and integrate on $I \setminus O_1$; in (b), k by u_1 and u by u_2 and integrate on O_1. Then, one adds the two relations which gives:

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1 - u_2)\{(h_2 - a(u_1,0))\xi_y + (b(u_2) - f_2)\xi\}\,dxdy \ge \\ \displaystyle \lim_{\epsilon \longrightarrow 0} \int_{I\times(I\setminus O_1)} (h_2 - a(u_1,0))\,\varphi(u_2)_y H'_\epsilon(\varphi(u_1) - \varphi(u_2))\xi dxdy. \end{cases} \tag{9}$$

From (8), we deduce that :

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1 - u_2)\{h_1(\xi_x + \xi_y) + (b(u_1) - f_1)\xi\}\,dxdy \le \\ \displaystyle\int_{I\times I} H_0(u_1 - u_2)a(u_2,0)\xi_x dxdy + \int_{I\times I} H_0(u_1 - u_2)h_1\xi_y dxdy \\ \displaystyle -\lim_{\epsilon \longrightarrow 0} \int_{(I\setminus O_2)\times I} (h_1 - a(u_2,0))\,\varphi(u_1)_x H'_\epsilon(\varphi(u_1) - \varphi(u_2))\xi dxdy. \end{cases} \tag{10}$$

From (9), we deduce that

$$\left\{\begin{array}{l}\displaystyle\int_{I\times I} H_0(u_1-u_2)\left\{h_2(\xi_x+\xi_y)+(b(u_2)-f_2)\xi\right\}dxdy \geq \\ \displaystyle\int_{I\times I} H_0(u_1-u_2)a(u_1,0)\xi_y dxdy+\int_{I\times I} H_0(u_1-u_2)h_2\xi_x dxdy \\ \displaystyle+\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1)}(h_2-a(u_1,0))\,\varphi(u_2)_y H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy.\end{array}\right. \tag{11}$$

Making the subtraction (10)-(11), one has:

$$\left\{\begin{array}{l}\displaystyle\int_{I\times I} H_0(u_1-u_2)\left\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_1)-b(u_2))\xi+(f_2-f_1)\xi\right\}dxdy \leq \\ \displaystyle\int_{I\times I} H_0(u_1-u_2)a(u_2,0)\xi_x dxdy+\int_{I\times I} H_0(u_1-u_2)h_1\xi_y dxdy \\ \displaystyle-\lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times I}(h_1-a(u_2,0))\,\varphi(u_1)_x H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy \\ \displaystyle-\int_{I\times I} H_0(u_1-u_2)a(u_1,0)\xi_y dxdy-\int_{I\times I} H_0(u_1-u_2)h_2\xi_x dxdy \\ \displaystyle-\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1)}(h_2-a(u_1,0))\,\varphi(u_2)_y H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy.\end{array}\right.$$

Which is equivalent to

$$\left\{\begin{array}{l}\displaystyle\int_{I\times I} H_0(u_1-u_2)\left\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_1)-b(u_2))\xi+(f_2-f_1)\xi\right\}dxdy \leq \\ \displaystyle-\int_{I\times I} H_0(u_1-u_2)[h_2-a(u_2,0)]\xi_x dxdy+\int_{I\times I} H_0(u_1-u_2)[h_1-a(u_1,0)]\xi_y dxdy \\ \displaystyle-\lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times I}(h_1-a(u_2,0))\,\varphi(u_1)_x H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy \\ \displaystyle-\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1)}(h_2-a(u_1,0))\,\varphi(u_2)_y H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy.\end{array}\right. \tag{12}$$

One has:

$$\left\{\begin{array}{l}\displaystyle\int_{I\times I} H_0(u_1-u_2)[h_1-a(u_1,0)]\xi_y dxdy = \int_{I\times(I\setminus O_1)} H_0(u_1-u_2)[h_1-a(u_1,0)]\xi_y dxdy\\ \displaystyle = \lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1)} H_\epsilon(\varphi(u_1)-\varphi(u_2))[h_1-a(u_1,0)]\xi_y dxdy =\\ \displaystyle -\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1)}[h_1-a(u_1,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))_y\xi dxdy.\end{array}\right. \tag{13}$$

In the same way, we have also

$$\left\{\begin{array}{l}\displaystyle\int_{I\times I} H_0(u_1-u_2)[h_2-a(u_2,0)]\xi_x dxdy = \int_{(I\setminus O_2)\times I} H_0(u_1-u_2)[h_2-a(u_2,0)]\xi_x dxdy\\ \displaystyle = \lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times I} H_\epsilon(\varphi(u_1)-\varphi(u_2))[h_2-a(u_2,0)]\xi_x dxdy =\\ \displaystyle -\lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times I}[h_2-a(u_2,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))_x\xi dxdy.\end{array}\right. \tag{14}$$

Using (13) and (14) in (12), we obtain

$$\left\{\begin{array}{l}\displaystyle\int_{I\times I} H_0(u_1-u_2)\left\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_1)-b(u_2))\xi+(f_2-f_1)\xi\right\}dxdy \leq\\ \displaystyle \lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times I}[h_2-a(u_2,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))_x\xi dxdy-\\ \displaystyle \lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1)}[h_1-a(u_1,0)]H_\epsilon(\varphi(u_1)-\varphi(u_2))_y\xi dxdy\\ \displaystyle -\lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times I}(h_1-a(u_2,0))\,\varphi(u_1)_x H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy\\ \displaystyle -\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1)}(h_2-a(u_1,0))\,\varphi(u_2)_y H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy,\end{array}\right.$$

which is equivalent to

$$\left\{\begin{array}{l}\displaystyle\int_{I\times I} H_0(u_1-u_2)\left\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_1)-b(u_2))\xi+(f_2-f_1)\xi\right\}dxdy \leq\\ \displaystyle \lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times(I\setminus O_1)} h_2 div_{xy} H_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy-\\ \displaystyle \lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times(I\setminus O_1)} h_1 div_{xy} H_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy.\end{array}\right. \tag{15}$$

Let us put

$$I = \lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2)\times(I\setminus O_1)}[h_2-h_1]div_{xy} H_\epsilon(\varphi(u_1)-\varphi(u_2))\xi dxdy.$$

According to (H_2),

$$\begin{cases} I = -\lim\limits_{\epsilon\longrightarrow 0}\int_{(I\backslash O_2)\times(I\backslash O_1)}[a(u_1,\varphi(u_1)_x) - a(u_2,\varphi(u_2)_y)](\varphi(u_1)_x - \varphi(u_2)_y)H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi \\ \leq \lim\limits_{\epsilon\longrightarrow 0}\int_{(I\backslash O_2)\times(I\backslash O_1)} M(u_1,u_2)(1+|\varphi(u_1)_x|^2+|\varphi(u_2)_y|^2)\,|\varphi(u_1)-\varphi(u_2)|\,H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi \\ -\lim\limits_{\epsilon\longrightarrow 0}\int_{(I\backslash O_2)\times(I\backslash O_1)}\Gamma(\varphi(u_1),\varphi(u_2))\varphi(u_1)_x H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi \\ -\lim\limits_{\epsilon\longrightarrow 0}\int_{(I\backslash O_2)\times(I\backslash O_1)}\widehat{\Gamma}(\varphi(u_1),\varphi(u_2))\varphi(u_2)_y H'_\epsilon(\varphi(u_1)-\varphi(u_2))\xi = \\ \lim\limits_{\epsilon\longrightarrow 0} I_1 - \lim\limits_{\epsilon\longrightarrow 0} I_2 - \lim\limits_{\epsilon\longrightarrow 0} I_3 \end{cases}$$

One shows easily that $\lim\limits_{\epsilon\longrightarrow 0} I_1 = 0$.
Set

$$F_\epsilon(z) = \int_0^z \Gamma(r,\varphi(u_2))H'_\epsilon(r-\varphi(u_2))dr.$$

We have then:

$$I_2 = \int_{(I\backslash O_2)\times(I\backslash O_1)} div_x F_\epsilon(\varphi(u_1))\xi dxdy = -\int_{(I\backslash O_2)\times(I\backslash O_1)} F_\epsilon(\varphi(u_1))\xi_x dxdy.$$

One has

$$F_\epsilon(z) = \frac{1}{\epsilon}\int_{min(z,\varphi(u_2))}^{min(z,\varphi(u_2)+\epsilon)} \Gamma(r,\varphi(u_2))dr.$$

The function $\Gamma \in C(\mathbb{R}^2)$, and acheave his maximum and minimum on any compact set of $\mathbb{R}$; in particular on $[\varphi(u_2),\varphi(u_2)+\epsilon]$ since $\|u_2\|_\infty$ is finite (see lemma 13 below). There exists m_ϵ and M_ϵ such that:

$$m_\epsilon \leq \frac{1}{\epsilon}\int_{min(z,\varphi(u_2))}^{min(z,\varphi(u_2)+\epsilon)} \Gamma(r,\varphi(u_2))dr \leq M_\epsilon.$$

According to the intermediate value theorem , there exists $r_1(\epsilon)$ and $r_2(\epsilon)$ in $[\varphi(u_2),\varphi(u_2)+\epsilon]$ such that:

$$m_\epsilon = \Gamma(r_1(\epsilon),\varphi(u_2))$$

and

$$M_\epsilon = \Gamma(r_2(\epsilon),\varphi(u_2)).$$

Since $r_1(\epsilon)$ and $r_2(\epsilon) \in [\varphi(u_2),\varphi(u_2)+\epsilon]$, there exists θ_1 and $\theta_2 \in]0,1[$ such that:

$$r_1(\epsilon) = \theta_1(\varphi(u_2)) + (1-\theta_1)(\varphi(u_2)+\epsilon)$$

and

$$r_2(\epsilon) = \theta_2(\varphi(u_2)) + (1-\theta_2)(\varphi(u_2)+\epsilon).$$

Consequently:

$$\lim_{\epsilon\longrightarrow 0} r_1(\epsilon) = \varphi(u_2) \text{ and } \lim_{\epsilon\longrightarrow 0} r_2(\epsilon) = \varphi(u_2).$$

One obtains then:

$$\lim_{\epsilon\longrightarrow 0} m_\epsilon = \Gamma(\varphi(u_2),\varphi(u_2)) = 0 \text{ and } \lim_{\epsilon\longrightarrow 0} M_\epsilon = \Gamma(\varphi(u_2),\varphi(u_2)) = 0.$$

Finally, one has $F_\epsilon \longrightarrow 0$ when $\epsilon \longrightarrow 0$, and then, one gets:
$\lim\limits_{\epsilon\longrightarrow 0} I_2 = 0$ and $\lim\limits_{\epsilon\longrightarrow 0} I_3 = 0$ (the limit of I_3 is shown by a similar way). Consequently, $I \leq 0$ and then, from (15), one deduces the following inequality:

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1-u_2)\left\{(h_1-h_2)(\xi_x+\xi_y)+(b(u_1)-b(u_2))\xi\right\} dxdy \leq \\ \displaystyle\int_{I\times I} H_0(u_1-u_2)(f_1-f_2)\xi dxdy \end{cases} \tag{16}$$

for any nonnegative $\xi \in D(I\times I)$.
Now, let $\xi \in D^+(I)$; let (ρ_n) be classical sequence of mollifiers in $\mathbb{R}$ such that $\rho_n(s) = \rho_n(-s)$.
One defines

$$\xi_n(x,y) = \xi(\frac{x+y}{2})\rho_n(\frac{x-y}{2});$$

then $\xi_n \in D^+(I\times I)$ for n large enough.
Let $(X,Y) = T(x,y) = (\frac{x+y}{2}, \frac{x-y}{2})$;
using (16), we deduce that

$$\begin{cases} \displaystyle\int_{T(I\times I)} (b(u_1)-b(u_2))^+\xi\rho_n dXdY + \int_{T(I\times I)} H_0(u_1-u_2)(h_1-h_2)\xi_X\rho_n dXdY \leq \\ \displaystyle\int_{T(I\times I)} H_0(u_1-u_2)(f_1-f_2)\xi\rho_n dXdY. \end{cases}$$

Passing to the limit when $n \to +\infty$ in the inequality above, we then obtain the Kato's inequality (3).
Moreover, the inequality (3) is still true for any nonnegative ξ belonging to $H^1(I)\cap L^\infty(I)$.

Theorem 2.2. *Let $f_1 \in L^\infty(I)$, $f_2 \in L^\infty(I)$. Let u_1, u_2 be entropy solutions with respect to $(SP)(b,a,\varphi,f_1)$, $(SP)(b,a,\varphi,f_2)$ respectively. Then:*

$$\begin{cases} \displaystyle\int_I H_0(u_1-u_2)\left\{(h_1-h_2)\xi_x+(b(u_1)-b(u_2))\xi dx\right\} \leq \\ \displaystyle\int_I H_0(u_1-u_2)(f_1-f_2)\xi dx \end{cases} \tag{17}$$

for all $\xi \in H^1(I)\cap L^\infty(I)$ and $\xi \geq 0$.

Proof: We consider two different variables x and y of $\mathbb{R}$ and we assume that $u_1 = u_1(x)$, $f_1 = f_1(x)$, and $u_2 = u_2(y)$, $f_2 = f_2(y)$. We use the same notation as that in the proof of theorem 7.
From (a) and (1), we deduce that inequality (10) is still true for any nonnegative $\xi \in D(\mathbb{R}\times\mathbb{R})$ when we replacing u_2 by any measurable function $v = v(y)$ satisfying

$$x \longmapsto H_\epsilon(\varphi(u_1)-\varphi(v))\xi \in H_0^1(I) \text{ for a.e. } y \in I,$$

and O_2 by $\{y \in I/;\varphi(v(y)) \in E\}$. In particular, let $\xi \in D(\overline{I}\times I)$, $\xi \geq 0$, let us replace u_2 by u_2^+ and O_2 by $O_2^+ = \{y \in I/;\varphi(u_2^+(y)) \in E\}$. Then we get from (10),

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1^+-u_2^+)\left\{h_1^+(\xi_x+\xi_y)+(b(u_1^+)-f_1)\xi\right\}dxdy \leq \\ \displaystyle\int_{I\times I} H_0(u_1^+-u_2^+)a(u_2^+,0)\xi_x dxdy+\int_{I\times I} H_0(u_1^+-u_2^+)h_1^+\xi_y dxdy \\ \displaystyle-\lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2^+)\times I}\left(h_1^+-a(u_2^+,0)\right)\varphi(u_1^+)_x H'_\epsilon(\varphi(u_1^+)-\varphi(u_2^+))\xi dxdy. \end{cases} \quad (18)$$

where $h^+ = a(u^+, \varphi(u^+)_x)$.
Now from (b) and (2), we deduce that for any nonnegative $\xi \in D(\overline{I}\times I)$, (9) and then (11) is still true when u_1 is replaced by u_1^+ and O_1 by $O_1^+ = \left\{x \in I/; \varphi(u_1^+(x)) \in E\right\}$. Then we get

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1^+-u_2)\left\{h_2(\xi_x+\xi_y)+(b(u_2)-f_2)\xi\right\}dxdy \geq \\ \displaystyle\int_{I\times I} H_0(u_1^+-u_2)a(u_1^+,0)\xi_y dxdy+\int_{I\times I} H_0(u_1^+-u_2)h_2\xi_x dxdy \\ \displaystyle+\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)}\left(h_2-a(u_1^+,0)\right)\varphi(u_2)_y H'_\epsilon(\varphi(u_1^+)-\varphi(u_2))\xi dxdy. \end{cases} \quad (19)$$

And since

$$H_0(u_1^+-u_2) = H_0(u_1^+-u_2^+)(1-H_0(u_2^-)) + H_0(u_2^-),$$

we get from (19)

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1^+-u_2^+)\left\{h_2^+(\xi_x+\xi_y)+(b(u_2^+)-(1-H_0(u_2^-))f_2)\xi\right\}dxdy \geq \\ \displaystyle-\int_{I\times I} H_0(u_2^-)\left\{h_2(\xi_x+\xi_y)+(b(u_2)-f_2)\xi\right\}dxdy+\int_{I\times I} H_0(u_2^-)a(u_1^+,0)\xi_y dxdy \\ \displaystyle+\int_{I\times I} H_0(u_1^+-u_2^+)(1-H_0(u_2^-))a(u_1^+,0)\xi_y dxdy+\int_{I\times I} H_0(u_1^+-u_2^+)h_2^+\xi_x dxdy \\ \displaystyle+\int_{I\times I} H_0(u_2^-)h_2\xi_x dxdy+\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)}\left(h_2-a(u_1^+,0)\right)\varphi(u_2)_y H'_\epsilon(\varphi(u_1^+)-\varphi(u_2))\xi dxdy \end{cases} \quad (20)$$

where $h^+ = a(u^+, \varphi(u^+)_x)$.
We know that

$$\begin{cases} \displaystyle\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)}\left(h_2-a(u_1^+,0)\right)\varphi(u_2)_y H'_\epsilon(\varphi(u_1^+)-\varphi(u_2))\xi dxdy = \\ \displaystyle\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} H_0(u_2^-)\left(h_2-a(u_1^+,0)\right)\varphi(u_2)_y H'_\epsilon(\varphi(u_1^+)-\varphi(u_2))\xi dxdy \\ \displaystyle\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)}\left(h_2^+-a(u_1^+,0)\right)\varphi(u_2^+)_y H'_\epsilon(\varphi(u_1^+)-\varphi(u_2^+))\xi dxdy, \end{cases} \quad (21)$$

since $1 = (1 - H_0(u_2^-)) + H_0(u_2^-)$ and $1 - H_0(u_2^-) = 0$ when $u_2 < 0$.
Using (21) in (20), we get

$$\begin{cases} \int_{I\times I} H_0(u_1^+ - u_2^+)\left\{h_2^+(\xi_x+\xi_y) + (b(u_2^+) - (1-H_0(u_2^-))f_2)\xi\right\} dxdy \geq \\ -\int_{I\times I} H_0(u_2^-)\left\{h_2(\xi_x+\xi_y) + (b(u_2) - f_2)\xi\right\} dxdy + \int_{I\times I} H_0(u_2^-)a(u_1^+,0)\xi_y dxdy \\ + \int_{I\times I} H_0(u_1^+ - u_2^+)(1-H_0(u_2^-))a(u_1^+,0)\xi_y dxdy + \int_{I\times I} H_0(u_1^+ - u_2^+)h_2^+\xi_x dxdy \\ + \int_{I\times I} H_0(u_2^-)h_2\xi_x dxdy + \lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} \left(h_2^+ - a(u_1^+,0)\right)\varphi(u_2^+)_y H_\epsilon'(\varphi(u_1^+) - \varphi(u_2^+))\xi dxdy \\ + \lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} H_0(u_2^-)\left(h_2 - a(u_1^+,0)\right)\varphi(u_2)_y H_\epsilon'(\varphi(u_1^+) - \varphi(u_2))\xi dxdy. \end{cases} \tag{22}$$

Making now the subtraction (18)-(22),we obtain

$$\begin{cases} \int_{I\times I} H_0(u_1^+ - u_2^+)\left\{(h_1^+ - h_2^+)(\xi_x+\xi_y) + (b(u_1^+) - b(u_2^+))\xi - (f_1 - (1-H_0(u_2^-))f_2)\xi\right\} \\ \leq \int_{I\times I} H_0(u_1^+ - u_2^+)a(u_2^+,0)\xi_x dxdy + \int_{I\times I} H_0(u_1^+ - u_2^+)h_1^+\xi_y dxdy \\ - \lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2^+)\times I} \left(h_1^+ - a(u_2^+,0)\right)\varphi(u_1^+)_x H_\epsilon'(\varphi(u_1^+) - \varphi(u_2^+))\xi dxdy \\ + \int_{I\times I} H_0(u_2^-)\left\{h_2(\xi_x+\xi_y) + (b(u_2) - f_2)\xi\right\} dxdy - \int_{I\times I} H_0(u_2^-)a(u_1^+,0)\xi_y dxdy \\ - \int_{I\times I} H_0(u_1^+ - u_2^+)(1-H_0(u_2^-))a(u_1^+,0)\xi_y dxdy - \int_{I\times I} H_0(u_1^+ - u_2^+)h_2^+\xi_x dxdy \\ - \int_{I\times I} H_0(u_2^-)h_2\xi_x dxdy - \lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} \left(h_2^+ - a(u_1^+,0)\right)\varphi(u_2^+)_y H_\epsilon'(\varphi(u_1^+) - \varphi(u_2^+))\xi dxdy \\ - \lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} H_0(u_2^-)\left(h_2 - a(u_1^+,0)\right)\varphi(u_2)_y H_\epsilon'(\varphi(u_1^+) - \varphi(u_2))\xi dxdy. \end{cases} \tag{23}$$

From (23), we have

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1^+ - u_2^+)\left\{(h_1^+ - h_2^+)(\xi_x + \xi_y) + (b(u_1^+) - b(u_2^+))\xi - (f_1 - (1 - H_0(u_2^-))f_2)\xi\right\} \\ \displaystyle\le -\int_{I\times I} H_0(u_1^+ - u_2^+)[h_2^+ - a(u_2^+,0)]\xi_x dxdy + \int_{I\times I} H_0(u_1^+ - u_2^+)[h_1^+ - a(u_1^+,0)]\xi_y dxdy \\ \displaystyle- \lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2^+)\times I} \left(h_1^+ - a(u_2^+,0)\right)\varphi(u_1^+)_x H_\epsilon'(\varphi(u_1^+) - \varphi(u_2^+))\xi dxdy \\ \displaystyle- \lim\nolimits_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} \left(h_2^+ - a(u_1^+,0)\right)\varphi(u_2^+)_y H_\epsilon'(\varphi(u_1^+) - \varphi(u_2^+))\xi dxdy \\ \displaystyle+ \int_{I\times I} H_0(u_2^-)\left\{h_2(\xi_x + \xi_y) + (b(u_2) - f_2)\xi\right\} dxdy - \int_{I\times I} H_0(u_2^-)a(u_1^+,0)\xi_y dxdy \\ \displaystyle+ \int_{I\times I} H_0(u_1^+ - u_2^+)H_0(u_2^-)a(u_1^+,0)\xi_y dxdy - \int_{I\times I} H_0(u_2^-)h_2\xi_x dxdy \\ \displaystyle- \lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} H_0(u_2^-)\left(h_2 - a(u_1^+,0)\right)\varphi(u_2)_y H_\epsilon'(\varphi(u_1^+) - \varphi(u_2))\xi dxdy. \end{cases} \tag{24}$$

Note the term

$$J = \begin{cases} \displaystyle -\int_{I\times I} H_0(u_1^+ - u_2^+)[h_2^+ - a(u_2^+,0)]\xi_x dxdy + \int_{I\times I} H_0(u_1^+ - u_2^+)[h_1^+ - a(u_1^+,0)]\xi_y dxdy \\ \displaystyle- \lim_{\epsilon\longrightarrow 0}\int_{(I\setminus O_2^+)\times I} \left(h_1^+ - a(u_2^+,0)\right)\varphi(u_1^+)_x H_\epsilon'(\varphi(u_1^+) - \varphi(u_2^+))\xi dxdy \\ \displaystyle- \lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} \left(h_2^+ - a(u_1^+,0)\right)\varphi(u_2^+)_y H_\epsilon'(\varphi(u_1^+) - \varphi(u_2^+))\xi dxdy \end{cases}$$

and

$$K = -\lim_{\epsilon\longrightarrow 0}\int_{I\times(I\setminus O_1^+)} H_0(u_2^-)\left(h_2 - a(u_1^+,0)\right)\varphi(u_2)_y H_\epsilon'(\varphi(u_1^+) - \varphi(u_2))\xi dxdy.$$

As in the proof of theorem 7 for $I \le 0$, we prove that $J \le 0$ and $K \le 0$. Consider now the term

$$L = \int_{I\times I} H_0(u_1^+ - u_2^+)H_0(u_2^-)a(u_1^+,0)\xi_y dxdy - \int_{I\times I} H_0(u_2^-)a(u_1^+,0)\xi_y dxdy.$$

It is easy to show that $L = 0$. Then, we obtain from (24)

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1^+ - u_2^+)\left\{(h_1^+ - h_2^+)(\xi_x + \xi_y) + (b(u_1^+) - b(u_2^+))\xi - (f_1 - (1 - H_0(u_2^-))f_2)\xi\right\} \\ \displaystyle\le \int_{I\times I} H_0(u_2^-)\left\{h_2\xi_y + (b(u_2) - f_2)\xi\right\} dxdy \end{cases} \tag{25}$$

for any nonnegative $\xi \in D(\overline{I} \times I)$.

Now let $\xi \in D(\mathbb{R})$, $\xi \ge 0$ such that $supp(\xi) \subset B$ where B is an interval satisfying the property:

$$\begin{cases} \text{either } B \cap \partial I = \emptyset \\ \text{or } B \subset\subset B' \cap \partial I \text{ is a part of the graph} \\ \text{of a Lipschitz continuous function .} \end{cases} \tag{26}$$

Then there exists a sequence of mollifiers ρ_n defined in $\mathbb{R}$ such that, for n large enough,

$$y \mapsto \rho_n(\frac{x-y}{2}) \in D(I) \quad \forall x \in B, \tag{27}$$

$$\chi_n(y) = \int_I \rho_n(\frac{x-y}{2})dx \text{ is an increasing sequence for } y \in B, \tag{28}$$

$$\chi_n(y) = 1 \text{ for any } y \in B \text{ such that } d(y, \mathbb{R} \setminus I) > c/n \tag{29}$$

where c is a positive constant depending on B. Then we can define the nonnegative function

$$\xi_n(x,y) = \xi(y)\rho_n(\frac{x-y}{2}) \in D(\overline{I} \times I).$$

By substituting ξ_n into (25), we obtain since $(\partial_x + \partial_y)\rho_n(\dfrac{x-y}{2}) = 0$

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1^+ - u_2^+)\left\{(h_1^+ - h_2^+)(\partial_x + \partial_y)\xi + (b(u_1^+) - b(u_2^+))\xi - (f_1 - (1 - H_0(u_2^-))f_2)\xi\right\}\rho_n \\ \displaystyle\le \int_{I\times I} H_0(u_2^-)\left\{h_2(\xi\rho_n)_y + (b(u_2) - f_2)(\xi\rho_n)\right\}dxdy \end{cases}$$

and therefore

$$\begin{cases} \displaystyle\int_{I\times I} H_0(u_1^+ - u_2^+)\left\{(h_1^+ - h_2^+)(\partial_x + \partial_y)\xi + (b(u_1^+) - b(u_2^+))\xi - (f_1 - (1 - H_0(u_2^-))f_2)\xi\right\}\rho_n \\ \displaystyle\le \int_I H_0(u_2^-)\left\{h_2(\xi\chi_n)_y + (b(u_2) - f_2)(\xi\chi_n)\right\}dy \end{cases} \tag{30}$$

where $\xi = \xi(y)$. When $n \to +\infty$, the integral at the left side of (30) converges to

$$\int_I H_0(u_1^+ - u_2^+)\left\{(h_1^+ - h_2^+)\xi_x + (b(u_1^+) - b(u_2^+))\xi - (f_1 - (1 - H_0(u_2^-))f_2)\xi\right\}dx \tag{31}$$

and then, any function in the integrand of this integral is now considered as a function of the variable x.
The integral of the right of (30) is nonnegative: In fact, u_2 is an entropy solution and therefore, from (b) we deduce that $M : D(\overline{I}) \mapsto \mathbb{R}$ defined by

$$\xi \mapsto M(\xi) = \int_I H_0(u_2^-)\left\{h_2\xi_x + (b(u_2) - f_2)\xi\right\}dx$$

is monotone. In particular, since $\xi\chi_n$ is an increasing sequence satisfying

$$0 \le \xi\chi_n \le \xi,$$

we deduce that $M(\xi\chi_n)$ is a bounded and increasing sequence and, therefore it converges. Then, from (30) and (31) we deduce that

$$\begin{cases} \displaystyle\int_I H_0(u_1^+ - u_2^+)\left\{(h_1^+ - h_2^+)\xi_x + (b(u_1^+) - b(u_2^+))\xi - (f_1 - (1 - H_0(u_2^-))f_2)\xi\right\}dx \\ \displaystyle\le \lim_{n\to+\infty}\int_I H_0(u_2^-)\left\{h_2(\xi\chi_n)_x + (b(u_2) - f_2)(\xi\chi_n)\right\}dx \end{cases} \tag{32}$$

for any nonnegative $\xi \in D(B)$.
Now in view of remark 2 and remark 4, inequality (32) is still true when u_1 is replaced

by $-u_2$, u_2 by $-u_1$, f_1 by $-f_2$, f_2 by $-f_1$, b by $\widetilde{b}$, φ by $\widetilde{\varphi}$ and a by $\widetilde{a}$. Then we have

$$\left\{\begin{array}{l}\displaystyle\int_I H_0((-u_1^-)-(-u_2^-))\left\{(h_1^- - h_2^-)\xi_x + (b(-u_1^-)-b(-u_2^-))\xi - ((1-H_0(u_1^+))f_1 - f_2)\xi\right\}\\ \displaystyle\leq -\lim_{n\to+\infty}\int_I H_0(u_1^+)\left\{h_1^+(\xi\chi_n)_x + (b(u_1^+)-f_1)(\xi\chi_n)\right\}dx\end{array}\right. \tag{33}$$

where $h_1^- = a(-u_1^-, \varphi(-u_1^-)_x)$ and $h_2^- = a(-u_2^-, \varphi(-u_2^-)_x)$.
It is easy to check that:

$$H_0((-u_1^-)-(-u_2^-))(1-H_0(u_1^+))H_0(u_2^-) + H_0(u_1^+ - u_2^+)H_0(u_1^+) = H_0(u_1-u_2), \tag{34}$$

$$H_0((-u_1^-)-(-u_2^-))H_0(u_2^-) + H_0(u_1^+ - u_2^+)(1-H_0(u_2^-))H_0(u_1^+) = H_0(u_1-u_2), \tag{35}$$

$$H_0(u_1^+ - u_2^+) = H_0(u_1^+ - u_2^+)H_0(u_1^+), \tag{36}$$

$$H_0((-u_1^-)-(-u_2^-)) = H_0((-u_1^-)-(-u_2^-))H_0(u_2^-), \tag{37}$$

$$H_0(u_1^+ - u_2^+)(h_1^+ - h_2^+) + H_0((-u_1^-)-(-u_2^-))(h_1^- - h_2^-) = H_0(u_1-u_2)(h_1-h_2) \tag{38}$$

and

$$\left\{\begin{array}{l}H_0(u_1^+ - u_2^+)(b(u_1^+)-b(u_2^+)) + H_0((-u_1^-)-(-u_2^-))(b(-u_1^-)-b(-u_2^-))\\ = H_0(u_1-u_2)(b(u_1)-b(u_2)).\end{array}\right. \tag{39}$$

Then adding (32) and (33) by using (34), (35), (36), (37), (38) and (39), we obtain

$$\left\{\begin{array}{l}\displaystyle\int_I H_0(u_1-u_2)\left\{(h_1-h_2)\xi_x + (b(u_1)-b(u_2))\xi - (f_1-f_2)\xi\right\}dx\\ \displaystyle\leq \lim_{n\to+\infty}\int_I H_0(u_2^-)\left\{h_2(\xi\chi_n)_x + (b(u_2)-f_2)(\xi\chi_n)\right\}dx\\ \displaystyle-\lim_{n\to+\infty}\int_I H_0(u_1^+)\left\{h_1^+(\xi\chi_n)_x + (b(u_1^+)-f_1)(\xi\chi_n)\right\}dx\end{array}\right. \tag{40}$$

for any nonnegative $\xi \in D(B)$.
Let ξ be nonnegative function of $D(B)$. Then $\xi\chi_{n'} \in D^+(B)$ for n' large enough, and from inequality (3) we have

$$\left\{\begin{array}{l}\displaystyle\int_I H_0(u_1-u_2)\left\{(h_1-h_2)(\xi\chi_{n'})_x + (b(u_1)-b(u_2))\xi\chi_{n'}\right\}dx-\\ \displaystyle-\int_I H_0(u_1-u_2)(f_1-f_2)\xi\chi_{n'}dx \leq 0\end{array}\right.$$

and therefore

$$\begin{cases} \int_I H_0(u_1-u_2)\left\{(h_1-h_2)\xi_x+(b(u_1)-b(u_2))\xi\right\}dx- \\ -\int_I H_0(u_1-u_2)(f_1-f_2)\xi dx \leq \\ \int_I H_0(u_1-u_2)\left\{(h_1-h_2)(\xi(1-\chi_{n'}))_x+(b(u_1)-b(u_2))\xi(1-\chi_{n'})\right\}dx- \\ -\int_I H_0(u_1-u_2)(f_1-f_2)\xi(1-\chi_{n'})dx, \end{cases} \tag{41}$$

and since $\xi(1-\chi_{n'})$ is nonnegative function of $D(B)$, from (40) we have

$$\begin{cases} \int_I H_0(u_1-u_2)\left\{(h_1-h_2)(\xi(1-\chi_{n'}))_x+(b(u_1)-b(u_2))\xi(1-\chi_{n'})\right\}dx- \\ -\int_I H_0(u_1-u_2)(f_1-f_2)\xi(1-\chi_{n'})dx \leq \\ \lim_{n\to+\infty}\int_I H_0(u_2^-)\left\{h_2(\xi(1-\chi_{n'})\chi_n)_x+(b(u_2)-f_2)(\xi(1-\chi_{n'})\chi_n)\right\}dx \\ -\lim_{n\to+\infty}\int_I H_0(u_1^+)\left\{h_1^+(\xi(1-\chi_{n'})\chi_n)_x+(b(u_1^+)-f_1)(\xi(1-\chi_{n'})\chi_n)\right\}dx. \end{cases} \tag{42}$$

Now since $Supp(\xi\chi_{n'})\subset I\cap B$, from (29) we deduce that for n large enough we have

$$\xi\chi_{n'}\chi_n=\xi\chi_{n'}.$$

Then we have

$$\begin{cases} \lim_{n\to+\infty}\int_I H_0(u_2^-)\left\{h_2(\xi(1-\chi_{n'})\chi_n)_x+(b(u_2)-f_2)(\xi(1-\chi_{n'})\chi_n)\right\}dx \\ -\lim_{n\to+\infty}\int_I H_0(u_1^+)\left\{h_1^+(\xi(1-\chi_{n'})\chi_n)_x+(b(u_1^+)-f_1)(\xi(1-\chi_{n'})\chi_n)\right\}dx= \\ \lim_{n\to+\infty}\int_I H_0(u_2^-)\left\{h_2(\xi\chi_n)_x+(b(u_2)-f_2)(\xi\chi_n)\right\}dx \\ -\lim_{n\to+\infty}\int_I H_0(u_1^+)\left\{h_1^+(\xi\chi_n)_x+(b(u_1^+)-f_1)(\xi\chi_n)\right\}dx- \\ \int_I H_0(u_2^-)\left\{h_2(\xi\chi_{n'})_x+(b(u_2)-f_2)(\xi\chi_{n'})\right\}dx+ \\ \int_I H_0(u_1^+)\left\{h_1^+(\xi\chi_{n'})_x+(b(u_1^+)-f_1)(\xi\chi_{n'})\right\}dx. \end{cases} \tag{43}$$

From (41), (42) and (43), we deduce that

$$\begin{cases} \int_I H_0(u_1-u_2)\left\{(h_1-h_2)\xi_x+(b(u_1)-b(u_2))\xi\right\}dx - \int_I H_0(u_1-u_2)(f_1-f_2)\xi dx \leq \\ \lim_{n\to+\infty}\int_I H_0(u_2^-)\left\{h_2(\xi\chi_n)_x+(b(u_2)-f_2)(\xi\chi_n)\right\}dx \\ -\lim_{n\to+\infty}\int_I H_0(u_1^+)\left\{h_1^+(\xi\chi_n)_x+(b(u_1^+)-f_1)(\xi\chi_n)\right\}dx- \\ \int_I H_0(u_2^-)\left\{h_2(\xi\chi_{n'})_x+(b(u_2)-f_2)(\xi\chi_{n'})\right\}dx+ \\ \int_I H_0(u_1^+)\left\{h_1^+(\xi\chi_{n'})_x+(b(u_1^+)-f_1)(\xi\chi_{n'})\right\}dx. \end{cases}$$

We check easily that the right and side of this inequality converges to 0 when $n'\to+\infty$, and therefore

$$\begin{cases} \int_I H_0(u_1-u_2)\left\{(h_1-h_2)\xi_x+(b(u_1)-b(u_2))\xi\right\}dx \\ \leq \int_I H_0(u_1-u_2)(f_1-f_2)\xi dx \end{cases} \tag{44}$$

for any nonnegative $\xi\in D(B)$.
Finally, let ξ be a nonnegative function of $D(\overline{I})$, let $I_0\subset\subset I$ be such that

$$I\subset I_0\cup(\cup_{j=1}^k B_j)$$

where the B_j are intervals satisfying (26), let $(\psi_j)_{j=0}^k$ be a partition of unity related to the above covering of I, and let $\xi_j=\xi\psi_j$ for $0\leq j\leq k$. Then by applying (3) for j=0 and (44) for $1\leq j\leq k$, we have

$$\begin{cases} \int_I H_0(u_1-u_2)\left\{(h_1-h_2)(\xi_j)_x+(b(u_1)-b(u_2))\xi_j\right\}dx \\ \leq \int_I H_0(u_1-u_2)(f_1-f_2)\xi_j dx \end{cases}$$

for $0\leq j\leq k$ and therefore

$$\begin{cases} \int_I H_0(u_1-u_2)\left\{(h_1-h_2)\xi_x+(b(u_1)-b(u_2))\xi\right\}dx \\ \leq \int_I H_0(u_1-u_2)(f_1-f_2)\xi dx \end{cases}$$

for any nonnegative $\xi\in D(\overline{I})$, which achieves the proof.
As a consequence of (17) we have

Corollary 2.1. *Let $f_1\in L^\infty(I)$, $f_2\in L^\infty(I)$. Let u_1, u_2 be entropy solutions with respect to $(SP)(b,a,\varphi,f_1)$, $(SP)(b,a,\varphi,f_2)$ respectively. Then*

$$\int_I (b(u_1)-b(u_2))^+dx\leq\int_I H_0(u_1-u_2)(f_1-f_2)dx\leq\int_I(f_1-f_2)^+dx, \tag{45}$$

and

$$\|b(u_1) - b(u_2)\|_{L^1(I)} \leq \|f_1 - f_2\|_{L^1(I)} . \quad (46)$$

Moreover, if $f_1 \leq f_2$ *almost everywhere in* I*, then*

$$b(u_1) \leq b(u_2) \text{ a.e. in } I. \quad (47)$$

and if $f_1 = f_2$ *almost everywhere in* I*, then*

$$b(u_1) = b(u_2) \text{ a.e. in } I. \quad (48)$$

Proof. Let $\xi = 1$ in the inequality (17). Then we get (45). From (45) we deduce (46), (47) and (48).

Remark 2.4. *The notion of uniqueness of solutions is the uniqueness of* $b(u)$ *due to possible degeneracy of* b*. On the other hand if* b *is strictly increasing, then the uniqueness of the solution* u *is equivalent to the uniqueness of the function* $b(u)$*.*

3. Existence of entropy solutions

In this part, we study the existence of entropy solution of the stationary problem $(SP)(b, a, \varphi, f)$.

Theorem 3.1. *(Existence of entropy solution). For all* $f \in L^\infty(I)$*, the stationary problem* $(SP)(b, a, \varphi, f)$ *has at least one entropy solution.*

Proof:

Step 1. We consider the following stationary problem $(SP)_{nm}$=$(SP)(b_{n,m}, a, \varphi, f)$

$$\begin{cases} b_{n,m}(u_{n,m}) - a(u_{n,m}, \varphi(u_{n,m})_x)_x = f & \text{on } I \\ \\ u_{n,m} = 0 & \text{on } \partial I \end{cases}$$

where $b_{n,m}(\sigma) = b(\sigma) + \frac{1}{n}\sigma^+ - \frac{1}{m}\sigma^-$ for all $\sigma \in \mathbb{R}$, $n, m \in \mathbb{N}^*$, $f^+ = sup(f, 0)$, $f^- = sup(-f, 0)$.

Remark 3.1. *Since the functions* $v \mapsto v^+$ *and* $v \mapsto v^-$ *are continuous, then* $b_{n,m}$ *is continuous. However* $b_{n,m}$ *is strictly increasing for* m, n *fixed.*

We establish now some preliminary results.

Lemma 3.1. *Let* $f \in L^\infty(I)$*, then:*
i) $\left\|b_{n,m}(u_{n,m})\right\|_p \leq \|f\|_p$ *for all* $1 \leq p \leq +\infty$*,*
ii) $\|u_{n,m}\|_\infty \leq C(b, \|f\|_\infty)$*;*
where C *is a positive constant.*

Proof. Let $p : \mathbb{R} \to \mathbb{R}$ a continuous function with $p' \geq 0$, $p(0) = 0$, p lipschitz function with compact support i.e $p \in P_0$.
We multiply $(SP)_{n,m}$ by $p(b_{n,m}(u_{n,m}))$ and integrate on I:

$$\begin{cases} \displaystyle\int_I p(b_{n,m}(u_{n,m}))b_{n,m}(u_{n,m})dx + \int_I a(u_{n,m}, \varphi(u_{n,m})_x)p(b_{n,m}(u_{n,m}))_x dx \\ \\ \displaystyle= \int_I fp(b_{n,m}(u_{n,m}))dx. \end{cases} \quad (49)$$

Consider the term

$$\int_I a(u_{n,m}, \varphi(u_{n,m})_x) p(b_{n,m}(u_{n,m}))_x dx. \tag{50}$$

By regularisation, one can always suppose that $b_{n,m}$ and $u_{n,m}$ are regular, and then we obtain from (50),

$$\int_I b'_{n,m}(u_{n,m}) p'(b_{n,m}(u_{n,m})) a(u_{n,m}, \varphi(u_{n,m})_x)(u_{n,m})_x dx. \tag{51}$$

$$(51) = \begin{cases} \displaystyle\int_{I\setminus E} b'_{n,m}(u_{n,m}) p'(b_{n,m}(u_{n,m}))[a(u_{n,m}, \varphi(u_{n,m})_x) - a(u_{n,m}, 0)](u_{n,m})_x dx \\ \displaystyle + \int_I b'_{n,m}(u_{n,m}) p'(b_{n,m}(u_{n,m})) a(u_{n,m}, 0)(u_{n,m})_x dx. \end{cases}$$

We know that $p' \geq 0$, $b'_{n,m} > 0$, $\varphi' > 0$ on $I \setminus E$ (by regularisation) and a is nondecreasing with respect to the second variable, then

$$\int_{I\setminus E} b'_{n,m}(u_{n,m}) p'(b_{n,m}(u_{n,m}))[a(u_{n,m}, \varphi(u_{n,m})_x) - a(u_{n,m}, 0)](u_{n,m})_x dx \geq 0$$

and by divergence theorem

$$\int_I b'_{n,m}(u_{n,m}) p'(b_{n,m}(u_{n,m})) a(u_{n,m}, 0)(u_{n,m})_x dx = \int_I div F_{n,m}(u_{n,m}) dx = 0$$

where $F_{n,m}(z) = \displaystyle\int_0^z p'(b_{n,m}(s)) b'_{n,m}(s) a(s, 0) ds$.
Consequently, one has from (49):

$$\int_I p(b_{n,m}(u_{n,m})) b_{n,m}(u_{n,m}) dx \leq \int_I f p(b_{n,m}(u_{n,m})) dx.$$

By approximation of p, one can takes $p(u) = |u|^{p-2} u$ $(1 \leq p < \infty)$, then we obtain from inequality above:

$$\begin{cases} \displaystyle\int_I |b_{n,m}(u_{n,m})|^{p-2} (b_{n,m}(u_{n,m}))^2 dx \leq \int_I f |b_{n,m}(u_{n,m})|^{p-2} (b_{n,m}(u_{n,m})) dx \\ \displaystyle\leq \left(\int_I (|b_{n,m}(u_{n,m})|^{p-1})^q dx\right)^{\frac{1}{q}} \left(\int_I |f|^p dx\right)^{\frac{1}{p}} \end{cases}$$

where $\frac{1}{p} + \frac{1}{q} = 1$; this inequality implies that

$$\left\|b_{n,m}(u_{n,m})\right\|_p \leq \|f\|_p \text{ for all } 1 \leq p < +\infty.$$

Since $f \in L^\infty(I)$, then by the preceding relation, we have also $\left\|b_{n,m}(u_{n,m})\right\|_\infty \leq \|f\|_\infty$; from where the proof of i).

For the proof of ii), we multiply $b_{n,m}(u_{n,m})$ by $p(b(u_{n,m}))$ where $p \in P_0$ and integrate on I, one has:

$$\begin{cases} \displaystyle\int_I p(b(u_{n,m}))b_{n,m}(u_{n,m})dx = \int_I p(b(u_{n,m}))b(u_{n,m})dx + \\ \displaystyle\frac{1}{n}\int_I (u_{n,m})^+ p(b(u_{n,m}))dx - \frac{1}{m}\int_I (u_{n,m})^- p(b(u_{n,m}))dx. \end{cases}$$

Now,

$$\begin{cases} \displaystyle\frac{1}{n}\int_I (u_{n,m})^+ p(b(u_{n,m}))dx - \frac{1}{m}\int_I (u_{n,m})^- p(b(u_{n,m}))dx = \\ \displaystyle\frac{1}{n}\int_{I\cap\{u_{n,m}\geq 0\}} (u_{n,m})^+ p(b(u_{n,m}))dx + \frac{1}{m}\int_{I\cap\{u_{n,m}\leq 0\}} (-(u_{n,m})^-)p(b(u_{n,m}))dx \geq 0. \end{cases}$$

This inequality implies that

$$\int_I p(b(u_{n,m}))b_{n,m}(u_{n,m})dx \geq \int_I p(b(u_{n,m}))b(u_{n,m})dx.$$

By approximation of p, we can take $p(u) = |u|^{p-2}\, u$ $(1 \leq p < +\infty)$, then we obtain from inequality above:

$$\|b(u_{n,m})\|_p \leq \|b_{n,m}(u_{n,m})\|_p \text{ for all } 1 \leq p < +\infty.$$

Thanks to i) and the fact that b is surjective, we obtain ii).

Lemma 3.2. *The sequence* $(u_{n,m})$ *converge when* $n, m \to +\infty$.

Proof. $(SP)_{nm} \Leftrightarrow b(u_{n,m}) - a(u_{n,m}, \varphi(u_{n,m})_x)_x = f^{n,m}$
where $f^{n,m} = f - \frac{1}{n}(u_{n,m})^+ + \frac{1}{m}(u_{n,m})^-$. $f^{n,m}$ is uniformely bounded on I due to lemma 13, then using comparison result (corollary 9), one has

$$\int_I H_0(u_{n,m} - u_{n,m'})(b(u_{n,m}) - b(u_{n,m'}))dx \leq \int_I H_0(u_{n,m} - u_{n,m'})(f^{n,m} - f^{n,m'})dx,$$

for $m, m' \in \mathbb{N}$ and for fixed $n \in \mathbb{N}$. Inequality above is equivalent to

$$\begin{cases} \displaystyle\int_I H_0(u_{n,m} - u_{n,m'})(b(u_{n,m}) - b(u_{n,m'}))dx \leq \\ \displaystyle\int_I H_0(u_{n,m} - u_{n,m'}) \left[\left(\frac{1}{n}(u_{n,m'})^+ - \frac{1}{m'}(u_{n,m'})^-\right) - \left(\frac{1}{n}(u_{n,m})^+ - \frac{1}{m}(u_{n,m})^-\right)\right] dx; \end{cases}$$

which is equivalent to

$$\int_I (b_{n,m}(u_{n,m}) - b_{n,m'}(u_{n,m'}))^+ dx \leq 0. \tag{52}$$

From (52), we deduce that

$$b_{n,m}(u_{n,m}) - b_{n,m'}(u_{n,m'}) \leq 0.$$

From the inequality above, we deduce when $m' \leq m$ that

$$b(u_{n,m}) - b(u_{n,m'}) + \frac{1}{n}\left((u_{n,m})^+ - (u_{n,m'})^+\right) + \frac{1}{m}\left((u_{n,m'})^- - (u_{n,m})^-\right) \leq 0.$$

It is easy to see that the three terms of the inequality above have the same sign, then they are negatives, which implies that $u_{n,m} \leq u_{n,m'}$ for $m' \leq m$ and n fixed.
Then $(u_{n,m})_m$ is nonincreasing. By the same method, we show that $(u_{n,m})_n$ is nonincreasing.
Since $(u_{n,m})$ is uniformly bounded then we deduce that

$$u_{n,m} \downarrow u_n \text{ when } m \to +\infty \text{ and } u_n \downarrow \underline{u} \text{ when } n \to +\infty.$$

Step 2. One has $h_{n,m} \in C(I)$ so that $h_{n,m}$ is finite on some fixed points of I; this implies that $h_{n,m} \in AC(I)$ since $f - b_{n,m}(u_{n,m}) \in L^1(I)$. Consequently, $h_{n,m}$ is uniformly bounded. Using (H_1), we deduce that $\varphi(u_{n,m})_x$ is uniformly bounded in $L^\infty(I)$ since $u_{n,m}$ is uniformly bounded in $L^\infty(I)$. We get then $\varphi(u_{n,m})$ uniformly bounded in $W^{1,\infty}(I)$.
Consequently, one has:

$$\begin{cases} u_{n,m} \overset{n,m\to\infty}{\longrightarrow} u \in L^\infty(I) \\ \\ b_{n,m}(u_{n,m}) \overset{n,m\to\infty}{\longrightarrow} b(u) \in L^\infty(I) \\ \\ \varphi(u_{n,m}) \overset{n,m\to\infty}{\rightharpoonup} \varphi(u) \in W^{1,\infty}(I). \end{cases} \tag{53}$$

Now, we Know by Bnilan and Tour work in 1994 (see [BT]) that $u_{n,m}$ is an entropy solution of $(SP)_{n,m}$ since $b_{n,m}$ does not degenerate; then it checks the following entropy inequalities:

$$(a_{nm}) \int_I H_0(u_{n,m} - k)\left\{(H(k) - h_{n,m})\xi_x + (f - b_{n,m}(u_{n,m}))\xi\right\} dx \geq 0$$
$$(b_{nm}) \int_I H_0(k - u_{n,m})\left\{(H(k) - h_{n,m})\xi_x + (f - b_{n,m}(u_{n,m}))\xi\right\} dx \leq 0,$$

now, passing to the limit when $n, m \to \infty$ in the two inequalities by using (53) and a pseudomonotony argument (see [L]), we obtain that there exists $\alpha_1 \in L^\infty(\mathbb{R})$, $\alpha_2 \in L^\infty(\mathbb{R})$, $\alpha_1 \in H^+(u - k)$, $\alpha_2 \in H^+(k - u)$ such that:

$$\int_I \alpha_1 \left\{(H(k) - h)\xi_x + (f - b(u))\xi\right\} dx \geq 0$$

and

$$\int_I \alpha_2 \left\{(H(k) - h)\xi_x + (f - b(u))\xi\right\} dx \leq 0,$$

which are equivalent to the entropy inequalities (see $[Bp], [T]$).

4. The A_b Operator associated

In this part, we define an operator associated to the evolution problem (CP) and show some useful properties of this operator. We define the operator A_b in $L^1(I)$ by:

$$\begin{cases} v \in A_b b(u) \text{ if and only if } b(u) \in L^1(I), v \in L^\infty(I) \text{and } u \text{ is entropy} \\ \\ \text{ solution of the stationary problem } (SP) \text{ with } f = v + b(u). \end{cases}$$

One show that
$A_b \subset \Big\{(b(u); -a(u, \varphi(u)_x)_x); \varphi(u) \in W_0^{1,\infty}(I), u \in L_0^\infty(I), b(u) \in L^1(I)\Big\}.$

We have the following result

Proposition 4.1. *The operator A_b defined above satisfies:*
(1) A_b *is* $T-$ *accretive in* $L^1(I)$.
(2) *For all* $\lambda > 0$, *the range* $R(I+\lambda A_b)$ *of* $I+\lambda A_b$ *is dense in* $L^1(I)$.
(3) *The domain* $D(A_b)$ *of* A_b *is dense in* $L^1(I)$.

Proof. The T-accretivity is a direct consequence of corollary 9.
According to theorem 11, $R(I+\lambda A_b) = L^\infty(I)$; which implies that $R(I+\lambda A_b)$ is dense in $L^1(I)$.
We have then to prove 3.
Let $D_0 = \{v \in L^1(I)/;\ \text{there exists } u \in L^\infty(I),\ \text{with } v = b(u)\}$.
$D(A_b) \subset D_0$. It is thus enough to prove that $\overline{D(A_b)} \supset D_0$ since D_0 is dense in $L^1(I)$.
Let $w_0 \in D_0$ and for $\lambda > 0$, consider the problem

$$b(u_\lambda) - \lambda a(u_\lambda, \varphi(u_\lambda)_x)_x = w_0$$

i.e

$$b(u_\lambda) - \lambda h_{\lambda,x} = w_0 \Leftrightarrow b(u_\lambda) = \lambda h_{\lambda,x} + w_0.$$

To prove that $b(u_\lambda) \longrightarrow w_0$ in $L^1(I)$ when $\lambda \to 0$, it is enough to prove that $\lambda h_{\lambda,x} \to 0$ in $D'(I)$.
Since $\lambda h_{\lambda,x}$ is bounded in $L^\infty(I)$, we have to prove only that

$$\lambda h_\lambda \to 0 \text{ in measure on } I \text{ for} \lambda \to 0. \tag{54}$$

To prove (54), consider $R > \|w_0\|_\infty$ and for $r > 0$, let

$$C(r) = \inf\left\{\frac{a(k,\xi).\xi}{|a(k,\xi)|}; |k| \le R, |a(k,\xi)| \ge r\right\}.$$

Using (H_1), we have

$$\lim_{r\to+\infty} C(r) = +\infty.$$

It is then clear that $C(r)$ is nondecreasing and $C(|h_\lambda|) \le \dfrac{a(u_\lambda, w_{\lambda,x})w_{\lambda,x}}{|a(u_\lambda, w_{\lambda,x})|}$, with $w_\lambda = \varphi(u_\lambda)$.
This implies that $C(|h_\lambda|)\,|h_\lambda| \le h_\lambda w_{\lambda,x}$. Let us fix now $\delta > 0$ and $r_0 > 0$ such that $C(r_0) \ge 0$. Let $\lambda > 0$ verifying $\lambda r_0 \le \delta$, one has:

$$C(\frac{\delta}{\lambda})\delta \le \lambda C(|h_\lambda|)\,|h_\lambda| \text{ if } \lambda\,|h_\lambda| > \delta.$$

The preceding inequality gives:

$$\int_{\{\lambda|h_\lambda|>\delta\}} C(\frac{\delta}{\lambda})\delta dx = C(\frac{\delta}{\lambda})\delta\,|\{\lambda\,|h_\lambda| > \delta\}| \le \int_{\{\lambda|h_\lambda|>\delta\}} \lambda C(|h_\lambda|)\,|h_\lambda|\,dx.$$

From the inequality above , we have

$$\int_{\{\lambda|h_\lambda|>\delta\}} C(\frac{\delta}{\lambda})\delta dx \le \int_{\{\lambda|h_\lambda|>\delta\}} \lambda h_\lambda w_{\lambda,x} dx;$$

which give then

$$\int_{\{\lambda|h_\lambda|>\delta\}} C(\frac{\delta}{\lambda})\delta dx \le \int_{\{|h_\lambda|>r_0\}} \lambda h_\lambda w_{\lambda,x} dx.$$

since $\{|h_\lambda| > r_0\} \supset \{\lambda\,|h_\lambda| > \delta\}$ (by using the fact that $\lambda r_0 \le \delta$).
So

$$\int_I \lambda h_\lambda w_{\lambda,x} dx = \int_{\{|h_\lambda|>r_0\}} \lambda h_\lambda w_{\lambda,x} dx + \int_{\{|h_\lambda|\le r_0\}} \lambda h_\lambda w_{\lambda,x} dx,$$

then

$$\begin{cases} \displaystyle\int_{\{\lambda|h_\lambda|>\delta\}} C(\frac{\delta}{\lambda})\delta dx \le \int_{\{|h_\lambda|>r_0\}} \lambda h_\lambda w_{\lambda,x} dx \le \\ \displaystyle\int_I \lambda h_\lambda w_{\lambda,x} dx - \int_{\{|h_\lambda|\le r_0\}} \lambda h_\lambda w_{\lambda,x} dx, \end{cases}$$

from the inequality above, we obtain

$$\begin{cases} \displaystyle\int_{\{\lambda|h_\lambda|>\delta\}} C(\frac{\delta}{\lambda})\delta dx \le \int_I \lambda h_\lambda w_{\lambda,x} dx - \int_{\{|h_\lambda|\le r_0\}} \lambda h_\lambda w_{\lambda,x} dx = \\ \displaystyle\int_I (w_0 - b(u_\lambda)) w_\lambda dx - \int_{\{|h_\lambda|\le r_0\}} \lambda h_\lambda w_{\lambda,x} dx, \end{cases}$$

from which, we get

$$\int_{\{\lambda|h_\lambda|>\delta\}} C(\frac{\delta}{\lambda})\delta dx \le \int_I (w_0 - b(u_\lambda)) w_\lambda dx - \int_{\{|h_\lambda|\le r_0\}} \lambda h_\lambda w_{\lambda,x} dx.$$

The inequality above is equivalent to

$$C(\frac{\delta}{\lambda})\delta\,|\{\lambda\,|h_\lambda| > \delta\}| \le \int_I (w_0 - b(u_\lambda)) w_\lambda dx - \int_{\{|h_\lambda|\le r_0\}} \lambda h_\lambda w_{\lambda,x} dx.$$

According to (H_1), we show that $C(\frac{\delta}{\lambda})\delta\,|\{\lambda\,|h_\lambda| > \delta\}|$ is bounded. Since $\lim_{r\to+\infty} C(r) = +\infty$, then necessarily

$$\lim_{\lambda\to 0} |\{\lambda\,|h_\lambda| > \delta\}| = 0.$$

Consequently, $b(u_\lambda) \to w_0$ in $L^1(I)$ for $\lambda \to 0$.

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