

Infinitely Many Small Solutions for a New Class of Sublinear-Quasilinear Schrödinger Equations

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ABSTRACT. We prove the existence of infinitely many small solutions for the quasilinear Schrödinger equation

$$-\Delta u + V(x)u + \tau \Delta \left(\sqrt{1 + u^2} \right) \frac{u}{2\sqrt{1 + u^2}} = W(x)f(x, u), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $\tau \geq 2$. Under significantly weakened hypotheses requiring only local boundedness of potentials, spatially varying growth conditions, and oscillation along sequences rather than uniformly, we prove the existence of a sequence of weak solutions converging to zero in $L^\infty(\mathbb{R}^N)$. Our approach combines variational methods with \mathbb{Z}_2 -genus theory and Moser iteration.

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1. Introduction

We consider the quasilinear Schrödinger equation

$$-\Delta u + V(x)u + \tau \Delta \left(\sqrt{1 + u^2} \right) \frac{u}{2\sqrt{1 + u^2}} = W(x)f(x, u), \quad x \in \mathbb{R}^N, \quad (1)$$

where $N \geq 3$, $\tau \geq 2$. This equation arises from seeking standing wave solutions $z(t, x) = e^{-iEt}u(x)$ of the time-dependent quasilinear Schrödinger equation

$$i\partial_t z = -\Delta z + K(x)z - l(x, |z|^2)z + \tau[\Delta \rho(|z|^2)]\rho'(|z|^2)z,$$

with $\rho(s) = \sqrt{1 + s}$. Such equations model various physical phenomena including plasma physics [10] and nonlinear optics [2].

In [8], the authors proved the existence of multiple small solutions under the following strong hypotheses:

- $V, W \in C(\mathbb{R}^N, \mathbb{R})$ with $0 < V_0 \leq V(x)$, $0 < W(x) \leq W_1$
- $|f(x, t)| \leq C_1|t|^{p-1}$ for $|t| \leq \delta$, with constant $C_1 > 0$, $1 \leq p < 2$
- $\liminf_{t \rightarrow 0} \inf_{x \in B_{r_0}(x_0)} \frac{F(x, t)}{t^2} > -\infty$
- $\limsup_{t \rightarrow 0} \inf_{x \in B_{r_0}(x_0)} \frac{F(x, t)}{t^2} = +\infty$, where $F(x, t) = \int_0^t f(x, s)ds$
- $Q(x) := W(x)^{2/(2-p)}V(x)^{p/(p-2)} \in L^1(\mathbb{R}^N)$

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These assumptions are too restrictive for many physical applications where potentials may have singularities, coefficients vary spatially, or oscillation occurs only locally.

Motivated by their work, in this paper we introduce significantly weakened assumptions that allow for more general physical models. Precisely, we introduce the following minimal assumptions:

- (A1) $V, W : \mathbb{R}^N \rightarrow \mathbb{R}$ are measurable functions. There exist constants $V_0, V_1 > 0$ such that $V_0 \leq V(x) \leq V_1$ for almost every $x \in \mathbb{R}^N$. $W(x) \geq 0$ almost everywhere, $W \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, and for some $r_1 > 0$,

$$\sup_{y \in \mathbb{R}^N} \int_{B_{r_1}(y)} W(x)^{\frac{2}{2-p}} dx < \infty.$$

- (A2) $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable in x for all t , and $t \mapsto f(x, t)$ is continuous for almost every $x \in \mathbb{R}^N$. $f(x, -t) = -f(x, t)$ for all $t \in \mathbb{R}$ and almost every $x \in \mathbb{R}^N$. There exist $\delta > 0$, $p \in [1, 2)$, and $C \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ such that

$$|f(x, t)| \leq C(x)|t|^{p-1} \quad \text{for all } |t| \leq \delta \text{ and almost every } x \in \mathbb{R}^N.$$

Moreover, for each $R > 0$, there exists $C_R > 0$ such that

$$\sup_{|t| \leq \delta} \int_{B_R(0)} |f(x, t)|^{\frac{2}{2-p}} dx \leq C_R.$$

- (A3) There exist $x_0 \in \mathbb{R}^N$, $r_0 > 0$, and sequences $\{\delta_n\} \subset (0, \delta)$, $\{M_n\} \subset \mathbb{R}^+$ with $\delta_n \rightarrow 0$, $M_n \rightarrow +\infty$, $M_n \delta_n^{2-p} \rightarrow \infty$. For each $n \in \mathbb{N}$, there exists a measurable set $E_n \subset B_{r_0}(x_0)$ with $|E_n| \geq \alpha_0 > 0$ such that

$$F(x, \delta_n) \geq M_n \delta_n^2 \quad \text{for all } x \in E_n.$$

There exists $\alpha > 0$, such that

$$F(x, t) \geq -\alpha t^2 \quad \text{for all } x \in B_{r_0}(x_0), |t| \leq \delta,$$

and $W(x) \geq W_0 > 0$ for almost every $x \in B_{r_0}(x_0)$.

Our main result is the following:

Theorem 1.1. *Assume hypotheses (A1)-(A3) hold. Then problem (1) possesses a sequence $\{u_n\}_{n=1}^\infty$ of weak solutions in the space*

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}$$

satisfying

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0 \quad \text{and} \quad I(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $I : X \rightarrow \mathbb{R}$ is the energy functional associated with (1).

2. Preliminary Results

Define the weighted Sobolev space

$$X = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 dx < \infty \right\}$$

with norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx \right)^{1/2}.$$

Since $V_0 \leq V(x) \leq V_1$, the norm $\|\cdot\|_X$ is equivalent to the standard H^1 norm. Therefore, X is a Hilbert space continuously embedded in $H^1(\mathbb{R}^N)$, and for any bounded domain $\Omega \subset \mathbb{R}^N$, the embedding $X \hookrightarrow L^s(\Omega)$ is compact for $1 \leq s < 2^*$.

Following [8], we introduce the auxiliary function $g : \mathbb{R} \rightarrow \mathbb{R}$:

$$g(s) = \begin{cases} \sqrt{1 - \frac{\tau s^2}{2(1+s^2)}}, & |s| < \frac{1}{\sqrt{\tau-1}}, \\ \frac{\sqrt{\tau-1}}{\sqrt{2\tau}|s|} + \frac{\sqrt{2}}{2\tau}, & |s| \geq \frac{1}{\sqrt{\tau-1}}, \end{cases}$$

extended as an even function. Let $G(t) = \int_0^t g(s)ds$ and denote its inverse by G^{-1} .

Lemma 2.1. *The inverse function G^{-1} satisfies:*

- (1) $\lim_{t \rightarrow 0} \frac{G^{-1}(t)}{t} = 1$
- (2) $\lim_{t \rightarrow +\infty} \frac{G^{-1}(t)}{t} = \sqrt{2\tau}$
- (3) $|t| \leq |G^{-1}(t)| \leq \sqrt{2\tau}|t|$ for all $t \in \mathbb{R}$
- (4) $-1 \leq \frac{t}{g(t)g'(t)} \leq 0$ for all $t \in \mathbb{R}$

Let $d \in C(\mathbb{R}, [0, 1])$ be an even cut-off function satisfying

$$d(t) = \begin{cases} 1, & |t| \leq \delta/2, \\ 0, & |t| \geq \delta, \end{cases} \quad \text{with } 0 \leq d(t) \leq 1 \text{ for all } t.$$

Define the truncated nonlinearity

$$\tilde{f}(x, t) = d(t)f(x, t), \quad \tilde{F}(x, t) = \int_0^t \tilde{f}(x, s)ds.$$

Consider the modified problem

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = W(x)\tilde{f}(x, u), \quad x \in \mathbb{R}^N. \quad (2)$$

The associated energy functional is

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} W(x)\tilde{F}(x, G^{-1}(v))dx.$$

Lemma 2.2. *Under hypotheses (A1)-(A2), $J \in C^1(X, \mathbb{R})$ and for all $\varphi \in X$,*

$$J'(v)\varphi = \int_{\mathbb{R}^N} \nabla v \cdot \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \varphi dx - \int_{\mathbb{R}^N} W(x) \frac{\tilde{f}(x, G^{-1}(v))}{g(G^{-1}(v))} \varphi dx.$$

If $v \in X$ is a critical point of J , then $u = G^{-1}(v) \in X$ is a weak solution of (2).

Proof. From (A2) and Lemma 2.1(3), $|\tilde{F}(x, G^{-1}(v))| \leq C(x)(\sqrt{2\tau})^p |v|^p/p$. By Hölder's inequality with exponents $\frac{2}{2-p}$ and $\frac{2}{p}$,

$$\int_{\mathbb{R}^N} W(x)|\tilde{F}(x, G^{-1}(v))|dx \leq \frac{(\sqrt{2\tau})^p}{p} \left(\int_{\mathbb{R}^N} (W(x)C(x))^{\frac{2}{2-p}} dx \right)^{\frac{2-p}{2}} \left(\int_{\mathbb{R}^N} |v|^2 dx \right)^{\frac{p}{2}}.$$

From (A1), $WC \in L^{\frac{2}{2-p}}_{\text{loc}}(\mathbb{R}^N)$, and by covering \mathbb{R}^N with balls of radius r_1 , the integral is finite for $v \in X$. Differentiability follows from standard arguments. \square

Lemma 2.3. *Under hypotheses (A1)-(A2), the functional J satisfies the Palais-Smale condition.*

Proof. Let $\{v_n\} \subset X$ satisfy $|J(v_n)| \leq M$ and $J'(v_n) \rightarrow 0$ in X^* . Consider $J(v_n) - \frac{1}{p}J'(v_n)v_n$:

$$J(v_n) - \frac{1}{p}J'(v_n)v_n = \left(\frac{1}{2} - \frac{1}{p}\right) \int |\nabla v_n|^2 dx + \frac{1}{2} \int V(x)|G^{-1}(v_n)|^2 dx - \frac{1}{p} \int V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n dx - \int W(x) \left(\tilde{F}(x, G^{-1}(v_n)) - \frac{1}{p} \frac{\tilde{f}(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n \right) dx.$$

Using Lemma 2.1(3) and the growth condition (A2), there exists $C_1 > 0$ such that

$$\left| \tilde{F}(x, G^{-1}(v_n)) - \frac{1}{p} \frac{\tilde{f}(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n \right| \leq C_1 C(x) |v_n|^p.$$

Thus,

$$M + \|v_n\|_X \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|v_n\|_X^2 - C_2 \int W(x)C(x)|v_n|^p dx.$$

Using Young's inequality with $\varepsilon > 0$:

$$W(x)C(x)|v_n|^p \leq \varepsilon V(x)|v_n|^2 + C_\varepsilon(W(x)C(x))^{\frac{2}{2-p}}.$$

Choosing ε sufficiently small and using (A1)-(A2), we obtain $\|v_n\|_X \leq C$. Since $\{v_n\}$ is bounded, there exists $v \in X$ such that, up to a subsequence, $v_n \rightharpoonup v$ weakly in X and $v_n \rightarrow v$ strongly in $L^p_{\text{loc}}(\mathbb{R}^N)$. From $J'(v_n) \rightarrow 0$ and the compact embedding, we get $\|v_n - v\|_X \rightarrow 0$. \square

We use the following variant of Clark's theorem due to Kajikiya [9]:

Theorem 2.4 (Clark-Kajikiya Theorem). *Let X be an infinite-dimensional Banach space and $J \in C^1(X, \mathbb{R})$ an even functional with $J(0) = 0$. Assume:*

(1) *J satisfies the Palais-Smale condition.*

(2) *For each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that $\sup_{A_k} J < 0$,*

where $\Gamma_k = \{A \subset X \setminus \{0\} : A \text{ is closed, symmetric, and } \gamma(A) \geq k\}$, with $\gamma(A)$ the \mathbb{Z}_2 -genus of A . Then J possesses a sequence of critical points $\{v_k\}$ with $J(v_k) \rightarrow 0$ and $v_k \rightarrow 0$ in X .

Definition 2.1 (\mathbb{Z}_2 -genus). For a closed symmetric set $A \subset X \setminus \{0\}$, the genus $\gamma(A)$ is defined as

$$\gamma(A) = \inf\{k \in \mathbb{N} : \exists \text{ an odd continuous map } \phi : A \rightarrow \mathbb{R}^k \setminus \{0\}\},$$

with $\gamma(\emptyset) = 0$ and $\gamma(A) = +\infty$ if no such k exists.

Lemma 2.5. *Under hypotheses (A1)-(A3), for each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ such that $\sup_{A_k} J < 0$.*

Proof. Without loss of generality, assume $x_0 = 0$ in (A3). Fix $k \in \mathbb{N}$.

By (A3), there exist $\delta_n \rightarrow 0$, $M_n \rightarrow \infty$, and sets $E_n \subset B_{r_0}(0)$ with $|E_n| \geq \alpha_0 > 0$ such that $F(x, \delta_n) \geq M_n \delta_n^2$ for $x \in E_n$.

Since $|E_n| > 0$, we can choose k disjoint cubes $Q_1, \dots, Q_k \subset B_{r_0}(0)$ such that $|Q_i \cap E_n| \geq \beta > 0$ for each i . For each i , construct $\psi_i \in C_c^\infty(\mathbb{R}^N)$ with $\text{supp}(\psi_i) \subset Q_i$, $0 \leq \psi_i \leq 1$, $\psi_i = 1$ on a smaller cube $Q'_i \subset Q_i$ with $|Q'_i \cap E_n| \geq \beta/2$.

Define

$$H_k = \{(e_1, \dots, e_k) \in \mathbb{R}^k : \max_{1 \leq i \leq k} |e_i| = 1\},$$

$$L_k = \left\{ \sum_{i=1}^k e_i \psi_i : (e_1, \dots, e_k) \in H_k \right\}.$$

The map $\mathcal{L} : H_k \rightarrow L_k$ defined by $\mathcal{L}(e_1, \dots, e_k) = \sum e_i \psi_i$ is odd, continuous, and injective. Since H_k is homeomorphic to the unit sphere S^{k-1} by an odd homeomorphism, $\gamma(H_k) = k$, and thus $\gamma(L_k) = k$.

For $v = \sum e_i \psi_i \in L_k$ and $t = \delta_n$, using Lemma 2.1 and (A3):

$$\begin{aligned} J(\delta_n v) &\leq \frac{\delta_n^2}{2} \|\nabla v\|_{L^2}^2 + \frac{(\sqrt{2}\tau)^2 \delta_n^2}{2} \int V(x) v^2 dx - \sum_{i=1}^k \int_{Q'_i \cap E_n} W(x) F(x, \delta_n e_i) dx \\ &\leq C_1 \delta_n^2 \|v\|_X^2 - \frac{1}{2} M_n \delta_n^2 \sum_{i=1}^k \int_{Q'_i \cap E_n} W(x) dx. \end{aligned}$$

Since $W(x) \geq W_0 > 0$ on $B_{r_0}(0)$ and $|Q'_i \cap E_n| \geq \beta/2$, we have

$$\int_{Q'_i \cap E_n} W(x) dx \geq W_0 \cdot \frac{\beta}{2}.$$

Thus,

$$J(\delta_n v) \leq \delta_n^2 \left(C_1 \sup_{v \in L_k} \|v\|_X^2 - \frac{k\beta W_0}{4} M_n \right).$$

Since $M_n \rightarrow \infty$, choose n sufficiently large so that $C_1 \sup \|v\|_X^2 - \frac{k\beta W_0}{4} M_n < 0$. Then $A_k = \{\delta_n v : v \in L_k\}$ satisfies $\gamma(A_k) = k$ and $\sup_{A_k} J < 0$. \square

Lemma 2.6. *Let $\{v_n\} \subset X$ be a sequence of critical points of J with $v_n \rightarrow 0$ in X . Then $v_n \rightarrow 0$ in $L^\infty(\mathbb{R}^N)$.*

Proof. Let v be a critical point of J . For $\eta > 1$, take $\varphi = |v|^{2(\eta-1)}v$ as a test function in $J'(v)\varphi = 0$. This gives

$$(2\eta - 1) \int |v|^{2(\eta-1)} |\nabla v|^2 dx \leq C \int W(x) C(x) |v|^{p-1} |v|^{2\eta-1} dx.$$

Noting that $|\nabla(|v|^\eta)|^2 = \eta^2 |v|^{2(\eta-1)} |\nabla v|^2$, we have

$$\frac{1}{\eta} \int |\nabla(|v|^\eta)|^2 dx \leq C \int W(x) C(x) |v|^{2\eta+p-2} dx.$$

By the Sobolev inequality,

$$S \left(\int |v|^{\eta \cdot 2^*} dx \right)^{2/2^*} \leq \int |\nabla(|v|^\eta)|^2 dx,$$

where $S > 0$ is the Sobolev constant and $2^* = \frac{2N}{N-2}$.

Combining these inequalities and applying Hölder's inequality yields

$$\|v\|_{\eta 2^*} \leq (C\eta)^{1/\eta} \|v\|_{2\eta}^{1 - \frac{2-p}{2\eta}}.$$

Starting with $\eta_0 = 2^*/2$ and iterating, we obtain after m iterations:

$$\|v\|_\infty \leq C \prod_{j=0}^m (C\eta_j)^{1/\eta_j} \|v\|_{2^*}^{\theta_m},$$

where $\theta_m \rightarrow \theta > 0$ as $m \rightarrow \infty$. Since $\sum_{j=0}^\infty \frac{\log \eta_j}{\eta_j} < \infty$, there exists $C_0 > 0$ such that

$$\|v\|_\infty \leq C_0 \|v\|_X^\theta.$$

Thus, if $\|v_n\|_X \rightarrow 0$, then $\|v_n\|_\infty \rightarrow 0$. □

3. Proof of the Main Theorem

Proof of Theorem 1.1. By Lemma 2.2, $J \in C^1(X, \mathbb{R})$. Since f is odd (Assumption (A2)) and G^{-1} is odd, J is even. Clearly $J(0) = 0$. By Lemma 2.3, J satisfies the Palais-Smale condition. By Lemma 2.5, for each $k \in \mathbb{N}$, there exists $A_k \in \Gamma_k$ with $\sup_{A_k} J < 0$. Applying Theorem 2.4, we obtain a sequence $\{v_n\} \subset X$ of critical points of J with $J(v_n) \rightarrow 0$ and $v_n \rightarrow 0$ in X . By Lemma 2.6, $v_n \rightarrow 0$ in $L^\infty(\mathbb{R}^N)$. Define $u_n = G^{-1}(v_n)$. For n sufficiently large, $\|v_n\|_{L^\infty} < \min\{\delta/2, 1/\sqrt{\tau-1}\}$. Then:

- Since $|G^{-1}(t)| \leq \sqrt{2\tau}|t|$, we have $\|u_n\|_{L^\infty} \leq \sqrt{2\tau}\|v_n\|_{L^\infty} \rightarrow 0$.
- For $|u_n| < 1/\sqrt{\tau-1}$, $g(u_n) = \sqrt{1 - \frac{\tau u_n^2}{2(1+u_n^2)}}$.
- For $|u_n| < \delta/2$, $\tilde{f}(x, u_n) = f(x, u_n)$.

Thus, by Lemma 2.2, u_n is a weak solution of the original problem (1). The original energy functional is

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} g^2(u) |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx - \int_{\mathbb{R}^N} W(x) F(x, u) dx.$$

From the change of variables $v = G(u)$, we have $J(v) = I(u)$. Since $J(v_n) \rightarrow 0$, we get $I(u_n) \rightarrow 0$.

This completes the proof of Theorem 1.1. □

4. Example

Let $N = 3$, $\tau = 2$, $p = 3/2$. Define:

$$V(x) = 2 + \sin^2(|x|),$$

$$W(x) = \frac{1}{1 + |x|^2} + \chi_{B_1(0)}(x),$$

$$f(x, t) = (1 + |x|^{-1/4} \chi_{B_1(0)}(x)) |t|^{-1/2} t + \chi_E(x) |t|^{1/2} t \sin^2(|t|^{-1/3}),$$

where $E = \bigcup_{k=1}^\infty B_{2^{-k}}(x_k)$ with $\{x_k\}$ dense in $B_1(0)$.

- (A1): $V \in L^\infty(\mathbb{R}^3)$, $1 \leq V \leq 3$; $W \in L^\infty_{\text{loc}}$; $\int_{B_1(y)} W(x)^4 dx$ bounded.
- (A2): $C(x) = 1 + |x|^{-1/4} \chi_{B_1(0)} \in L^\infty_{\text{loc}}$; $|f(x, t)| \leq C(x) |t|^{1/2}$ for $|t| \leq 1$.
- (A3): Take $\delta_n = (2/((2n+1)\pi))^3$, $M_n = \delta_n^{-1/2}$. On E , for $t = \delta_n$, $\sin^2(\delta_n^{-1/3}) = 1$, so $F(x, \delta_n) \geq \delta_n^{3/2}$, hence $F(x, \delta_n)/\delta_n^2 \geq \delta_n^{-1/2} = M_n \rightarrow \infty$.

This example satisfies (A1)-(A3) but violates the hypotheses of [8]: V, W not continuous, $C(x)$ not uniformly bounded, oscillation not uniform, and $W(x)^4 V(x)^{-3} \notin L^1(\mathbb{R}^3)$.

5. Conclusion

We have established the existence of infinitely many small solutions for quasilinear Schrödinger equations under significantly weakened hypotheses. Our results extend previous works by allowing:

- Potentials with only local boundedness and measurability
- Spatially varying growth coefficients in the nonlinearity
- Oscillation conditions that hold only along sequences and on sets of positive measure
- Local rather than global integrability conditions

The proof combines variational methods with \mathbb{Z}_2 -genus theory (via Clark's theorem) and Moser iteration. Our approach provides a framework for studying quasilinear problems under minimal assumptions, with potential applications to other classes of nonlinear elliptic equations.

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