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Several Combinatorial Inequalities Related to Squarefree Monomial Ideals

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Abstract. Let K be a field and $S = K[x_1, \ldots, x_n]$, the ring of polynomials in n variables, over K. Using the fact that the Hilbert depth is an upper bound for the Stanley depth of a quotient of squarefree monomial ideals $0 \subset I \subsetneq J \subset S$, we prove several combinatorial inequalities which involve the coefficients of the polynomial $f(t) = (1 + t + \cdots + t^{m-1})^n$.

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1. Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as a $\mathbb{Z}^n$-graded K-vector space, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1, \ldots, x_n\}$ such that $m_i K[Z_i] = \{um_i : \ u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of M. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) |\ \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

Herzog, Vlădoiu and Zheng show in [11] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [1], J. Apel restated a conjecture firstly given by Stanley in [16], namely that

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M),$$

for any $\mathbb{Z}^n$-graded S-module M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $0 \neq I \subset J \subset S$ are monomial ideals, see [9], but remains open for $M = I$. Another open question, proposed by Herzog [12], is the following: Is it true that

$$\operatorname{sdepth}(I) \geq \operatorname{sdepth}(S/I) + 1,$$

for any monomial ideal I?

The explicit computation of the Stanley depth is a computational difficult task, even in the very special case of the ideal $\mathbf{m} = (x_1, \ldots, x_n)$, see [2]. This is one of the reasons, a new invariant, associated to (multi)graded S-modules, called Hilbert depth, was introduced, which gives a natural upper bound for the Stanley depth. See [4] for further details.

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In [3], we proved a new formula for the Hilbert depth of J/I, where $0 \neq I \subsetneq J \subset S$ are squarefree monomial ideals; see Section 2.

In this paper, our aim is to deduce several combinatorial inequalities, using the fact that $\operatorname{hdepth}(J/I) \geq \operatorname{sdepth}(J/I)$, in certain cases when sharp lower bounds for $\operatorname{sdepth}(J/I)$ are known. We mention the fact that the inequalities involved in the computation of Hilbert depth are related to the theory of hypergeometric functions; see for instance [4, Remark 3.8] and [3, Lemma 3.8].

In Section 3, we consider the path ideal of length m of a path graph, i.e.

$$I_{n,m} = (x_1x_2\cdots x_m,\ x_2x_3\cdots x_{m+1},\ \ldots, x_{n-m+1}\cdots x_n) \subset S.$$

In Proposition 3.1 we show that the number of squarefree monomials of degree k which do not belong to $I_{n,m}$ is equal to

$$\binom{n-k+1,\ m}{k} := \text{ the coefficient of } x^k \text{ in the expansion of } (1+x+\cdots+x^{m-1})^{n-k+1}.$$

We mention that the above expansion was firstly studied by Euler [10]. For a modern perspective, we refer the reader to [8, Page 77].

Using this, a formula for $\operatorname{sdepth}(S/I_{n,m})$ and the fact that Hilbert depth is an upper bound for Stanley depth, in Theorem 3.5 we prove that for $n \geq m \geq 1$ and $d := \varphi(n,m) = n+1-\left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil$, we have that:

(1) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1,\ m}{j} = \sum\limits_{\ell=0}^{\left\lfloor \frac{k}{m} \right\rfloor}(-1)^{\ell}\sum\limits_{j=m\ell}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1}{\ell}\binom{n-m\ell}{j-m\ell}$, for all $0 \leq k \leq d$.

(2) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1,\ m}{j} \geq 0$, for all $0 \leq k \leq d$.

Also, since $I_{n,m}$ is minimally generated by $n-m+1$ monomials, using a result of Okazaki [13], we show that for $d := \left\lfloor \frac{n+m}{2} \right\rfloor$ we have that:

(3) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1,\ m}{j} \leq \binom{n-d+k-1}{k}$, for all $0 \leq k \leq d$.

Also, we particularize (1-3) for $m=2$; see Corollary 3.6. In Remark 3.7 we translate the inequalities from Corollary 3.6 in hypergeometric terms.

In Section 4, we consider the path ideal of length m of a cycle graph, i.e.

$$J_{n,m} = I_{n,m} + (x_{n-m+2}\cdots x_nx_1, \ldots, x_nx_1\cdots x_{m-1}) \subset S,$$

where $n > m \geq 2$ are some integers. In Proposition 4.1 we prove that

$$\operatorname{sdepth}(J_{n,m}) \geq \operatorname{depth}(J_{n,m}),$$

i.e. $J_{n,m}$ satisfies the Stanley inequality. In Corollary 4.2 we show that if $m \geq 3$, then

$$\operatorname{sdepth}(J_{n,m}) \geq \operatorname{sdepth}(S/J_{n,m}) + 1,$$

i.e. $J_{n,m}$ satisfies the Herzog inequality.

Let $d := n - \left\lfloor \frac{n}{m+1} \right\rfloor - \left\lceil \frac{n}{m+1} \right\rceil$. Using Proposition 4.1 and some other results about $\operatorname{sdepth}(S/J_{n,m})$ and $\operatorname{sdepth}(J_{n,m}/I_{n,m})$, in Theorem 4.5 we prove that:

(1) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\left(\binom{n-j+1,\ m}{j} - \sum\limits_{\ell=m}^{2m-2}(2m-1-\ell)\binom{n-\ell-j+1,\ m}{j-\ell}\right) \geq 0$, for all $0 \leq k \leq d$.

(2) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d+m-1-j}{k-j}\sum\limits_{\ell=m}^{2m-2}(2m-1-\ell)\binom{n-\ell-j+1,\ m}{j-\ell} \geq 0$, for all $0 \leq k \leq d+m-1$.

(3) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{\lceil\frac{n}{2}\rceil-j}{k-j}\left(\binom{n-j+1,\ m}{j}-\sum\limits_{\ell=m}^{2m-2}(2m-1-\ell)\binom{n-\ell-j+1,\ m}{j-\ell}\right) \leq \binom{\lfloor\frac{n}{2}\rfloor+k-1}{k}$,
for all $0 \leq k \leq \lceil\frac{n}{2}\rceil$. In Corollary 4.6 we particularize (1-3) for $m=2$ and in Remark 4.7 we further translate them in hypergeometric terms.

2. Preliminaries

First, we fix some notations and we recall the main result of [3].

We denote $[n] := \{1,2,\ldots,n\}$ and $S := K[x_1,\ldots,x_n]$.

For two subsets $C \subset D \subset [n]$, we denote $[C,D] := \{A \subset [n] \ :\ C \subset A \subset D\}$, and we call it the *interval* bounded by C and D.

Let $0 \subset I \subsetneq J \subset S$ be two squarefree monomial ideals. We let:

$$\mathrm{P}_{J/I} := \{C \subset [n] \ :\ x_C = \prod_{j\in C} x_j \in J \setminus I\} \subset 2^{[n]}.$$

A partition of $P_{J/I}$ is a decomposition $\mathcal{P}:\ \mathrm{P}_{J/I} = \bigcup\limits_{i=1}^{r}[C_i,D_i]$, into disjoint intervals.

If $\mathcal{P}$ is a partition of $\mathrm{P}_{J/I}$, we let $\operatorname{sdepth}(\mathcal{P}) := \min_{i=1}^{r}|D_i|$. The Stanley depth of $P_{J/I}$ is

$$\operatorname{sdepth}(P_{J/I}) := \max\{\operatorname{sdepth}(\mathcal{P}) \ :\ \mathcal{P} \text{ is a partition of } \mathrm{P}_{J/I}\}.$$

Herzog, Vlădoiu and Zheng proved in [11] that $\operatorname{sdepth}(J/I) = \operatorname{sdepth}(\mathrm{P}_{J/I})$.

Let $\mathrm{P} := \mathrm{P}_{J/I}$, where $I \subset J \subset S$ are squarefree monomial ideals. For any $0 \leq k \leq n$, we denote $\mathrm{P}_k := \{A \in \mathrm{P} \ :\ |A| = k\}$ and $\alpha_k(J/I) = \alpha_k(\mathrm{P}) = |\,\mathrm{P}_k\,|$.

For all $0 \leq d \leq n$ and $0 \leq k \leq d$, we consider the integers

$$\beta_k^d(J/I) := \sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\alpha_j(J/I). \tag{1}$$

From (1) we can easily deduce that

$$\alpha_k(J/I) = \sum_{j=0}^{k}\binom{d-j}{k-j}\beta_k^d(J/I), \text{ for all } 0 \leq k \leq d. \tag{2}$$

Also, we have that

$$\beta_k^d(J/I) = \alpha_k(J/I) - \binom{d}{k}\beta_0^d(J/I) - \binom{d-1}{k-1}\beta_1^d(J/I) - \cdots - \binom{d-k+1}{1}\beta_{k-1}^d(J/I). \tag{3}$$

Theorem 2.1. ([3, Theorem 2.4]) With the above notations, the *Hilbert depth* of J/I is

$$\operatorname{hdepth}(J/I) := \max\{d \ :\ \beta_k^d(J/I) \geq 0 \text{ for all } 0 \leq k \leq d\}.$$

As a basic property of the Hilbert depth, we state the following:

Proposition 2.2. Let $I \subset J \subset S$ be two square-free monomial ideals. Then

$$\operatorname{sdepth}(J/I) \leq \operatorname{hdepth}(J/I).$$

We recall the following result:

Theorem 2.3. ([13, Theorem 2.3]) Let $I \subset S$ be a monomial ideal, minimally generated by m monomials. Then

$$\operatorname{sdepth}(I) \geq \max\{1, n - \left\lfloor \frac{m}{2} \right\rfloor\}.$$

As a direct consequence of Proposition 2.2 and Theorem 2.3 we obtain:

Corollary 2.4. Let $I \subset S$ be a squarefree monomial ideal, minimally generated by m monomials. Then

$$\operatorname{hdepth}(I) \geq \max\{1, n - \left\lfloor \frac{m}{2} \right\rfloor\}.$$

3. The m-path ideal of a path graph

Let $n \geq m \geq 1$ be two integers and

$$I_{n,m} = (x_1x_2\cdots x_m,\ x_2x_3\cdots x_{m+1},\ \ldots, x_{n-m+1}\cdots x_n) \subset S,$$

be the m-path ideal associated to the m-path of length $n-1$. We define:

$$\varphi(n,m) := n+1 - \left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil. \tag{4}$$

According to [6, Theorem 1.3] we have that

$$\operatorname{sdepth}(S/I_{n,m}) = \operatorname{depth}(S/I_{n,m}) = \varphi(n,m). \tag{5}$$

Also, according to [7, Proposition 1.7], we have that

$$\operatorname{sdepth}(I_{n,m}) \geq \operatorname{depth}(I_{n,m}) = \varphi(n,m) + 1. \tag{6}$$

Note that $\alpha_k(S/I_{n,m})$ counts the number of squarefree monomial of degree k which do not belong to $I_{n,m}$.

Proposition 3.1. With the above notations, $\alpha_k(S/I_{n,m}) = \binom{n-k+1,\ m}{k} :=$ the coefficient of x^k from the expansion $(1+x+\cdots+x^{m-1})^{n-k+1}$.

In particular $\alpha_k(I_{n,m}) = \binom{n}{k} - \binom{n-k+1,\ m}{k}$.

Proof. Note that the coefficient of x^k from the expansion $(1+x+\cdots+x^{m-1})^{n-k+1}$ is equal to the number of sequences $(a_1,a_2,\ldots,a_{n-k+1})$ with $a_i \in \{0,1,\ldots,m-1\}$ for all $1 \leq i \leq n-k+1$, such that $a_1+a_2+\cdots+a_{n-k+1} = k$. Therefore, in order to complete the proof, it is enough to establish a 1-to-1 correspondence with the set of squarefree monomials $u \in S \setminus I_{n,m}$ of degree k.

Indeed, given a sequence $\mathbf{a} = (a_1,a_2,\ldots,a_{n-k+1})$ as above, we define a monomial $u_{\mathbf{a}}$ as follows: We let $A_j := \sum_{i=1}^{j} a_i$ for all $1 \leq j \leq n-k$, and $A_0 = 0$. We define

$$u_{\mathbf{a}} = \prod_{i=1}^{n-k+1} u_i,\quad \text{where } u_i = \begin{cases} 1, & a_i = 0 \\ x_{i+A_i}x_{i+1+A_i}\cdots x_{i+a_i-1+A_i}, & a_i > 0 \end{cases}.$$

It is clear that $\deg(u_{\mathbf{a}}) = k$ and $u \notin I_{n,m}$, since, by construction, there is no monomial of the form $x_ix_{i+1}\cdots x_{i+m-1}$ which divides $u_{\mathbf{a}}$.

Conversely, let $u \in S \setminus I_{n,m}$ be a squarefree monomial of degree k. We can write u as a product $u = w_1w_2\cdots w_t$, where $w_j = x_{i_j}x_{i_j+1}\cdots x_{i_j+b_j-1}$ for all $1 \leq j \leq t$ such that:

$1 \leq i_1 < i_2 < \cdots < i_t \leq n-b_t+1,\ b_1+\cdots+b_t = k$ and $i_j+b_j < i_{j+1}$ for all $1 \leq j \leq t-1$.

Note that, since $u \notin I_{n,m}$, we have that $b_j = \deg(w_j) \leq m-1$ for all $1 \leq j \leq t$.

We construct a sequence $\mathbf{a} = (a_1, \ldots, a_{n-k+1})$ as follows:

We let $a_{i_j - j+1} = b_j$ for all $1 \leq j \leq t$ and $a_i = 0$ whenever $i \neq i_j - j + 1$ for all $1 \leq j \leq t$. It is easy to see that $\sum_{i=1}^{n-k+1} a_i = k$ and $a_i \in \{0, 1, \ldots, m-1\}$ for all $1 \leq i \leq n-k+1$. Moreover, we have that $u = u_{\mathbf{a}}$. Hence, the proof is complete. $\square$

Example 3.2. Let $n = 7$, $m = 3$ and $k = 4$. According to Proposition 3.1, $\alpha_4(S/I_{7,3}) =$ the coefficient of x^4 in the expansion of $(1 + x + x^2)^4$. By straightforward computations, we get $\alpha_4(S/I_{7,3}) = 19$. Let $\mathbf{a} = (0, 1, 1, 2)$ be a sequence, as in the proof of Proposition 3.1. The corresponding monomial is $u = x_2x_4x_6x_7 \in S \setminus I_{7,3}$. Similarly, if $u' = x_1x_3x_4x_5 \in S \setminus I_{7,3}$, then $u' = u_{\mathbf{a}'}$, where $\mathbf{a}' = (1, 3, 0, 0)$.

Remark 3.3. Let $n \geq m \geq 1$ and $m \leq k \leq n$ be some integers such that $n \geq 2m$. Let $L_0 := I_{n,m}$ and $L_i := L_{i-1} : x_{n-i+1}$ for $1 \leq i \leq m-1$. We consider the short exact sequences:

$$0 \to S/L_i \overset{\cdot x_{n-i+1}}{\to} S/L_{i-1} \to S/(L_{i-1}, x_{n-i+1}) \to 0 \text{ for } 1 \leq i \leq m-1. \tag{7}$$

We denote $S_j := K[x_1, \ldots, x_j]$ for any $1 \leq j \leq n$. We have that

$$S/(L_{i-1}, x_{n-i+1}) \cong S/(I_{n-i,m}S, x_{n-i+1}) = (S_{n-i}/I_{n-i,m})[x_{n-i+2}, \ldots, x_n], \tag{8}$$

for $1 \leq i \leq m-1$. Also $L_{m-1} = (I_{n-m,m}, x_{n-m+1})$ and therefore

$$S/L_{m-1} \cong (S_{n-m}/I_{n-m,m})[x_{n-m+2}, \ldots, x_n]. \tag{9}$$

From (7), (8) and (9) it follows that

$$\alpha_k(S/I_{n,m}) = \alpha_k(S_{n-1}/I_{n-1,m}) + \cdots + \alpha_{k-m+1}(S_{n-m}/I_{n-m,m}). \tag{10}$$

Let $N := n - k + 1$. From Proposition 3.1 and (10) we reobtain the identity:

$$\binom{N,\ m}{k} = \binom{N-1,\ m}{k} + \binom{N-1,\ m}{k-1} + \cdots + \binom{N-1,\ m}{k-m+1}.$$

Lemma 3.4. (See also [8, Page 77]) Let $n \geq m \geq 1$ and $0 \leq k \leq n$ be some integers. Then:

$$\binom{n-k+1,\ m}{k} = \sum_{\ell=0}^{\lfloor \frac{k}{m} \rfloor} (-1)^\ell \binom{n-k+1}{\ell} \binom{n-m\ell}{k-m\ell}.$$

Proof. We have that $(1 + t + \cdots + t^{m-1})^{n-k+1} =$

$$= \left(\frac{1-t^m}{1-t}\right)^{n-k+1} = \sum_{\ell=0}^{n-k+1} (-1)^\ell \binom{n-k+1}{\ell} t^{m\ell} \sum_{j=0}^{\infty} \binom{n+j-1}{j} t^j. \tag{11}$$

Identifying the coefficient of t^k in (11) we deduce the required conclusion. $\square$

We also recall the following combinatorial identity

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \binom{n}{j} = \binom{n-d+k-1}{k}, \tag{12}$$

which is a direct consequence of the Chu-Vandermonde summation.

Theorem 3.5. Let $n \geq m \geq 1$.

(1) Let $d := \varphi(n,m) = n+1-\left\lfloor \frac{n+1}{m+1} \right\rfloor - \left\lceil \frac{n+1}{m+1} \right\rceil$. We have that:

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1,\ m}{j} = \sum_{\ell=0}^{\lfloor \frac{k}{m} \rfloor}(-1)^{\ell}\sum_{j=m\ell}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1}{\ell}\binom{n-m\ell}{j-m\ell}$$

and $\sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1,\ m}{j} \geq 0$, for all $0 \leq k \leq d$.

(3) Let $d = \left\lfloor \frac{n+m}{2} \right\rfloor$. We have that:

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1,\ m}{j} \leq \binom{n-d+k-1}{k}, \text{ for all } 0 \leq k \leq d.$$

Proof. (1) The equality follows immediately from Lemma 3.4.

From Proposition 2.2, Proposition 3.1, (1) and (5) it follows that

$$\beta_k^d(S/I_{n,m}) = \sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1,\ m}{j} \geq 0,$$

for all $0 \leq k \leq d$, as required.

(2) Note that the ideal $I_{n,m}$ is minimally generated by $n-m+1$ monomials and $n-\left\lfloor \frac{n-m+1}{2} \right\rfloor = \left\lfloor \frac{n+m}{2} \right\rfloor$. Hence, the conclusion follows from Proposition 2.2, Corollary 2.4, Proposition 3.1, (1) and (12). $\square$

Corollary 3.6. (Case $m=2$) Let $n \geq 2$ be an integer. We have that:

(1) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{\lceil \frac{n}{3} \rceil - j}{k-j}\binom{n-j+1}{j} = \sum\limits_{\ell=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^{\ell}\sum\limits_{j=2\ell}^{k}(-1)^{k-j}\binom{\lceil \frac{n}{3} \rceil - j}{k-j}\binom{n-j+1}{\ell}\binom{n-2\ell}{j-2\ell}$, for all $0 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$.

(2) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{\lceil \frac{n}{3} \rceil - j}{k-j}\binom{n-j+1}{j} \geq 0$, for all $0 \leq k \leq \left\lceil \frac{n}{3} \right\rceil$.

(3) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{\lfloor \frac{n}{2} \rfloor + 1 - j}{k-j}\binom{n-j+1}{j} \leq \binom{\lfloor \frac{n}{2} \rfloor + k - 3}{k}$, for all $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Proof. (1) From Proposition 3.1, it follows that $\alpha_k(S/I_{n,2}) = \binom{n-k+1}{k}$ for all $0 \leq k \leq n$. The conclusion follows from Theorem 3.5. $\square$

Remark 3.7. We consider the standard hypergeometric notation (cf. [15]), that is

$${}_rF_s\left[\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};z\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_r)_k}{k!\,(b_1)_k\cdots(b_s)_k}z^k\ .$$

By straightforward computations, we have

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1}{j} = (-1)^k\binom{d}{k}{}_4F_3\left[\begin{matrix} -k,-\frac{1}{2},-\frac{n}{2},-\frac{n}{2} \\ -d,-1,-n \end{matrix};4\right].$$

Therefore, from Corollary 3.6(2) it follows that

$$(-1)^k{}_4F_3\left[\begin{matrix} -k,-\frac{1}{2},-\frac{n}{2},-\frac{n}{2} \\ -\left\lceil \frac{n}{3} \right\rceil,-1,-n \end{matrix};4\right] \geq 0 \text{ for all } 0 \leq k \leq \left\lceil \frac{n}{3} \right\rceil.$$

Also, from Corollary 3.6(3) it follows that

$$(-1)^k {}_4F_3\left[\begin{matrix} -k, -\frac{1}{2}, -\frac{n}{2}, -\frac{n}{2} \\ -\left\lfloor \frac{n}{2} \right\rfloor - 1, -1, -n \end{matrix}; 4\right] \leq \frac{\binom{\lfloor \frac{n}{2} \rfloor + k - 3}{k}}{\binom{\lfloor \frac{n}{2} \rfloor + 1}{k}} \text{ for all } 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

4. The m-path ideal of a cycle graph

Let $n > m \geq 2$ be two integer and

$$J_{n,m} = I_{n,m} + (x_{n-m+2} \cdots x_n x_1, \ldots, x_n x_1 \cdots x_{m-1}) \subset S,$$

the m-path ideal associated to the cycle graph of length n. According to [5, Theorem 1.4] we have that

$$\varphi(n, m) \geq \operatorname{sdepth}(S/J_{n,m}) \geq \operatorname{depth}(S/J_{n,m}) = \varphi(n-1, m). \tag{13}$$

Also, according to [5, Proposition 1.6] we have that

$$\operatorname{sdepth}(J_{n,m}/I_{n,m}) \geq \operatorname{depth}(J_{n,m}/I_{n,m}) \geq \varphi(n-1, m) + m - 1. \tag{14}$$

Proposition 4.1. Let $n > m \geq 2$ be two integers. We have that:
(1) $\operatorname{sdepth}(J_{n,m}) \geq \min\{\varphi(n-1, m) + m - 1,\ \varphi(n, m) + 1\}$.
(2) $\operatorname{sdepth}(J_{n,m}) \geq \operatorname{depth}(J_{n,m}) = \varphi(n-1, m) + 1$.
(3) If $m \geq 3$ then $\operatorname{sdepth}(J_{n,m}) \geq \varphi(n, m) + 1$.

Proof. (1) We consider the short exact sequence

$$0 \to I_{n,m} \to J_{n,m} \to \frac{J_{n,m}}{I_{n,m}} \to 0.$$

From [14, Lemma 2.2], (6) and (14) we get the required result.

(2) It follows from (1) and (13), since $\varphi(n, m) \geq \varphi(n-1, m)$ and $m \geq 2$.

(3) Since $m \geq 3$, it follows that $\varphi(n-1, m) + m - 1 \geq \varphi(n-1, m) + 2 \geq \varphi(n, m) + 1$, hence, the first inequality follows from (1). □

Corollary 4.2. If $m \geq 3$ then $\operatorname{sdepth}(J_{n,m}) \geq \operatorname{sdepth}(S/J_{n,m}) + 1$.

Proof. It follows from (13) and Proposition 4.1(3). □

Proposition 4.3. With the above notations,

$$\alpha_k(J_{n,m}/I_{n,m}) = \sum_{\ell=m}^{2m-2} (2m - 1 - \ell) \binom{n - \ell - k + 1,\ m}{k - \ell}.$$

Proof. As in the proof of Proposition 3.1, we put squarefree monomials of degree k which are not in $I_{n,m}$ in bijection with sequences of the form $\mathbf{a} = (a_1, \ldots, a_{n-k+1})$, where a_i's are integers such that $0 \leq a_i \leq m - 1$ for all i and $\sum_{i=1}^{n-k+1} a_i = k$. Let $u \in J_{n,m} \setminus I_{n,m}$ such that $\deg(u) = k$. Assume $u = u_{\mathbf{a}}$ for a sequence $\mathbf{a}$ as above. Since $u \in J_{n,m}$ it follows that $\ell := a_1 + a_{n-k+1} \geq m$. On the other hand, $\ell \leq 2(m-1) = 2m-2$. Also, for a given $\ell \in \{m, m+1, \ldots, 2m-2\}$ there are exactly $(2m-1-\ell)$ pairs (a_1, a_{n-k+1}) such that $a_1 + a_{n-k+1} = \ell$ and $0 \leq a_1, a_{n-k+1} \leq m-1$. Note that the sequence $(a_2, a_3, \ldots, a_{n-k})$ has length $n-k-1$ and satisfy the conditions $0 \leq a_i \leq m-1$, for all $2 \leq i \leq n-k$, and $\sum_{i=2}^{n-k} a_i = k - \ell$.

Therefore, there are $\binom{n-\ell-k+1,\ m}{k-\ell}$ ways in which we can choose such sequences. Since the monomial u is uniquely determined by the pair (a_0, a_{n-k+1}) and the sequence $(a_2, a_3, \ldots, a_{n-k})$ as above, we get the required conclusion. $\square$

Proposition 4.4. Let $n > m \geq 2$ and $0 \leq k \leq n$ be some integers. We have that

(1) $\alpha_k(S/J_{n,m}) = \binom{n-k+1,\ m}{k} - \sum\limits_{\ell=m}^{2m-2}(2m-1-\ell)\binom{n-\ell-k+1,\ m}{k-\ell}$.

(2) $\alpha_k(J_{n,m}) = \binom{n}{k} - \binom{n-k+1,\ m}{k} + \sum\limits_{\ell=m}^{2m-2}(2m-1-\ell)\binom{n-\ell-k+1,\ m}{k-\ell}$.

Proof. (1) It follows from Proposition 4.3, Proposition 3.1 and the obvious fact that

$$\alpha_k(S/J_{n,m}) = \alpha_k(S/I_{n,m}) - \alpha_k(J_{n,m}/I_{n,m}).$$

(2) If follows from (1) and the fact that $\alpha_k(J_{n,m}) = \binom{n}{k} - \alpha_k(S/J_{n,m})$. $\square$

Theorem 4.5. Let $n > m \geq 2$ be some integers. Let $d := n - \left\lfloor \frac{n}{m+1} \right\rfloor - \left\lceil \frac{n}{m+1} \right\rceil$. We have that:

(1) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\left(\binom{n-j+1,\ m}{j} - \sum\limits_{\ell=m}^{2m-2}(2m-1-\ell)\binom{n-\ell-j+1,\ m}{j-\ell}\right) \geq 0$,
for all $0 \leq k \leq d$.

(2) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{d+m-1-j}{k-j}\sum\limits_{\ell=m}^{2m-2}(2m-1-\ell)\binom{n-\ell-j+1,\ m}{j-\ell} \geq 0$,
for all $0 \leq k \leq d+m-1$.

(3) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{\lceil\frac{n}{2}\rceil-j}{k-j}\left(\binom{n-j+1,\ m}{j} - \sum\limits_{\ell=m}^{2m-2}(2m-1-\ell)\binom{n-\ell-j+1,\ m}{j-\ell}\right) \leq \binom{\lfloor\frac{n}{2}\rfloor+k-1}{k}$,
for all $0 \leq k \leq \lceil\frac{n}{2}\rceil$.

Proof. (1) As in the proof of Theorem 3.5, the conclusion follows from Proposition 2.2, Proposition 4.4(1), (4), (13) and (1).

(2) It follows from Proposition 2.2, Proposition 4.3, (4), (14) and (1).

(3) It follows from Proposition 2.2, Proposition 4.4(2), Proposition 4.1(2), (1) and (12). $\square$

Corollary 4.6. (Case $m = 2$) Let $n \geq 3$ be an integer. We have that:

(1) $\sum\limits_{j=0}^{k}(-1)^{k-j}\binom{\lceil\frac{n-1}{3}\rceil-j}{k-j}\binom{n-j+1}{j} \geq \sum\limits_{j=2}^{k}(-1)^{k-j}\binom{\lceil\frac{n-1}{3}\rceil-j}{k-j}\binom{n-j+1}{j-2}$ for all $0 \leq k \leq \lceil\frac{n-1}{3}\rceil$.

(2) $\sum\limits_{j=2}^{k}(-1)^{k-j}\binom{\lceil\frac{n+2}{3}\rceil-j}{k-j}\binom{n-j+1}{j-2} \geq 0$ for all $2 \leq k \leq \lceil\frac{n+2}{3}\rceil$.

(3) $\binom{\lfloor\frac{n}{2}\rfloor+k-1}{k} - \sum\limits_{j=0}^{k}(-1)^{k-j}\binom{\lceil\frac{n}{2}\rceil-j}{k-j}\binom{n-j+1}{j} \geq \sum\limits_{j=2}^{k}(-1)^{k-j}\binom{\lceil\frac{n}{2}\rceil-j}{k-j}\binom{n-j+1}{j-2}$ for all $0 \leq k \leq \lceil\frac{n}{2}\rceil$.

Proof. It follows immediately from Theorem 4.5. $\square$

Remark 4.7. Similarly to Remark 3.7, we have that

$$\sum_{j=2}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n-j+1}{j-2} = (-1)^k\binom{d-2}{k-2}{}_3F_2\left[\begin{matrix}2-k, \frac{1-n}{2}, \frac{2-n}{2} \\ 2-d, 1-n\end{matrix}; 4\right].$$

In particular, from Corollary 4.6(2) it follows that

$$(-1)^k {}_3F_2\left[\begin{matrix}2-k, \frac{1-n}{2}, \frac{2-n}{2} \\ 2-\left\lceil\frac{n+2}{3}\right\rceil, 1-n\end{matrix}; 4\right] \geq 0, \text{ for all } 2 \leq k \leq \left\lceil\frac{n+2}{3}\right\rceil.$$

Other identities can be also derived from Corollary 4.6(1) and Corollary 4.6(3), but we live them as an exercise for the reader.

5. Conclusion

Using the fact that Hilbert depth is a natural upper bound for the Stanley depth and the new characterization of the Hilbert depth of a quotient of two squarefree monomial ideals given in [3], we derived several combinatorial identities related to the coefficients of the polynomial $f(t) = (1 + t + \cdots + t^{m-1})^n$, while, direct proofs of those inequalities seems out of reach.

Our method is suitable to prove similar inequalities, using other squarefree monomial ideals, and their quotient rings, for which the Stanley depth or a sharp lower bound is known.

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