Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 53(1), 2026, Pages 123–135, DOI: 10.52846/ami.v53i1.2108
ISSN: 1223-6934

$\mathcal{I}$-Convergence of Fractional Difference Sequences of Bi-complex Numbers

Tapasi Deb and Binod Chandra Tripathy

Abstract. This study uses the fractional difference operator Δ^α for $\alpha \notin \{0, -1, -2, ...\}$ to establish new classes of fractional difference sequences $Z[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]$ for $Z \in \{c_0, c, \ell_\infty\}$. Solidity and other properties are examined. The spaces have Schauder bases and are $\mathbb{BC}$-submodules. To investigate their topological properties and other characteristics, we employ the generalized fractional difference operators $\Delta^{(\tilde{\alpha})}$ and $\Delta^{(-\tilde{\alpha})}$, for a positive proper fraction $\tilde{\alpha}$. Metrix transformations between the spaces $Z[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$, for $Z \in \{c_0, c, \ell_\infty\}$ and the basic sequence spaces $Z[\mathbb{BC}, \mathcal{I}]$ for $Z \in \{c_0, c, \ell_\infty\}$ are also explained.

2020 *Mathematics Subject Classification.* 40A05, 40A35, 40G15, 46A35, 46A45.
Key words and phrases. Bi-complex, Ideal, $\mathcal{I}$-convergence, Fractional difference.

1. Introduction

1.1. Bi-complex Numbers. For many years, bi-complex numbers have been the focus of in-depth investigation and analysis. The project probably started with the work of the Italian schools of Segre [15], Spampinato [16], and Dragoni [6]. Price [12] conducted the most thorough analysis of bi-complex numbers. Srivastava and Srivastava [17], Wagh [20], Rochan and Shapiro [13], Kumar and Tripathy [11], and numerous others have conducted subsequent research on sequences of bi-complex numbers. Segre [15] provided the following definition of the bi-complex number ξ:

$$\begin{aligned}\xi &= z_1 + i_2 z_2,\\ &= (a_1 + i_1 a_2) + i_2(a_3 + i_1 a_4),\\ &= a_1 + i_1 a_2 + i_2 a_3 + i_1 i_2 a_4,\end{aligned}$$

where, $z_1, z_2 \in \mathbb{C}$. i_1 and i_2 are two distinct imaginary units of $\mathbb{BC}$ and
$i_1^2 = i_2^2 = -1$; $i_1 i_2 = i_2 i_1$.
$\mathbb{BC}$ (or $\mathbb{C}_2$) denotes the set of all bi-complex numbers, defined as

$$\mathbb{BC} = \{z_1 + i_2 z_2 : z_1, z_2 \in\},$$

where, $\mathbb{C} = \{a_1 + i_1 a_2 : a_1, a_2 \in \mathbb{R}\}$. The following are the definitions of the three conjugation types in $\mathbb{BC}$:
1. i_1- conjugation $\xi^* : \xi^* = \bar{z}_1 + i_2 \bar{z}_2$,
2. i_2-conjugation $\bar{\xi} : \bar{\xi} = z_1 - i_2 z_2$, and
3. $i_1 i_2-$conjugation $\xi^{'} : \xi^{'} = \bar{z}_1 - i_2 \bar{z}_2$, $\bar{z}$ is conjugate of z in $\mathbb{C}$.
In $\mathbb{BC}$ the only idempotent elements are $0, 1, \frac{1+i_1 i_2}{2}$ and $\frac{1-i_1 i_2}{2}$.
$\frac{1+i_1 i_2}{2}$, and $\frac{1-i_1 i_2}{2}$ are denoted by e_1 and e_2 and they satisfies: $e_1 + e_2 = 1, e_1 e_2 = 0$.

Received February 14, 2025. Accepted December 27, 2025.

Furthermore, every bi-complex number $\xi = z_1 + i_2 z_2 \in \mathbb{BC}$, can be expressed as $\xi = \mu_1 e_1 + \mu_2 e_2$, where $\mu_1 = z_1 - i_1 z_2$ and $\mu_2 = z_1 + i_1 z_2$ and $\mathbb{BC}$ can be represented as

$$\mathbb{BC} = A_1(i_1)e_1 + A_2(i_1)e_2,$$

where, $A_1(i_1) = \{z_1 - i_1 z_2 : z_1, z_2 \in \mathbb{C}\}$ and $A_2(i_1) = \{z_1 + i_1 z_2 : z_1, z_2 \in \mathbb{C}\}$.
The Euclidean norm $\|\cdot\|_{\mathbb{BC}}$ on $\mathbb{BC}$ is defined as

$$\|\xi\|_{\mathbb{BC}} = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}.$$

Let $\xi,\ \mu \in \mathbb{BC}$, then $\|\xi \cdot \mu\|_{\mathbb{BC}} \leq \sqrt{2}\|\xi\|_{\mathbb{BC}} \cdot \|\mu\|_{\mathbb{BC}}$.
One may refer to [12] for the details on $\mathbb{BC}$.

Here, $\omega(\mathbb{BC})$ is for all sequences of $\mathbb{BC}$. A subspace of $\omega(\mathbb{BC})$ is a sequence space. The bounded, convergent, and null sequence spaces of $\mathbb{BC}$ are $\ell_\infty(\mathbb{BC}), c(\mathbb{BC})$, and $c_0(\mathbb{BC})$, respectively, and are normed by

$$\|\xi\|_\infty = \sup_k \|\xi_k\|_{\mathbb{BC}}.$$

Kizmaz proposed the concept of difference sequence spaces [9] as follows:

$$Z(\Delta) = \{(x_k) \in \omega : (\Delta x_k) \in Z\}$$

where, $\Delta x_k = x_k - x_{k+1}$, for all $n \in \mathbb{N}$. He has studied different properties of difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$, and $c_0(\Delta)$.

Fractional difference operator, introduced by P. Baliarsingh [1], denoted by Δ^α, is defined as

$$\Delta^\alpha x_k = \sum_{i=1}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i},\ \forall\ n \in \mathbb{N}, \alpha \in \mathbb{Q}.$$

1.2. Ideal Convergence. Schoenberg [14] and Fast [7] each introduced the idea of statistical convergence independently. Afterward, it was examined in more detail out of the perspective of sequence space and connected to the Summability theory by Bera and Tripathy [4], and numerous others. A generalization of statistical convergence is $\mathcal{I}-$convergence. Kostyrko, Šalát, and Wilczyński were the first to present it [10].

Let $\mathcal{I} \subseteq 2^X$, $X \neq \phi$. Then $\mathcal{I}$ is called an ideal if

1. If $A \in \mathcal{I}$, $B \subseteq A$, $\implies B \in \mathcal{I}$.
2. If $A, B \in \mathcal{I}$, $\implies A \cup B \in \mathcal{I}$.

$\mathcal{I}$ is a nontrivial ideal if $\mathcal{I} \neq 2^X$. An ideal $\mathcal{I}(\neq \phi)$ is admissible ideal $\{x\} \in \mathcal{I}$ for each $x \in X$. If $\mathcal{F} \subset 2^X$ is closed under finite intersections and supersets and excludes the empty set, then $\mathcal{F}$ is filter.
For every $\mathcal{I}$ there is a $\mathcal{F}(\mathcal{I})$ corresponding to $\mathcal{I}$, such that $\mathcal{F}(\mathcal{I}) = \{K \subseteq \mathbb{N} : \mathbb{N}\backslash K \in \mathcal{I}\}$.

Example 1.1. *Here are some examples of ideals: :*
1. The class I_f of all finite subsets of $2^{\mathbb{N}}$ is admissible ideal of natural numbers, which is non trivial.
2. Let $\mathcal{I}_\delta = \{N \in 2^{\mathbb{N}} : \delta(N) = 0\}$. Hence $\mathcal{I}_\delta$ is an admissible ideal of natural numbers, which is non trivial, where δ is the statistical density of sequences.
3. let $\mathcal{I}_d = \{N \in 2^{\mathbb{N}} : d(N) = 0\}$. Hence, $\mathcal{I}_d$ is an ideal of $\mathbb{N}$, where d is the logarithmic density of sequences.

Ideals and filters are interconnected due to their complementary nature. Specifically, given a filter, an ideal can often be constructed, and vice versa. One may refer to [10] for the details on ideals.

Definition 1.1. [5] $\xi = (\xi_k) \in \omega(\mathbb{BC})$ is called $\mathcal{I}-$convergent to $\zeta \in \mathbb{BC}$, if for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : \|\xi_k - \zeta\|_{\mathbb{BC}} \geq \varepsilon\} \in \mathcal{I},$$

and written as $\mathcal{I} - \lim \xi_k = \zeta$.

Definition 1.2. [5] $\xi = (\xi_k) \in \omega(\mathbb{BC})$ is called $\mathcal{I}$-null if $\zeta = 0$ and it is written as $\mathcal{I} - \lim \xi_k = 0$.

Definition 1.3. [5] $\xi = (\xi_k) \in \omega(\mathbb{BC})$ is called $\mathcal{I}$-bounded if there exists $M > 0$ such that

$$\{k \in \mathbb{N} : \|\xi_k\|_{\mathbb{BC}} > M\} \in \mathcal{I}.$$

Definition 1.4. [5] Let $\xi = (\xi_k),\ \mu = (\mu_k) \in \omega(\mathbb{BC})$. We say $\xi_k = \mu_k$ for almost all k relative to $\mathcal{I}(a.a.k.r.\mathcal{I})$, if

$$\{k \in \mathbb{N} : \xi_k \neq \mu_k\} \in \mathcal{I}.$$

In [5], the $\mathcal{I}$ convergent sequence spaces of bi-complex numbers are defined as:

$$c[\mathbb{BC}, \mathcal{I}] = \{(\xi_k) \in \omega(\mathbb{BC}) : (\xi_k) \text{ is } \mathcal{I}-\text{ convergent sequence of bi-complex numbers}\}$$

$$c_0[\mathbb{BC}, \mathcal{I}] = \{(\xi_n) \in \omega(\mathbb{BC}) : (\xi_n) \text{ is } \mathcal{I}-\text{ null sequences of bi-complex numbers}\}$$

$$\ell_\infty[\mathbb{BC}, \mathcal{I}] = ((\xi_k) \in \omega(\mathbb{BC}) : \xi_n) \text{ is } \mathcal{I}-\text{ bounded sequences of bi-complex numbers}\}$$

$$\ell_p[\mathbb{BC}, \mathcal{I}] = \{(\xi_n) \in \omega(\mathbb{BC}) : \sum_{i=1}^{\infty} \|\xi_{k_i}\|_{\mathbb{BC}}^p < \infty, \text{ for some } \{k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I})\}$$

$$cs[\mathbb{BC}, \mathcal{I}] = \{(\xi_n) \in \omega(\mathbb{BC}) : \sum_{i=0}^{\infty} \xi_i \text{ is } \mathcal{I}-\text{convergent, for some } \{k_1 < k_2 < \cdots\} \in \mathcal{F}(\mathcal{I})\}$$

2. New classes of fractional difference sequence spaces of bi-complex numbers

The classes of new sequence spaces are defined in this section as

$$\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}], c[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}], \text{ and } c_0[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]$$

as $\mathcal{I}-$ bounded, $\mathcal{I}-$ convergent, and $\mathcal{I}-$ null fractional difference sequences of bi-complex numbers using fractional difference operator Δ^α. These sequence spaces are defined as:

$$\begin{aligned}
\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}] &= \{\xi = (\xi_k) \in \omega(\mathbb{BC}) : (\Delta^\alpha \xi_k) \in \ell_\infty[\mathbb{BC}, \mathcal{I}]\}, \\
c[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}] &= \{\xi = (\xi_k) \in \omega(\mathbb{BC}) : (\Delta^\alpha \xi_k) \in c[\mathbb{BC}, \mathcal{I}]\}, \text{ and} \\
c_0[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}] &= \{\xi = (\xi_k) \in \omega(\mathbb{BC}) : (\Delta^\alpha \xi_k) \in c_0[\mathbb{BC}, \mathcal{I}]\}.
\end{aligned}$$

where, $\Delta^\alpha \xi_k = \sum\limits_{i=1}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} \xi_{k+i}$

Theorem 2.1. *The classes of sequences* $Z[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]$ *are subspaces of* $\omega(\mathbb{BC})$, *for* $Z \in \{\ell_\infty, c, c_0\}$.

Proof. Let $\xi=(\xi_k),\eta=(\eta_k)\in c[\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}}]$ and $a,b\in\mathbb{F}$.
Therefore,

$$(\Delta^\alpha\xi_k),(\Delta^\alpha\eta_k)\in c[\mathbb{BC},\mathcal{I}]$$

for some ideals I_1 and I_2 respectively.

So,

$$I_1=\{k\in\mathbb{N}:\|(\Delta^\alpha\xi_k)-\zeta_1\|_{\mathbb{BC}}\geq\varepsilon\}\in\mathcal{I}.$$
$$I_2=\{k\in\mathbb{N}:\|(\Delta^\alpha\eta_k)-\zeta_2\|_{\mathbb{BC}}\geq\varepsilon\}\in\mathcal{I}.$$

Therefore,

$$I=I_1\cup I_2=\{k\in\mathbb{N}:\|(\Delta^\alpha\xi_k+\Delta^\alpha\eta_k)-(\zeta_1+\zeta_2)\|_{\mathbb{BC}}\geq\varepsilon\}\in\mathcal{I}.$$

Now,

$$\begin{aligned}\Delta^\alpha(a\xi+b\eta)&=\Delta^\alpha(a\xi_k+b\eta_k)\\&=\Delta^\alpha(a\xi_k)+\Delta^\alpha(b\eta_k)\\&=a\Delta^\alpha(\xi_k)+b\Delta^\alpha(\eta_k)\in c[\mathbb{BC},\mathcal{I}].\end{aligned}$$

Therefore, $a\xi+b\eta\in c[\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}}]$, is a subspace of $\omega(\mathbb{BC})$

Similarly, we can show that the classes of sequences $c_0[\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}}]$ and $\ell_\infty[\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}}]$ are subspaces of $\omega(\mathbb{BC})$. □

Theorem 2.2. *The sequence spaces* $Z[\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}}]$ *are* $\mathbb{BC}-$ *submodule for* $Z\in\{\ell_\infty,c,c_0\}$.

Proof. Let $\xi=(\xi_k)\in c[\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}}]$.

Therefore, $(\Delta^\alpha\xi_k)\in c[\mathbb{BC},\mathcal{I}]$ and

$$I=\{k:\|\Delta^\alpha\xi_k-\zeta\|_{\mathbb{BC}}\geq\varepsilon\}\in\mathcal{I}.$$

Let $a\in\mathbb{BC}$, then $(\Delta^\alpha a\xi_k)=a(\Delta^\alpha\xi_k)$ and for all k in I,

$$\|\Delta^\alpha a\xi_k-a\zeta\|_{\mathbb{BC}}\geq\varepsilon$$
$$\text{or, }\mathcal{I}-\lim_{k\to\infty}\Delta^\alpha a\xi_k=a\zeta.$$

Hence, $c(\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}})$ is the $\mathbb{BC}-$ submodule.
The remaining claims can be proven in a similar manner. □

Remark 2.1. The sequence spaces $Z[\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}}]$ are not $\mathbb{BC}-$solid, for $Z\in\{\ell_\infty,c,c_0\}$.

Example 2.1. The sequence space $c_0[\mathbb{BC},\mathcal{I},\Delta^\alpha,\|\cdot\|_{\mathbb{BC}}]$ is not $\mathbb{BC}-$solid. Let $\xi=(\xi_k)\in\omega(\mathbb{BC})$ defined as

$$\xi_k=\begin{cases}e_1e_2, & \text{if } k=i^2, i\in\mathbb{N};\\ e_1+e_2, & \text{otherwise.}\end{cases}$$

$$\Delta\xi_k=\begin{cases}e_1+e_2, & \text{if } k=i^2-1, i\in\mathbb{N};\\ -(e_1+e_2), & k=i^2, i\in\mathbb{N};\\ e_1e_2, & \text{otherwise.}\end{cases}$$

So $(\Delta\xi_k)\in Z[\mathbb{BC},\mathcal{I}]$ and $(\xi_k)\in c_0[\mathbb{BC},\mathcal{I},\Delta,\|\cdot\|_{\mathbb{BC}}]$ with respect to $\mathcal{I}=\mathcal{I}_\delta$.

Let, $(\eta_k)\in\omega(\mathbb{BC})$, defined as

$$\eta_k=\begin{cases}e_1e_2, & \text{if } k=i^2, i\in\mathbb{N};\\ (-1)^k(e_1+e_2), & \text{otherwise.}\end{cases}$$

Therefore, $(\Delta\eta_k) = \{-(e_1+e_2), 2(e_1+e_2), -(e_1+e_2), (e_1+e_2), -2(e_1+e_2), 2(e_1+e_2), -2(e_1+e_2),$
$(e_1+e_2), -(e_1+e_2), 2(e_1+e_2), -2(e_1+e_2), ...\}$ which is not $\mathcal{I}-$ convergent.

Therefore, $\eta \notin c_0[\mathbb{BC}, \mathcal{I}, \Delta, \|\cdot\|_{\mathbb{BC}}]$.
But $\|\eta_k\|_{\mathbb{BC}} \leq \|\xi_k\|_{\mathbb{BC}}$ for $a.a.k.r.\mathcal{I}$. So, $c_0[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]$ is not $\mathbb{BC}$-solid.

Example 2.2. The sequence spaces $c[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]$ and $\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]$ are not $\mathbb{BC}$-solid.
$\xi = (\xi_k) \in Z[\mathbb{BC}, \mathcal{I}, \Delta, \|\cdot\|_{\mathbb{BC}}]$ for $Z \in \{c, \ell_\infty\}$ is defined as

$$\xi_k = \begin{cases} e_1 e_2, & \text{if } k = i^3, i \in \mathbb{N}; \\ k\ i_1 i_2, & \text{otherwise.} \end{cases}$$

and $\eta = (\eta_k) \in \omega(\mathbb{BC})$ defined as

$$\eta_k = \begin{cases} e_1 e_2, & \text{if } k = i^3, i \in \mathbb{N}; \\ (-1)^k\ k\ i_1 i_1, & \text{otherwise.} \end{cases}$$

Here, $\|\xi_k\|_{\mathbb{BC}} \leq \|\eta_k\|_{\mathbb{BC}}$ for all k. Now

$$\Delta\xi_k = \begin{cases} -(k+1)\ i_1 i_2, & \text{if } k = i^3, i \in \mathbb{N}; \\ k\ i_1 i_2, & \text{if } k = i^3 - 1, i \in \mathbb{N}; \\ -i_1 i_2, & \text{otherwise.} \end{cases}$$

and so $\Delta\xi_k$ is $\mathcal{I}$-convergent to $-i_1 i_2$ and $\mathcal{I}$-bounded with respect to $\mathcal{I} = \mathcal{I}_\delta$. Now,

$$\begin{aligned} \Delta\eta &= (\Delta\eta_k) \\ &= \{-2i_1i_2, 5i_1i_2, -7i_1i_2, 9i_1i_2, -11i_1i_2, 13i_1i_2, -7i_1i_2, 9i_1i_2, -19i_1i_2, 21i_1i_2, ...\} \\ &\notin Z[\mathbb{BC}, \mathcal{I}] \text{for} Z \in \{c, \ell_\infty\} \end{aligned}$$

and so $\eta \notin Z[\mathbb{BC}, \mathcal{I}, \Delta, \|\cdot\|_{\mathbb{BC}}]$, for $Z \in \{c, \ell_\infty\}$.
Therefore, $Z[\mathbb{BC}, \mathcal{I}, \Delta, \|\cdot\|_{\mathbb{BC}}]$ is not $\mathbb{BC}$-solid $Z \in \{c, \ell_\infty\}$.

The following is the decomposition theorem.

Theorem 2.3.

$$Z[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}] = Z[A_1(i_1), \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}] + Z[A_1(i_2), \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]$$

where, $Z \in \{\ell_\infty, c, c_0\}$.
Proof. Let $\xi = (\xi_k) \in Z[\mathbb{BC}, \mathcal{I}, \Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]$ for $Z = c_0$.
Then $(\Delta^\alpha \xi_k) \in c_0[\mathbb{BC}, \mathcal{I}]$. Now, $\xi_k = \mu_k e_1 + \mu_k e_2$.
As Δ^α has linearity, so

$$\Delta^\alpha \xi_k = \Delta^\alpha \mu_k e_1 + \Delta^\alpha \mu_k e_2.$$

$\Delta^\alpha(\xi_k) \in c_0[\mathbb{BC}, \mathcal{I}]$, so $(\Delta^\alpha \mu_{1k})$ and $(\Delta^\alpha \mu_{2k})$ are also ideal convergent sequences of bi-complex numbers and therefore belong to $A_1(i_1)$ and $A_1(i_2)$, respectively.
Hence, the theorem is proved. □

3. Matrix Representation

Baliarsingh and Dutta [3, 2] introduced a generalized fractional difference operators $\Delta^{(\tilde{\alpha})}$ and its inverse $\Delta^{(-\tilde{\alpha})}$ for a positive proper fraction $\tilde{\alpha}$. as

$$\Delta^{(\tilde{\alpha})}x_k = \sum_{i=1}^{\infty}(-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)}x_{k-i},$$

$$\Delta^{(-\tilde{\alpha})}x_k = \sum_{i=1}^{\infty}(-1)^i \frac{\Gamma(-\tilde{\alpha}+1)}{i!\Gamma(-\tilde{\alpha}-i+1)}x_{k-i}, \text{ respectively.}$$

It is more convenient to express $\Delta^{(\tilde{\alpha})}$ as a infinite lower triangular matrix,

$$\Delta^{(\tilde{\alpha})}_{nk} = \begin{cases} (-1)^{n-k}\frac{\Gamma(\tilde{\alpha}+1)}{(n-k)!\Gamma(\tilde{\alpha}-n+k+1)}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n. \end{cases}$$

or,

$$\Delta^{(\tilde{\alpha})} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\tilde{\alpha} & 1 & 0 & 0 & \cdots \\ \frac{\tilde{\alpha}(\tilde{\alpha}-1)}{2!} & -\tilde{\alpha} & 1 & 0 & \cdots \\ -\frac{\tilde{\alpha}(\tilde{\alpha}-1)(\tilde{\alpha}-2)}{3!} & \frac{\tilde{\alpha}(\tilde{\alpha}-1)}{2!} & -\tilde{\alpha} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

And the inverse matrix $\Delta^{(-\tilde{\alpha})}_{nk}$ is

$$\Delta^{(-\tilde{\alpha})}_{nk} = \begin{cases} (-1)^{n-k}\frac{\Gamma(\tilde{-\alpha}+1)}{(n-k)!\Gamma(\tilde{-\alpha}-n+k+1)}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k > n. \end{cases}$$

or,

$$\Delta^{(\tilde{\alpha})} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \tilde{\alpha} & 1 & 0 & 0 & \cdots \\ \frac{\tilde{\alpha}(\tilde{\alpha}+1)}{2!} & \tilde{\alpha} & 1 & 0 & \cdots \\ \frac{\tilde{\alpha}(\tilde{\alpha}+1)(\tilde{\alpha}+2)}{3!} & \frac{\tilde{\alpha}(\tilde{\alpha}+1)}{2!} & \tilde{\alpha} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Dutta and Baliarsingh [1] proved some equalities:

$$\Delta^{(\tilde{\alpha})} \circ \Delta^{(\tilde{\beta})} = \Delta^{(\tilde{\alpha}+\tilde{\beta}\ )} = \Delta^{(\tilde{\beta})} \circ \Delta^{(\tilde{\alpha})}.$$

Considering the identity operator as I_Δ, we get

$$\Delta^{(\tilde{\alpha})} \circ \Delta^{(-\tilde{\alpha})} = \Delta^{(-\tilde{\alpha})} \circ \Delta^{(\tilde{\alpha})} = \Delta^{(\tilde{\alpha}-\tilde{\alpha}\ )} = I_\Delta.$$

Theorem 3.1. *The classes of sequences $Z[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ for $Z \in \{\ell_\infty, c, c_0\}$ are sequence spaces of $\omega(\mathbb{BC})$, where*

$$Z[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}] = \{\xi = (\xi_k) \in \omega(\mathbb{BC}) : (\Delta^{(\tilde{\alpha})}\xi_k) \in Z[\mathbb{BC}, \mathcal{I}]\}.$$

Proof. Here, $Z[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}] = \{\xi = (\xi_k) \in \omega(\mathbb{BC}) : (\Delta^{(\tilde{\alpha})}\xi_k) \in Z[\mathbb{BC}, \mathcal{I}]\} \subset \omega(\mathbb{BC})$.

Let, $\xi = (\xi_k),\ \eta = (\eta_k) \in \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}],\ a \in \mathbb{F}$.

So, $(\Delta^{(\tilde{\alpha})}\xi_k), (\Delta^{(\tilde{\alpha})}\eta_k) \in \ell_\infty[\mathbb{BC}, \mathcal{I}]$.
Then combine

$$\sup_{k\in K_1\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}\xi_k\|_{\mathbb{BC}} \leq \infty, \text{ and}$$
$$\sup_{k\in K_2\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}\xi_k\|_{\mathbb{BC}} \leq \infty$$

with the facts

$$\|\xi_k + \eta_k\|_{\mathbb{BC}} \leq \|\xi_k\|_{\mathbb{BC}} + \|\eta_k\|_{\mathbb{BC}} \text{ and linearity of } \Delta^{(\tilde{\alpha})},$$

we have

$$\sup_{k\in K_1\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}(a\xi+\eta)\| \leq \sup_{k\in K_1\cap K_2\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}\xi_k\|_{\mathbb{BC}} + \sup_{k\in K_1\cap K_2\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}\eta_k\|_{\mathbb{BC}} \leq \infty.$$

Therefore,

$$(\Delta^{(\tilde{\alpha})}(a\xi+\eta)) \in \ell_\infty[\mathbb{BC}, \mathcal{I}] \to (a\xi+\eta) \in \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}].$$

That is, $\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ is a subspace of $\omega(\mathbb{BC})$.
Similarly, we can show that $c[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ and $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ are also sequence spaces of $\omega(\mathbb{BC})$. □

Theorem 3.2. *Define the function* $d_{\ell_\infty[\mathbb{BC},\mathcal{I}]}$ *by*

$$d_{\ell_\infty[\mathbb{BC},\mathcal{I}]} : \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{C}_2}] \times \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}] \to [0,\infty),$$
$$d_{\ell_\infty[\mathbb{BC},\mathcal{I}]}(\xi,\eta) = \sup_{k\in K\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}\xi_k - \Delta^{(\tilde{\alpha})}\eta_k\|_{\mathbb{BC}}. \tag{1}$$

Then, $(\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}], d_{\ell_\infty[\mathbb{BC},\mathcal{I}]})$ *is a complete metric space.*
We have $(c[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}],\ d_{\ell_\infty[\mathbb{BC},\mathcal{I}]})$ *and* $(c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}], d_{\ell_\infty[\mathbb{BC},\mathcal{I}]})$ *are also complete metric spaces.*

Proof. One can easily show that $d_{\ell_\infty[\mathbb{BC},\mathcal{I}]})$ is a metric on the space $\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$.
To show $\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ is complete. Let $(\xi^n) \in \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$, be an arbitrary Cauchy sequence for
$(\xi^n) = (\xi^n_k)_{k\in\mathbb{N}}$. So, $\exists K \in \mathcal{F}(\mathcal{I})$ such that

$$\sup_{k\in K\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}\xi^n_k - \Delta^{(\tilde{\alpha})}\xi^r_k\|_{\mathbb{BC}} \leq \varepsilon,\ \forall n, r \in K.$$

Then for any fixed $k \in K$,

$$\|\Delta^{(\tilde{\alpha})}\xi^n_k - \Delta^{(\tilde{\alpha})}\xi^r_k\|_{\mathbb{BC}} \leq \varepsilon,\ \forall n, r \in K.$$

In this case, for any fixed k, $(\Delta^{(\tilde{\alpha})}\xi^1_k,\ \Delta^{(\tilde{\alpha})}\xi^2_k,\ \Delta^{(\tilde{\alpha})}\xi^3_k, \cdots, \Delta^{(\tilde{\alpha})}\xi^r_k, \cdots)$ is a Cauchy sequence, so it converges to a point say $\Delta^{(\tilde{\alpha})}\xi^*_k$.
Define

$$(\Delta^{(\tilde{\alpha})}\xi^*_k) = (\Delta^{(\tilde{\alpha})}\xi^*_1,\ \Delta^{(\tilde{\alpha})}\xi^*_2,\ \Delta^{(\tilde{\alpha})}\xi^*_3, ...)$$

with infinitely many limits $\Delta^{(\tilde{\alpha})}\xi^*_1,\ \Delta^{(\tilde{\alpha})}\xi^*_2,\ \Delta^{(\tilde{\alpha})}\xi^*_3, ...$ and show $(\Delta^{(\tilde{\alpha})}\xi^*_k) \in \ell_\infty[\mathbb{BC}, \mathcal{I}]$ and
so $(\xi^*_k) \in \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$, and $(\xi^n_k) \to (\xi^*_k)$ as $n \to \infty, n \in K \in \mathcal{F}(\mathcal{I})$.

Taking limit as $r \to \infty$ and fixing k, using the continuity of Euclidian norm function, we get $\forall n \in K$, we get

$$\|\Delta^{(\tilde{\alpha})}\xi_k^n - \Delta^{(\tilde{\alpha})}\xi_k^*\|_{\mathbb{BC}} \ \le \varepsilon$$

and so,

$$d_{\ell_\infty[\mathbb{BC},\mathcal{I}]}(\xi^n, \xi^*) = \sup_{k\in K\in\mathcal{F}(\mathcal{I})} \|(\Delta^{(\tilde{\alpha})}\xi_k^n) - (\Delta^{(\tilde{\alpha})}\xi_k^*)\|_{\mathbb{BC}} \ \le \varepsilon.$$

Therefore, $(\xi_k^n) \in \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ converges to $(\xi_k^*) \in \omega(\mathbb{BC})$.
On the other hand, since $(\xi_k^n) \in \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ for each $n \in \mathbb{N}$, $\exists t_n \in (0,\infty)$, such that

$$\|\Delta^{(\tilde{\alpha})}\xi_k^n\|_{\mathbb{BC}} \ \le t_n, \forall k, n \in K,$$

where, n is independent of k.
Hence

$$(\xi_k^*) \in \ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}].$$

This completes the theorem.
Similarly, we can show that $(c[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}],\ d_{\ell_\infty[\mathbb{BC},\mathcal{I}]})$ and $(c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}], d_{\ell_\infty[\mathbb{BC},\mathcal{I}]})$ are also complete metric spaces. □

Theorem 3.3. $\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}], c[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ *and* $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ *are complete normed spaces with norm*

$$\|\xi\|_{\ell_\infty[\mathbb{BC},\mathcal{I}]} = \|(\xi_k)\|_{\ell_\infty[\mathbb{BC},\mathcal{I}]} = \sup_{k\in K\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}\xi_k\|_{\mathbb{BC}}.$$

Proof. This proof is immediate from the above theorem 5. □

Theorem 3.4. $\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}], c[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ *and* $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ *are BK-spaces where the norm is defined by*

$$\|\xi\|_{\ell_\infty[\mathbb{BC},\mathcal{I}]} = \|(\xi_k)\|_{\ell_\infty[\mathbb{BC},\mathcal{I}]} = \sup_{k\in K\in\mathcal{F}(\mathcal{I})} \|\Delta^{(\tilde{\alpha})}\xi_k\|_{\mathbb{BC}}.$$

Proof. Let $\|\xi^n - \xi\|_{\ell_\infty[\mathbb{BC},\mathcal{I}]}$ is $\mathcal{I}-$convergent to 0, as $n \to \infty,\ n \in K \in \mathcal{F}(\mathcal{I})$.
Then for given $\varepsilon > 0$,

$$\begin{aligned}
&\|\xi^n - \xi\|_{\ell_\infty[\mathbb{BC},\mathcal{I}]} < \varepsilon,\ \forall n \in K,\ K \in \mathcal{F}(\mathcal{I}),\\
&\Longrightarrow \sup_{k\in K,\ K\in\mathcal{F}(\mathcal{I})} \|\xi_k^n - \xi_k\|_{\mathbb{BC}} < \varepsilon,\ \forall n \in K,\ K \in \mathcal{F}(\mathcal{I}),\\
&\Longrightarrow \|\xi_k^n - \xi_k\|_{\mathbb{BC}},\ \forall n \in K \in \mathcal{F}(\mathcal{I})
\end{aligned}$$

Therefore, $\|\xi_k^n - \xi_k\|_{\mathbb{BC}}$ is $\mathcal{I}-$ convergent to 0, as k tends to 0, $\forall n \in K \in \mathcal{F}(\mathcal{I})$ □

Theorem 3.5. *The sequence spaces* $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}], c[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ *and* $\ell_\infty[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ *are isomorphic to* c_0, c *and* ℓ_∞ *respectively.*

Proof. We will demonstrate the outcome for the space $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$. Other results can be obtained similarly.
We define a mapping

$$T : c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}] \to c_0,\ \text{by}\ \xi \to \eta = T\xi.$$

Now, for $T\xi = (0, 0, 0, ...)$, $\xi = (0, 0, 0, ...)$ as T is linear. Thus, T is injective.
Let $\eta \in c_0$ and using the inverse operator $\Delta^{(-\tilde{\alpha})}$, we define $\xi = (\xi_k) = (\Delta^{(-\tilde{\alpha})}\eta_k)$, where

$$(\Delta^{(\tilde{\alpha})}\xi_k) = (\sum_{i=1}^{\infty}(-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)}(\Delta^{(-\tilde{\alpha})}\eta_{k-i})) = (\eta_k).$$

As, $\eta \in c_0$, and $\eta = (\eta_k) = (\Delta^{(\tilde{\alpha})}\xi_k)$. Therefore,

$$\xi \in c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}].$$

Thus T is surjective.
Therefore, $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ is isomorphic to c_0. □

Schauder Basis: A Schauder basis is a sequence (x_k) of a normed space $(X, \|\cdot\|)$ if there is a distinct sequence of scalars (a_k) for each $u \in X$ such that

$$\lim_{n\to\infty}\left\|u - \sum_{k=1}^{n} a_k x_k\right\| = 0.$$

Define the sequence ${}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}} = ({}^{(k)}\gamma_n^{\Delta^{(-\tilde{\alpha})}})_{n\in\mathbb{N}}$ defined by

$${}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}} = \begin{cases} (-1)^{n-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(n-k)!\Gamma(-\tilde{\alpha}-n+k+1)}, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n. \end{cases}$$

Theorem 3.6. *The sequence $({}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}})$ is Schauder basis for $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ and every $\xi \in c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ has a distinc way of expressing in the form*

$$\xi = \sum_k \lambda_k^{\Delta^{(\tilde{\alpha})}}\, {}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}}$$

where,

$$\lambda_k^{\Delta^{(\tilde{\alpha})}} = (\Delta^{(\tilde{\alpha})}\xi)_k\ , \quad \textit{for each } k \in \mathbb{N}.$$

Proof. Applying the definition of $\Delta^{(\tilde{\alpha})}$ and ${}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}}$ we get

$$\Delta^{(\tilde{\alpha})}\,{}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}} = e^{(k)} \in c_0.$$

Now, set $\{e^{(k)} : n \in \mathbb{N}\}$ is the basis for c_0. As $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ is isomorphic to c_0, and $T : c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}] \to c_0$ is by $\xi \to \eta = T\xi$ is onto, the basis of $c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ is the inverse image of the basis of c_0. i.e

$$\lim_{n\to\infty}\left\|\xi - \sum_{k=1}^{n}\lambda_k^{\Delta^{(\tilde{\alpha})}}\,{}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}}\right\|_{\ell_\infty[\mathbb{BC},\mathcal{I}]} = 0,$$

$\xi \in c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$.

Uniqueness: Let us assume that $\xi = \sum_k \beta_k^{\Delta^{(\tilde{\alpha})}}\,{}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}}$, then

$$\begin{aligned}(\Delta^{(\tilde{\alpha})}\xi)_k &= (\Delta^{(\tilde{\alpha})}\sum_k \beta_k^{\Delta^{(\tilde{\alpha})}}\,{}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}})_k \\ &= \sum_k \beta_k^{\Delta^{(\tilde{\alpha})}}(\Delta^{(\tilde{\alpha})}\,{}^{(k)}\gamma^{\Delta^{(-\tilde{\alpha})}})_k \\ &= \sum_k \beta_k^{\Delta^{(\tilde{\alpha})}} e^{(k)} = \beta_k^{\Delta^{(\tilde{\alpha})}},\end{aligned}$$

which contradicts our assumption that $\lambda_k^{\Delta^{(\tilde{\alpha})}} = (\Delta^{(\tilde{\alpha})}\xi)_k$, for each $k \in \mathbb{N}$. □

4. Dual Properties

Theorems establishing the $\alpha-$, $\beta-$, and $\gamma-$ duals of the ideal convergent fractional differences of bi-complex numbers are developed and proven in this section. For $\alpha-$duals, the notation α differs from the operator $\Delta^{(\tilde{\alpha})}$. Initially, the study on dual computation was started by Köthe and Toeplitz [8] and got the $\alpha-$dual whose elements can be represented as sequences. The duals of the new spaces will be obtained using the following results (Lemma 4.1 to Lemma 4.4) attributed to Stielglitz and Tietz [18]. Here, S stands for the collection of all finite subsets of $\mathbb{N}$.

Lemma 4.1. *$A = (a_{nm}) \in (c_0, \ell_1) = (c, \ell_1)$ if and only if*

$$\sup_{M \in S} \sum_n \Big| \sum_{m \in M} a_{nm} \Big| < \infty \tag{2}$$

Lemma 4.2. *$A = (a_{nm}) \in (c_0, c)$ if and only if*

$$\sup_{n \in \mathbb{N}} \sum_m |a_{nm}| < \infty, \text{ and} \tag{3}$$

$$\lim_{n\to\infty} a_{nm} \text{ exists, for any } m \in \mathbb{N}. \tag{4}$$

Lemma 4.3. *$A = (a_{nm}) \in (c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$ if and only if Equation (3) holds.*

Lemma 4.4. *$A = (a_{nm}) \in (c, c)$ if and only if Equation (2) and (3) hold and $\lim\limits_{n\to\infty} \sum\limits_m a_{nm}$ exists.*

Theorem 4.5. *Let*

$$\mathcal{U}_1^{\Delta^{(-\tilde{\alpha})}} = \Big\{(\mu_k) \in \omega(\mathbb{BC}) : \sup_{K \in S \cap 2^{\mathcal{F}(\mathcal{I})}} \sum_n \Big\| \sum_{k \in K} \sum_{j=k}^{n} (-1)^{n-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)} \mu_k \Big\|_{\mathbb{BC}} < \infty \Big\},$$

then,

$$c_0[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]^\alpha = c[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]^\alpha = \mathcal{U}_1^{\Delta^{(-\tilde{\alpha})}}.$$

Proof. Let, $(\mu_n) \in \omega(\mathbb{BC})$ and $\xi = (\xi_n)$ is defined as $(\xi_n) = (\Delta^{(-\tilde{\alpha})}\eta_n)$, Then

$$\mu_n \xi_n = \sum_{i=0}^{n} \sum_{j=i}^{n} (-1)^{n-i} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)} \mu_n \eta_i = (\mathcal{U}^{\Delta^{(-\tilde{\alpha})}} \eta_n), \text{for each } n \in \mathbb{N}.$$

where, $\mathcal{U}^{\Delta^{(-\tilde{\alpha})}} = (\mu_{nk}^{\Delta^{(-\tilde{\alpha})}})$ is a matrix defined by

$$\mu_{nk}^{\Delta^{(-\tilde{\alpha})}} = \begin{cases} \sum\limits_{j=k}^{n} (-1)^{n-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)} \mu_n, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n. \end{cases}$$

Therefore, we conclude that

$$\mu\xi = (\mu_n \xi_n) \in \ell_1[\mathbb{BC}, \mathcal{I}], \text{ whenever } \xi \in Z[\mathbb{BC}, \mathcal{I}, \Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}] \text{ for } Z \in \{c_0, c\}$$

if and only if

$$(\mathcal{U}^{\Delta^{(-\tilde{\alpha})}}\eta_n) \in \ell_1[\mathbb{BC},\mathcal{I}], \text{ whenever } y \in Z[\mathbb{BC},\mathcal{I}], \quad \text{for } Z \in \{c_0, c\}.$$

Therefore, $(\mu_n) \in Z[\mathbb{BC},\mathcal{I},\Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ for $Z \in \{c_0, c\}$ if and only if

$$\mathcal{U}^{\Delta^{(-\tilde{\alpha})}} \in (c_o[\mathbb{BC},\mathcal{I}], \ell_1[\mathbb{BC},\mathcal{I}]) = (c[\mathbb{BC},\mathcal{I}], \ell_1[\mathbb{BC},\mathcal{I}]).$$

Thus by using 4.1, we get

$$c_0[\mathbb{BC},\mathcal{I},\Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]^\alpha = c[\mathbb{BC},\mathcal{I},\Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]^\alpha = \mathcal{U}_1^{\Delta^{(-\tilde{\alpha})}}.$$

□

Theorem 4.6. *Define the sets* $\mathcal{U}_2^{\Delta^{(-\tilde{\alpha})}}$, $\mathcal{U}_3^{\Delta^{(-\tilde{\alpha})}}$, *and* $\mathcal{U}_4^{\Delta^{(-\tilde{\alpha})}}$ *by*

$$\mathcal{U}_2^{\Delta^{(-\tilde{\alpha})}} = \Big\{(\mu_k) \in \omega(\mathbb{BC}) : \sup_{n\in\mathbb{N}\cap K,\ K\in\mathcal{F}(\mathcal{I})} \sum_{k\in K\in\mathcal{F}(\mathcal{I})} \|a_{nk}^{\Delta^{(-\tilde{\alpha})}}\|_{\mathbb{BC}} < \infty\Big\};$$

$$\mathcal{U}_3^{\Delta^{(-\tilde{\alpha})}} = \Big\{(\mu_k) \in \omega(\mathbb{BC}) : \lim_{n\to\infty, n\in\mathbb{N}\cap K,\ K\in\mathcal{F}(\mathcal{I})} a_{nk}^{\Delta^{(-\tilde{\alpha})}} \text{ exists, for all } k \in K \in \mathcal{F}(\mathcal{I})\Big\};$$

$$\mathcal{U}_4^{\Delta^{(-\tilde{\alpha})}} = \Big\{(\mu_k) \in \omega(\mathbb{BC}) : \lim_{n\to\infty, n\in\mathbb{N}\cap K,\ K\in\mathcal{F}(\mathcal{I})} \sum_{k\in K\in\mathcal{F}(\mathcal{I})} a_{nk}^{\Delta^{(-\tilde{\alpha})}} \text{ exists}\Big\},$$

where,

$$A^{\Delta^{(-\tilde{\alpha})}} = a_{nk}^{\Delta^{(-\tilde{\alpha})}} = \begin{cases} \sum\limits_{j=k}^{n} (-1)^{n-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)}\mu_n, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k > n. \end{cases}$$

Then,

$$c_0[\mathbb{BC},\mathcal{I},\Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]^\beta = \mathcal{U}_2^{\Delta^{(-\tilde{\alpha})}} \cap \mathcal{U}_3^{\Delta^{(-\tilde{\alpha})}} \text{ and}$$
$$c[\mathbb{BC},\mathcal{I},\Delta^\alpha, \|\cdot\|_{\mathbb{BC}}]^\beta = \mathcal{U}_2^{\Delta^{(-\tilde{\alpha})}} \cap \mathcal{U}_3^{\Delta^{(-\tilde{\alpha})}} \cap \mathcal{U}_4^{\Delta^{(-\tilde{\alpha})}}.$$

Proof. For the space $c_0[\mathbb{BC},\mathcal{I},\Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$, we will demonstrate the result.

Let, $(\mu_k) \in \omega(\mathbb{BC})$ and $\xi = (\xi_k)$ be as $(\xi_k) = (\Delta^{(-\tilde{\alpha})}\eta_k)$, then we have

$$\begin{aligned}\sum_{k=0}^{n} \mu_k\xi_k &= \sum_{k=0}^{n} \mu_k \sum_{i=0}^{k}\sum_{j=i}^{k} (-1)^{k-i} \frac{\Gamma(-\tilde{\alpha}+1)}{(k-j)!\Gamma(-\tilde{\alpha}-k+j+1)}\eta_i \\ &= \sum_{k=0}^{n} \Big[\sum_{i=0}^{k}\sum_{j=i}^{k} (-1)^{k-i} \frac{\Gamma(-\tilde{\alpha}+1)}{(k-j)!\Gamma(-\tilde{\alpha}-k+j+1)}\mu_i\Big]\eta_k \\ &= (A^{\Delta^{(-\tilde{\alpha})}}\eta)_n, \text{ for each } n \in \mathbb{N}.\end{aligned}$$

Hence, $\mu\xi = (\mu_k\xi_k) \in cs[\mathbb{BC},\mathcal{I}]$ whenever $(\xi_k) \in c_0[\mathbb{BC},\mathcal{I},\Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]$ if and only if $A^{\Delta^{(-\tilde{\alpha})}} \in (c_0[\mathbb{BC},\mathcal{I}], c[\mathbb{BC},\mathcal{I}])$.

Then using Lemma 4.2, we can conclude $c_0[\mathbb{BC},\mathcal{I},\Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}] = \mathcal{U}_2^{\Delta^{(-\tilde{\alpha})}} \cap \mathcal{U}_3^{\Delta^{(-\tilde{\alpha})}}$. □

Theorem 4.7. $c_0[\mathbb{BC},\mathcal{I},\Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]^\gamma = c[\mathbb{BC},\mathcal{I},\Delta^{(\tilde{\alpha})}, \|\cdot\|_{\mathbb{BC}}]^\gamma = \mathcal{U}_2^{\Delta^{(-\tilde{\alpha})}}$.

Proof. Following a similar process and using lemma 4.2, one can easily prove the theorem. □

5. Conclusion

This article extends existing findings on fractional order difference sequence spaces of bi-complex numbers to $\mathcal{I}-$convergent Fractional order sequence spaces of bi-complex numbers. The findings in this publication offer a fresh viewpoint in addition to generalizing the earlier research conducted by other authors ([1], [2], [3]).

6. Acknowledgement

The DST/INSPIRE Fellowship/[IF220239], Ministry of Science and Technology, Government of India, Department of Science and Technology, Technology Bhawan, New Mehrauli Road, New Delhi-110016, provided financial support for the first author's work.

References

[1] P. Baliarsingh, Some new difference sequence spaces of fractional order and their dual spaces, *Applied Mathematical Computing* **219** (2013), 9737–9742.

[2] P. Baliarsingh, S. Dutta, A unifying approach to the difference operators and their applications, *Bol. Soc. Paran. Mat.* **33** (2015), 49–57.

[3] P. Baliarsingh, S. Dutta, On the classes of fractional order difference sequence spaces and their matrix transformations, *Applied Mathematical Computing* **250** (2015), 665–674.

[4] S. Bera, B.C. Tripathy, Statistically convergent difference sequences of bi-complex numbers, *Journal of Applied Analysis*, (2024). https://doi.org/10.1515/jaa-2023-0133

[5] T. Deb, B.C. Tripathy, I-Monotonic Convergence of Sequence of Bi-complex Numbers, *Journal of Applied Analysis and Computation* **31** (2025), no. 1, 103-111. https://doi.org/10.1515/jaa-2024-0080

[6] G.S. Dragoni, Sulle funzioni olomorfe di una variable bicomplessa, *Reale Acad. d'Italia Mem. Class Sci. Fic. Mat. Nat.* **5** (1934), 597-665.

[7] H. Fast, Sur la convergence statistique, *Colloquium Mathematicum* **2** (1951), no. 3-4, 241–244.

[8] G. Köthe, O. Toeplitz, Linear Raume mit unendlich vielen koordinaten and Ringe unenlicher Matrizen, *Journal für die reine und angewandte Mathematik* **171** (1934), 193–226.

[9] H. Kızmaz, On certain sequence spaces, *Canadian Mathematical Society* **24** (1981), 169–176.

[10] P. Kostyrko, T. Šalàt, W. Wilczyński, I-convergence, *Real Analysis Exchange* **26** (2000-2001), 669-686.

[11] S. Kumar, B.C. Tripathy, Almost Convergence Double Sequences of Bi-complex Numbers, *Filomat* **38** (2024), no. 11, 3957–3970.

[12] G.B. Price, *An introduction to multicomplex space and function*, Marcel Dekker Inc., 1991.

[13] D. Rochon, M. Shapiro, On algebraic properties of bi-complex and hyperbolic numbers, *Anal. Univ. Oradea, fasc. Math.* **11** (2004), 71-110.

[14] I.J. Schoenberg, The integrability of certain functions and related summability methods, *The American Mathematical Monthly* **66** (1959), no. 5, 361–375.

[15] C. Segre, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Mathematische Annalen* **40** (1892), 413-467.

[16] N. Spampinato, Estensione nel campo bicomplesso di due teoremi, del levi-Civita e del severi, per le funzione olomorfe di due variablili complesse. I.II, *Atti della Reale Accademia dei Lincei, Rendiconti* **22** (1935), no. 6, 38-43.

[17] R.K. Srivastava, N.K. Srivastava, On a class of entire bicomplex sequences, *South East Asian Journal of Mathematics and Mathematical Sciences* **5** (2007), no.3, 47–68.

[18] M. Stieglitz, H. Tietz, Matrixtransformationen von Folgenräumen eine Ergebnisübersicht, *Math. Z.* **154** (1977), 1–16.

[19] B.C. Tripathy, M. Sen, S. Nath, I-convergence in probabilistic n-normed space, *Soft Computing* **16** (2012), 1021-1027. DOI 10.1007/s00500-011-0799-8.

[20] M.A. Wagh, On certain spaces of bi-complex sequences, *International journal of Chemistry, Mathematics and Physics* **7** (2014), no. 1, 1–6.

(Tapasi Deb) Tripura University, Department of Mathematics, Suryamaninagar, Agartala 799022, India
E-mail address: tapasideb.dnr@gmail.com

(Binod Chandra Tripathy) Tripura University, Department of Mathematics, Suryamaninagar, Agartala 799022, India
E-mail address: tripathybc@gmail.com, tripathybc@yahoo.com