Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 53(1), 2026, Pages 136–144, DOI: 10.52846/ami.v53i1.2113
ISSN: 1223-6934

Local Metric Dimension of Cayley Graphs of Commutative Rings

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Abstract. Let p and q be prime numbers. In this paper, for a positive integer n, we investigate the local metric dimension for Cayley graphs $\mathrm{Cay}(\mathbb{Z}_{qp^n}, Z^*(\mathbb{Z}_{qp^n}))$ in the case that $q \in \{p, 2, 3\}$, where $Z^*(\mathbb{Z}_{qp^n})$ is the set of all non-zero zero-divisor of $\mathbb{Z}_{qp^n}$.

2020 *Mathematics Subject Classification.* Primary 05C25; Secondary 05C12.
Key words and phrases. Cayley graph; Zero-divisor; Metric dimension; Local metric dimension.

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1. Introduction

Cayley graphs are one of the oldest and most important algebraic graphs that have established a suitable link between algebraic structures and graph theory. Let Ω be a finite group with identity element e and S be a non-empty subset of Ω such that $e \notin S$ and $S = S^{-1} = \{s^{-1} \mid s \in S\}$. Then the Cayley graph of Ω with respect to S, denoted by $\mathrm{Cay}(\Omega, S)$ is a simple graph with the vertex set Ω and two distinct vertices x and y are adjacent if and only if $x^{-1}y \in S$. Moreover, if S is finite, then $\mathrm{Cay}(\Omega, S)$ is $|S|$-regular.

Let R be a finite commutative ring with non-zero identity. To study the algebraic properties of R, some researchers focus on Cayley graph $\mathrm{Cay}((R, +), S)$, where S is an special subset of R. For instance, whenever $S = U(R)$ is the set of all unit elements of R, the graph $\mathrm{Cay}((R, +), U(R))$, is called the unitary Cayley graph of R. For more information on unitary Cayley graphs, we refer the reader to [20, 5, 23, 24] and [1]. Also for more information about this context and related graphs, see [13, 10, 11, 33, 21, 9, 22, 18].

Another direction is that whenever we put $S = Z^*(R)$, the set of all non-zero zero-divisor of R. In this situation, the Cayley graph $\mathrm{Cay}((R, +), Z^*(R))$, is called the Cayley graph of R with respect to its zero-divisors (see [2] and its introduction).

On the other hand, the concept of metric dimension was introduced by Slater [32], and Harary and Melter [19]. Let G be a simple, and connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For any two distinct vertices x and y in $V(G)$, the distance $\mathrm{d}_G(x, y)$ (or $\mathrm{d}(x, y)$ for short) between x and y is the length of a shortest path between them. We say that the vertices $x, y \in V(G)$ are distinguished by a vertex $w \in V(G)$ if $\mathrm{d}(x, w) \neq \mathrm{d}(y, w)$. A subset $W \subseteq V(G)$ is a metric generator for G if every pair of vertices of G can be distinguished by some vertex in W. A metric

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Received February 22, 2025. Accepted March 13, 2025.

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generator of minimum cardinality is a metric basis, and its cardinality is the metric dimension of G, denoted by $\dim(G)$. Now, suppose that $W = \{w_1, w_2, \cdots, w_\ell\}$ is an ordered set of vertices of G. For any vertex x of G, the metric representation $r(x|W)$ of x with respect to W is the ℓ-tuple $(\mathrm{d}(x, w_1), \mathrm{d}(x, w_2), \cdots, \mathrm{d}(x, w_\ell))$. Thus, W is a metric generator for G if and only if, for every pair of distinct vertices x and y of G, we have $r(x|W) \neq r(y|W)$. It is usually not necessary to distinguish all pairs of vertices, but only adjacent pairs. For this reason, the local metric dimension was introduced in [28]. A subset W of $V(G)$ is called a local metric generator for G if, for every adjacent vertices x and y of G, we have $r(x|W) \neq r(y|W)$. The local metric dimension of G, denoted by $\dim_\ell(G)$, is the smallest order of a local metric generator for G.

For a positive integer m, assume that $\mathbb{Z}_m$ is the ring of integers modulo m. In [30], the second present author together with A. Rezaei and M. Afkhami proved that

$$\dim_\ell(\mathrm{Cay}(\mathbb{Z}_n, \{\pm 1, \pm 3\})) = \begin{cases} 2 & \text{if } n > 7 \text{ and } n \equiv 1, 3, 5 \bmod 6, \\ 1 & \text{if } n > 7 \text{ and } n \equiv 0, 2, 4 \bmod 6. \end{cases}$$

Several papers are devoted to the study of metric dimensions of algebraic graphs. We refer the reader to [29, 17, 4, 30] for the metric dimension of various types of Cayley graphs and to [15, 25, 14, 26, 3, 16, 27] for the metric dimension of some other algebraic graphs.

Additionally, the concept of total graphs was introduced in [6]. Let R be a commutative ring with non-zero identity. The total graph of R, denoted by $T(\Gamma(R))$ is an undirected graph with vertex set R and two vertex x and y are adjacent if $x+y \in Z(R)$. Note that, in general, $T(\Gamma(R))$ is not regular. By using a method similar that they used in [31, Theorem 5], one can show that, for even positive integer m, the Cayley graph $\mathrm{Cay}(\mathbb{Z}_m, Z^*(\mathbb{Z}_m))$ is isomorphic to $T(\Gamma(\mathbb{Z}_m))$. Clearly, for even number $m = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, the Jacobson radical of $\mathbb{Z}_m$ is equal to $J = < 2p_2 \cdots p_t >$. Hence, in view of [15, Theorem 2.6], we have $\dim(\mathrm{Cay}(\mathbb{Z}_m, Z^*(\mathbb{Z}_m))) = (|J| - 1)|\mathbb{Z}_m/J|$.

Let p be a prime number and q be a positive integer. In this paper, we determine local metric dimension of the unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_{qp^n}, Z^*(\mathbb{Z}_{qp^n}))$ with respect to its zero-divisors in the following situations:

$$\dim_\ell(\mathrm{Cay}(\mathbb{Z}_{qp^n}, Z^*(\mathbb{Z}_{qp^n}))) = \begin{cases} p^{n+1} - p & \text{if } q = p, \\ p - 1 & \text{if } q \in \{2, 3\} \text{ and } n = 1, \\ qp^n - qp & \text{if } q \in \{2, 3\} \text{ and } n > 2. \end{cases}$$

Now, we recall some basic definitions and notations on graph theory from [7]. Let G be a simple finite graph with the vertex set $V(G)$ and the edge set $E(G)$. For any two distinct vertices x and y in $V(G)$, the adjacency of x and y will be denoted by $x \sim y$. The diameter of G, denoted by $\mathrm{diam}(G)$, is

$$\mathrm{diam}(G) = \mathrm{Sup}\{\mathrm{d}(x, y) : \ x \text{ and } y \text{ are distinct vertices of } G\}.$$

The neighbourhood of a vertex x of G, denoted by $N(x)$, is defined to be

$$\{y \mid y \in V(G) \text{ and } y \sim x \}.$$

For a subset A of $V(G)$, by $G[A]$, we mean the subgraph of G induced by A. A subset H of $V(G)$ is called a clique if $G[H]$ is a complete graph. Also, we use the notation K_n to denote the complete graph with n vertices. Suppose that, for each $0 \leq i \leq n$, G_i

is an induced subgraph of G such that $\{V(G_1), \cdots, V(G_n)\}$ is a partition for $V(G)$. Then we denote G by $G = \dot{\bigcup}_{0\leq i\leq n} G_i$.

For a commutative ring R, we denote the unitary Cayley graph of R with respect to its zero-divisors briefly by $\mathcal{C}_R$. Finally, we use $\mathbb{N}_0$ and $\mathbb{N}$ to denote the sets of non-negative integers and positive integers, respectively.

2. Local metric dimension of $\mathcal{C}_{\mathbb{Z}_{p^n}}$

Recall that for a disconnected graph G with components $G_1, \cdots, G_t$, and T isolated vertices of G, we have $\dim_\ell(G) = \sum_{i=1}^{t} \dim_\ell(G_i)$ and

$$\dim(G) = \sum_{i=1}^{t} \dim(G_i) + \begin{cases} 0 & \text{if } T = \emptyset, \\ |T| - 1 & \text{otherwise.} \end{cases}$$

Also we have that $\dim_\ell(K_n) = n - 1$.

We begin with the following theorem.

Theorem 2.1. *Assume that* $p \geq 2$ *is a prime number and* n *is a positive integer with* $n > 1$. *Then* $\dim(\mathcal{C}_{\mathbb{Z}_{p^n}}) = \dim_\ell(\mathcal{C}_{\mathbb{Z}_{p^n}}) = p^n - p$.

Proof. Clearly $Z^*(\mathbb{Z}_{p^n}) = \{rp \mid 1 \leqslant r \leqslant p^{n-1} - 1\}$. Hence $|Z^*(\mathbb{Z}_{p^n})| = p^{n-1} - 1$. Now, for each $i \in \mathbb{N}$ with $0 \leq i \leq p - 1$, we put

$$X_i := \{pk + i \mid k \in \mathbb{N} \text{ and } 0 \leq pk + i \leq p^n - 1\}.$$

It is easy to see that $|X_i| = p^{n-1}$ and that $X_0, X_1, \cdots, X_{p-1}$ provide a partition for $\mathbb{Z}_{p^n}$. Moreover, for each $0 \leq i, j \leq p - 1$ with $i \neq j$, we have the following facts:

(i) $\mathcal{C}[X_i] \cong K_{p^{n-1}}$ because, for every distinct vertices $pk + i$ and $pk' + i$ in X_i, we have

$$pk + i - (pk' + i) = p(k + k') \in Z^*(\mathbb{Z}_{p^n}).$$

(ii) There is no edge in $\mathcal{C}_{\mathbb{Z}_{p^n}}$ with one end in X_i and the other end in X_j.

Thus $\mathcal{C}_{\mathbb{Z}_{p^n}} = \dot{\bigcup}_{0\leq i\leq p-1} \mathcal{C}[X_i]$. Hence

$$\dim_\ell(\mathcal{C}_{\mathbb{Z}_{p^n}}) = \sum_{i=0}^{p-1} \dim_\ell(X_i) = p \dim_\ell(K_{p^{n-1}}) = p(p^{n-1} - 1) = p^n - p.$$

□

Remark 2.1. Since, for a prime number p, $\mathbb{Z}_p$ is a field, the graph $\mathcal{C}_{\mathbb{Z}_p}$ consists of p isolated vertices. Hence $\dim(\mathcal{C}_{\mathbb{Z}_p}) = p - 1$ and $\dim_\ell(\mathcal{C}_{\mathbb{Z}_p}) = 0$.

3. Local metric dimension of $\mathcal{C}_{\mathbb{Z}_{2p^n}}$

Throughout this section, we shall assume that $p > 2$ is a prime number. Let n be a positive integer. In this section, for convenience, we shall use A and B to denote the sets of odd and even integers in $\mathbb{Z}_{2p^n}$, respectively. Clearly, for any two distinct vertices x and y in A (or B), we have $x \sim y$ in $\mathcal{C}_{\mathbb{Z}_{2p^n}}$. Hence we have the following lemma.

Lemma 3.1. *The graphs* $\mathcal{C}_{\mathbb{Z}_{2p^n}}[A]$ *and* $\mathcal{C}_{\mathbb{Z}_{2p^n}}[B]$ *are cliques.*

Proposition 3.2. *Let x and y be two arbitrary distinct vertices in $\mathcal{C}_{\mathbb{Z}_{2p^n}}$. Then*

$$\mathrm{d}(x,y)=\begin{cases} 1 & \text{if } x,y\in A \text{ or } x,y\in B,\\ 1 & \text{if } x\in A \text{ and } y\in B \text{ (or } y\in A \text{ and } x\in B\text{) and } p|x-y,\\ 2 & \text{otherwise.}\end{cases}$$

Proof. The first claim follows from Lemma 3.1. So, without loss of generality, we may assume that $x\in A$ and $y\in B$. Now, we consider the following two cases:

Case 1. $p\mid x-y$. Then $x\sim y$ in $\mathcal{C}_{\mathbb{Z}_{2p^n}}$, and so $\mathrm{d}(x,y)=1$.

Case 2. $p\nmid x-y$. Hence $\mathrm{d}(x,y)\neq 1$. In the following, we determine a common neighbourhood for x and y. To achieve this, we consider the following three situations:

(1) If $y<p^n$, then $y+p^n<2p^n$. Hence $x\sim(y+p^n)\sim y$, and so $\mathrm{d}(x,y)=2$.

(2) If $y>p^n$, then $0<y-p^n<2p^n$. Hence $x\sim(y-p^n)\sim y$, and so $\mathrm{d}(x,y)=2$.

(3) If $y=p^n$, then $x\neq 0$ because $p\nmid x-y$. Hence $x\sim 0\sim y$, and so $\mathrm{d}(x,y)=2$. □

Proposition 3.3. *Assume that $p>2$ is a prime number and that $x\in\mathbb{N}_0$ with $0\le x\le 2p-1$. Then, for any even number k with $0\le x+kp\le 2p^n-1$, in graph $\mathcal{C}_{\mathbb{Z}_{2p^n}}$, we have that $N(x)=N(x+kp)$.*

Proof. Suppose that k is an even number with $0\le x+kp\le 2p^n-1$. Then $y\in N(x+kp)$ if and only if $p\mid x+kp-y$ or $2\mid x+kp-y$ (or both of them). Since k is even, we can conclude that $y\in N(x+kp)$ if and only if $p\mid x-y$ or $2\mid x-y$. Thus $y\in N(x+kp)$ if and only if $y\in N(x)$, and so $N(x+kp)=N(x)$. □

Theorem 3.4. *Suppose that $p>2$ is a prime number and $n\ge 2$ is an integer. Then $\dim_\ell(\mathcal{C}_{\mathbb{Z}_{2p^n}})=2p^n-2p$.*

Proof. For any integer x with $0\le x\le 2p-1$, we put

$$Y_x:=\{x+kp\mid k \text{ is an even number with } 0\le x+kp\le 2p^n-1\}.$$

Clearly $|Y_x|=p^{n-1}$. It is easy to see that $\{Y_0,Y_1,Y_2,\cdots,Y_{2p-1}\}$ forms a partition for $\mathbb{Z}_{2p^n}$. We put

$$W:=\bigcup_{x=0}^{2p-1}(Y_x\backslash\{x\}).$$

Then $|W|=2p(p^{n-1}-1)=2p^n-2p$.

Now, we show that W is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{2p^n}}$. To do this, suppose that x and y are two arbitrary adjacent vertices in $\mathcal{C}_{\mathbb{Z}_{2p^n}}$. Without loss of generality, we may assume that $x,y\notin W$ and that $x<y$. Hence $0\le x<y\le 2p-1$. Since $x\sim y$, we have $p\mid y-x$ or $2\mid y-x$.

First assume that $p\mid y-x$. Since $0\le x<y\le 2p-1$, we have $y-x=p$. Then $x\sim x+2$ and $y\not\sim x+2$. Since Y_x's form a partition for $\mathbb{Z}_{2p^n}$, there exists $w\in W$ such that $N(w)=N(x+2)$. Hence, by Proposition 3.2, $\mathrm{d}(x,w)=1$ and $\mathrm{d}(y,w)=2$. Thus $r(x|W)\neq r(y|W)$.

Now, suppose that $2\mid y-x$. Since $0\le x<y\le 2p-1$, we have $y-x=2\alpha$, for some integer $\alpha<p$. Then $x\sim x+p$ and $y\not\sim x+p$. Again, since Y_x's form a partition for $\mathbb{Z}_{2p^n}$, there exists $w\in W$ such that $N(w)=N(x+p)$. Hence, by Proposition 3.2, $\mathrm{d}(x,w)=1$ and $\mathrm{d}(y,w)=2$. Thus $r(x|W)\neq r(y|W)$.

Theses imply that W is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{2p^n}}$.

Now, we assume that W' is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{2p^n}}$ with $|W'| < 2p^n - 2p$, and we seek a contradiction. Since $|W'| < 2p^n - 2p$ and Y_x's form a partition for $\mathbb{Z}_{2p^n}$, in view of Proposition 3.3, there exist $0 \leq x \leq 2p-1$ and $y, y' \in Y_x \backslash W'$ such that $N(y) = N(y')$. Hence $y \sim y'$ and $\mathrm{d}(y,w) = \mathrm{d}(y',w)$ for all $w \in W'$. Thus $r(y|W') = r(y'|W')$ which is a contradiction. Hence

$$\dim_\ell(\mathcal{C}_{\mathbb{Z}_{2p^n}}) = |W| = 2p^n - 2p.$$

□

Theorem 3.5. *Suppose that $p > 2$ is a prime number. Then $\dim_\ell(\mathcal{C}_{\mathbb{Z}_{2p}}) = p - 1$.*

Proof. In view of Proposition 3.2, it is easy to see that, for any vertex a in A, there exists unique vertex b in B such that $\mathrm{d}(a,b) = 1$.

We claim that $W = \{2k \mid k = 0, 1, \cdots, p-2\}$ is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{2p}}$. To achieve this, we suppose that x and y are two adjacent vertices in $\mathcal{C}_{\mathbb{Z}_{2p}}$. Without loss of generality, we may assume that $x, y \notin W$ and that $x \in A$ and $y \in B$. Hence $x = 2p - 2$. So, for all $w \in W$, $\mathrm{d}(x,w) = 1$ and $\mathrm{d}(y,w) = 2$. Thus $r(x|W) \neq r(y,W)$. Hence $\dim_\ell(\mathcal{C}_{\mathbb{Z}_{2p}}) \leq |W| = p - 1$.

Now, we assume that $W' \subseteq \mathcal{C}_{\mathbb{Z}_{2p}}$ with $|W'| < p - 1$ is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{2p}}$ and we seek a contradiction. Then, in view of Proposition 3.2, there exist $x', y' \in A \backslash W'$ and $x'', y'' \in B \backslash W'$ such that $x' \sim x''$ and $y' \sim y''$. Therefore, for all $w \in W' \bigcap A$ and $w' \in W' \bigcap B$, we have $\mathrm{d}(x'', w) = 2 = \mathrm{d}(y'', w)$ and $\mathrm{d}(x'', w') = 1 = \mathrm{d}(y'', w')$. Note that $x'' \sim y''$. So $r(x''|W') = r(y''|W')$ which is a contradiction. Thus

$$\dim_\ell(\mathcal{C}_{\mathbb{Z}_{2p}}) = |W| = p - 1.$$

□

4. Local metric dimension of $\mathcal{C}_{\mathbb{Z}_{3p^n}}$

In this section, we determine the local metric dimension of $\mathcal{C}_{\mathbb{Z}_{3p^n}}$, where p is a prime with $p > 3$. At first, we assume that $n \neq 1$.

Theorem 4.1. *Suppose that $p > 3$ is a prime number and $n > 1$ is an integer. Then $\dim_\ell(\mathcal{C}_{\mathbb{Z}_{3p^n}}) = 3p^n - 3p$.*

Proof. For any integer i with $0 \leq i \leq p-1$, we consider the following sets:

$$A^i := \{3i + 3\ell p \mid \ell \in \mathbb{N}_0 \text{ and } 3i + 3\ell p < 3p^n\},$$
$$B^i := \{1 + 3i + 3\ell p \mid \ell \in \mathbb{N}_0 \text{ and } 1 + 3i + 3\ell p < 3p^n\},$$
$$C^i := \{2 + 3i + 3\ell p \mid \ell \in \mathbb{N}_0 \text{ and } 2 + 3i + 3\ell p < 3p^n\}.$$

Clearly $|A^i| = |B^i| = |C^i| = p^{n-1}$. Suppose that $A = \cup_{i=0}^{p-1} A^i$, $B = \cup_{i=0}^{p-1} B^i$ and $C = \cup_{i=0}^{p-1} C^i$. Then $|A| = |B| = |C| = p^n$. It is easy to see that the induced subgraphs $\mathcal{C}_{\mathbb{Z}_{3p^n}}[A]$, $\mathcal{C}_{\mathbb{Z}_{3p^n}}[B]$ and $\mathcal{C}_{\mathbb{Z}_{3p^n}}[C]$ are cliques.

On the other hand, $|Z^*(\mathbb{Z}_{3p^n})| = p^{n-1}(p+2) - 1$. Hence the graph $\mathcal{C}_{\mathbb{Z}_{3p^n}}$ is $(p^{n-1}(p+2) - 1)$-regular.

We have the following two situations:

(I) $p \equiv 1 \mod 3$. Hence $p = 3k + 1$ for some natural number k. Thus, for any integer i with $0 \leqslant i \leqslant p - 1$, the sets A^i , B^{k+i} and C^{2k+i} from the complete graph $K_{3p^{n-1}}$ in which the subscripts in $k + i$ and $2k + i$ are real modulo p.

(II) $p \equiv 2 \bmod 3$. Hence $p = 3k + 2$ for some natural number k. Thus, for any integer i with $0 \leqslant i \leqslant p-1$, the sets A^i , B^{2k+1+i} and C^{k+i} from the complete graph $K_{3p^{n-1}}$ in which the subscripts in $2k+1+i$ and $k+i$ are real modulo p.

Now, we put

$$W := \mathbb{Z}_{3p^n} - \{3i, 3i+1, 3i+2 \mid i = 0, 1, \cdots, p-1\}.$$

Then $|W| = 3p^n - 3p$. We claim that W is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{3p^n}}$. To achieve this, we suppose that x and y are two adjacent vertices in $\mathcal{C}_{\mathbb{Z}_{3p^n}}$. Without loss of generality, we may assume that $x, y \notin W$. So we have the following two cases:

Case 1. $x, y \in A$. (The case in which $x, y \in B$ or $x, y \in C$ has quite similar argument and we omit them.) Since we only removed one member from each A^i, does not exist i with $0 \leqslant i \leqslant p-1$ such that $x, y \in A^i$. Hence there exist $0 \leqslant i, j \leqslant p-1$ with $i \neq j$ such that $x \in A^i$ and $y \in A^j$. Now, in view of (I) and (II), there exists $0 \leqslant t \leqslant p-1$ such that $A^i \cup B^t$ is a clique. But there is not any adjacency between vertices in A^j and vertices in B^t. These imply that $\mathrm{d}(x, w) \neq \mathrm{d}(y, w)$ for all $w \in W \cap B^t$, and so $r(x|W) \neq r(y|W)$.

Case 2. $x \in A$ and $y \in B$. (The case in which $x \in A$, $y \in C$ or $x \in B$, $y \in C$ has quite similar argument and we omit them.) By (I) and (II), there exists $w \in (A \cap W) \backslash A^i$ such that $\mathrm{d}(x, w) = 1$ and $\mathrm{d}(y, w) = 2$. Hence $r(x|W) \neq r(y|W)$.

Thus W is a local metric generator for $\mathbb{Z}_{3p^n}$.

Now, we assume that $W' \subseteq \mathbb{Z}_{3p^n}$ with $|W'| < 3p^n - 3p$ is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{3p^n}}$ and we seek a contradiction. Then there exists i with $1 \leq i \leq p-1$ such that one of the following inequalities holds:

(i) $|A^i \cap W'| \leq p^{n-1} - 2$,
(ii) $|B^i \cap W'| \leq p^{n-1} - 2$,
(iii) $|C^i \cap W'| \leq p^{n-1} - 2$.

So, without loss of generality, we may assume that $x, y \in A^i \backslash W'$. Thus $x \sim y$ and $N(x) = N(y)$. Therefore $r(x|W') = r(y|W')$ which is a contradiction. Hence $\dim_\ell(\mathcal{C}_{\mathbb{Z}_{3p^n}}) = |W| = 3p^n - 3p$. ☐

Now we focus on determining $\dim_\ell(\mathcal{C}_{\mathbb{Z}_{3p}})$, where $p > 3$ is a prime number. Throughout the rest of this section, assume that

$$\begin{aligned} &A := \{0, 3, 6, \cdots, 3(p-1)\}, \\ &B := \{1, 4, 7, \cdots, 3(p-1)+1\} \text{ and} \\ &C := \{2, 5, 8, \cdots, 3(p-1)+2\}. \end{aligned}$$

Also we assume that $p \equiv s \bmod 3$, where $s \in \{1, 2\}$. Hence there exists $k \in \mathbb{N}$ such that $p = 3k + s$.

Lemma 4.2. *There exist p triangles T in $\mathcal{C}_{\mathbb{Z}_{3p}}$ such that*

$$|T \cap A| = |T \cap B| = |T \cap C| = 1.$$

Proof. Clearly, for any integer i with $0 \leq i \leq p-1$, subgraph

$$T_i := \{3i, 3(k+i) + s, 3(2k+i) + 2s\}$$

is a triangle and

$$|T \cap A| = |T \cap B| = |T \cap C| = 1.$$

All these members are real modulo $3p$. ☐

Note that the other triangles in $\mathcal{C}_{\mathbb{Z}_{3p}}$ consist entirely of elements A (B or C).

Remark 4.1. Assume that u and v are two adjacent vertices in $\mathcal{C}_{\mathbb{Z}_{3p}}$. Then we have one of the following cases:
(i) $\{u, v\} \subseteq A$;
(ii) $\{u, v\} \subseteq B$;
(iii) $\{u, v\} \subseteq C$;
(iv) There exists an integer i with $0 \leq i \leq p-1$ such that $\{u, v\} \subseteq T_i$.
Moreover, for any arbitrary vertex α in $\mathcal{C}_{\mathbb{Z}_{3p}}$, there exists a unique integer i with $0 \leq i \leq p-1$ such that $\alpha \in T_i$.

Theorem 4.3. *Suppose that $p > 3$ is a prime number. Then* $\dim_\ell(\mathcal{C}_{\mathbb{Z}_{3p}}) = p-1$.

Proof. We put

$$W := \{3i, 3(k+j)+s, 3(2k+\ell)+2s \mid i = 0, 1, \cdots, k-1,\ j = k, \cdots, 2k-1,\ \ell = 2k, \cdots, 3k+s-2\}.$$

All these members are real modulo $3p$. It is easy to see that for any integer i with $0 \leq i \leq p-2$, $|W \cap T_i| = 1$ and that $W \cap T_{p-1} = \emptyset$.

We claim that the set W is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{3p}}$. To achieve this, we assume that x and y are two adjacent vertices in $\mathcal{C}_{\mathbb{Z}_{3p}}$ and that $x, y \notin W$. We have the following different cases:

Case 1. $x, y \in A$. (The case in which $x, y \in B$ or $x, y \in C$ has quite similar arguments and we omit them.) Hence there exist distinct integers i and i' with $0 \leq i, i' \leq p-1$ such that $x \in T_i$ and $y \in T_{i'}$. Without loss of generality, we may assume that $T_i \cap W \neq \emptyset$. Suppose that $w \in T_i \cap W$. Then $\mathrm{d}(x, w) = 1 \neq \mathrm{d}(y, w)$, and so $r(x|W) \neq r(y|W)$, and so W is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{3p}}$.

Case 2. $\{x, y\} \subseteq T_i$ for some integer i with $0 \leq i \leq p-1$. Without loss of generality, we may assume that $x \in A$ and $y \in B$. (The other cases have quite similar arguments and we omit them.) Choose $w \in W \cap A$. Then $\mathrm{d}(x, w) = 1 \neq \mathrm{d}(y, w)$, and so $r(x|W) \neq r(y|W)$. Hence W is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{3p}}$.

Thus $\dim_\ell(\mathcal{C}_{\mathbb{Z}_{3p}}) \leq |W| = p-1$.

Now, we assume that $|W'| < p-1$ is a local metric generator for $\mathcal{C}_{\mathbb{Z}_{3p}}$, and we seek a contradiction. By Lemma 4.2, there exist two distinct integers i' and j' such that $0 \leq i', j' \leq p-1$ and that $T_{i'} \cap W' = \emptyset = T_{j'} \cap W'$. Clearly, $3i' \in T_{i'}$ and $3j' \in T_{j'}$ and that $3i' \sim 3j'$. Then we have the following facts:
(i) For all $w' \in W' \cap A$, $\mathrm{d}(3i', w') = 1 = \mathrm{d}(3j', w')$;
(ii) For all $w' \in W' \cap B$, $\mathrm{d}(3i', w') = 2 = \mathrm{d}(3j', w')$;
(iii) For all $w' \in W' \cap C$, $\mathrm{d}(3i', w') = 2 = \mathrm{d}(3j', w')$.
Hence $r(3i'|W') = r(3j'|W')$ which is a contradiction. Therefore

$$\dim_\ell(\mathcal{C}_{\mathbb{Z}_{3p}}) = p-1.$$

□

Conflict of interest. The authors declare that they have no conflict of interest.

References

[1] G. Aalipour, S. Akbari, Some properties of a Cayley graph of a commutative ring, *Comm. Algebra* **42** (2014), no. 4, 1582–1593.

[2] G. Aalipour, S. Akbari, On the Cayley graph of a commutative ring with respect to its zero-divisors, *Comm. Algebra* **44** (2016), no. 4, 1443–1459.

[3] N. Abachi, M. Adlifard, M. Bakhtyiari, On the strong metric dimension of a total graph of nonzero annihilating ideals, *Bull. Aust. Math. Soc.* **105** (2022), no. 3, 431–439.

[4] M. Abas, T. Vetrík, Metric dimension of Cayley digraphs of split metacyclic groups, *Theoret. Comput. Sci.* **809** (2020), 61–72.

[5] R. Akhtar, T. Jackson-Henderson, R. Karpman, M. Boggess, I. Jiménez, A. Kinzel, D. Pritikin, On the unitary Cayley graph of a finite ring, *Electron. J. Combin.* **16** (2009), no. 1, Research Paper 117, 13 pp.

[6] D.F. Anderson, A. Badawi, The total graph of a commutative ring, *J. Algebra* **320** (2008), no. 7, 2706–2719.

[7] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, American Elsevier Publishing Co., Inc., New York, 1976.

[8] M. Bašić, A. Ilić, On the clique number of integral circulant graphs, *Appl. Math. Lett.* **22** (2009), no. 9, 1406–1411.

[9] M. Bašić, A. Ilić, On the automorphism group of integral circulant graphs, *Electron. J. Combin.* **18** (2011), no. 1, Paper 68, 21 pp.

[10] P. Berrizbeitia, R.E. Giudici, Counting pure k-cycles in sequences of Cayley graphs, *Discrete Math.* **149** (1996), no. 1-3, 11–18.

[11] P. Berrizbeitia, R.E. Giudici, On cycles in the sequence of unitary Cayley graphs, *Discrete Math.* **282** (2004), no. 1-3, 239–243.

[12] D.X. Charles, E.Z. Goren, K.E. Lauter, Families of Ramanujan graphs and quaternion algebras, *Groups and symmetries*, 53–80, CRM Proc. Lecture Notes, 47, Amer. Math. Soc., Providence, RI, 2009.

[13] I.J. Dejter, R.E. Giudici, On unitary Cayley graphs, *J. Combin. Math. Combin. Comput.* **18** (1995), 121–124.

[14] D. Dolžan, The metric dimension of the annihilating-ideal graph of a finite commutative ring, *Bull. Aust. Math. Soc.* **103** (2021), no. 3, 362–368.

[15] D. Dolžan, The metric dimension of the total graph of a finite commutative ring, *Canad. Math. Bull.* **59** (2016), no. 4, 748–759.

[16] Sh. Ebrahimi, R. Nikandish, A. Tehranian, H. Rasouli, Metric dimension of complement of annihilator graphs associated with commutative rings, *Appl. Algebra Engrg. Comm. Comput.* **34** (2023), no. 6, 995–1011.

[17] M. Fehr, S. Gosselin, O.R. Oellermann, The metric dimension of Cayley digraphs, *Discrete Math.* **306** (2006), no. 1, 31–41.

[18] E.D. Fuchs, Longest induced cycles in circulant graphs, *Electron. J. Combin.* **12** (2005), Research Paper 52, 12 pp.

[19] F. Harary, R.A. Melter, On the metric dimension of a graph, *Ars Combin.* **2** (1976), 191–195.

[20] W. Klotz, T. Sander, Some properties of unitary Cayley graphs, *Electron. J. Combin.* **14** (2007), no. 1, Research Paper 45, 12 pp.

[21] A. Ilić, M. Bašić, On the chromatic number of integral circulant graphs, *Comput. Math. Appl.* **60** (2010), no. 1, 144–150.

[22] A. Ilić, M. Bašić, New results on the energy of integral circulant graphs, *Appl. Math. Comput.* **218** (2011), no. 7, 3470–3482.

[23] D. Kiani, M. Molla Haji Aghaei, Y. Meemark, B. Suntornpoch, Energy of unitary Cayley graphs and gcd-graphs, *Linear Algebra Appl.* **435** (2011), no. 6, 1336–1343.

[24] X. Liu, S. Zhou, Spectral properties of unitary Cayley graphs of finite commutative rings, *Electron. J. Combin.* **19** (2012), no. 4, Paper 13, 19 pp.

[25] X. Ma, M. Feng, K. Wang, The strong metric dimension of the power graph of a finite group, *Discrete Appl. Math.* **239** (2018), 159–164.

[26] X. Ma, L. Zhai, Strong metric dimensions for power graphs of finite groups, *Comm. Algebra* **49** (2021), no. 11, 4577–4587.

[27] S. Nithya, V. Prisci, On the metric dimension of extended zero divisor graphs, *Gulf J. Math.* **16** (2024), no. 1, 205–214.

[28] F. Okamoto, B. Phinezy, P. Zhang, The local metric dimension of a graph. Math. Bohem. **135** (2010), no. 3, 239–255.

[29] O.R. Oellermann, C.D. Pawluck, A. Stokke, The metric dimension of Cayley digraphs of abelian groups, *Ars Combin.* **81** (2006), 97–111.

[30] A. Rezaei, K. Khashyarmanesh, M. Afkhami, On the metric dimension of Cayley graphs, *AKCE Int. J. Graphs Comb.* **19** (2022), no. 2, 118–124.

[31] D. Sinha, P. Garg, A. Singh, Some properties of unitary addition Cayley graphs, *Notes on Number Theory and Discrete Mathematics* **17** (2011) 3, 49–59.

[32] P.J. Slater, Leaves of trees, Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1975), 549–559, Congress. Numer., No. XIV, Utilitas Math., Winnipeg, MB, 1975.

[33] W. So, Integral circulant graphs, *Discrete Math.* **306** (2006), no. 1, 153–158.

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