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Approximation properties: Modified Szász-Durrmeyer Type Operators via General-Appell Polynomials

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Abstract. We present a new sequence of modified Szász-Durrmeyer type of sequence of operators via general-Appell Polynomials to investigate the approximation properties of Lebesgue integrable functions ($L_1[0,\infty)$). In addition, we are study estimates in view of test functions and central moments. Next, convergence rate is discussed using the Korovkin theorem and Voronovskaja type theorem. Moreover, direct approximation results via modulus of continuity of first and second order, Peetre's K-functional, Lipschitz type space, and the r^{th} order Lipschitz type maximal functions are investigated. In subsequent section, we present weighted approximation results and statistical approximation theorems are discussed.

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1. Introduction

Approximation theory has been a cornerstone of mathematical analysis for a long time, its influence extending across pure mathematics, computational science, and practical domains such as engineering and computer graphics. Weierstrass (1885) [1] formulated a pivotal and widely recognized result in this field, the Weierstrass approximation theorem, which demonstrates that any continuous function defined on a closed interval can be uniformly approximated using polynomials. In 1912, Bernstein [2] devised a remarkable and elegant proof of Weierstrass's approximation theorem using a sequence of polynomials. The emergence of fractional calculus has brought a trans-formative perspective to operator theory within the realm of approximation. The convergence of fractional calculus and operator theory constitutes an expanding area of research with applications that extend beyond the field of pure mathematics into interdisciplinary fields such as quantum mechanics, control theory, signal processing, and material science.

Szász in 1950 [3], introduced a sequence of operators over the infinite length of interval, i.e., $[0,\infty)$ as:

$$P_\kappa(\tilde{g};u)=\sum_{i=0}^{\infty}\tilde{g}\left(\frac{i}{\kappa}\right)p_{\kappa,i}(u), \tag{1}$$

where $u\in[0,\infty)$, $\tilde{g}\in C[0,\infty)$ and $p_{\kappa,i}(u)=e^{-\kappa u}\frac{(\kappa u)^i}{i!}$.

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In 1962, Schurer [4] constructed a new sequence of Bernstein operators [2] which is denoted as $B_{\kappa+l} : C[0, 1 + l] \to C[0, 1]$ and defined by:

$$B_{\kappa+l}(\tilde{g}; u) = \sum_{j=0}^{\kappa+l} \tilde{g}\left(\frac{j}{\kappa}\right) \left(\begin{array}{c} \kappa+l \\ j \end{array} \right) u^j (1-u)^{\kappa+l-j}, u \in [0, 1+l], \tag{2}$$

where $l \in \mathbb{N} \cup \{0\}$ and $\tilde{g} \in C[0, 1 + l]$. But these sequences of operators given in (2) are restricted to $C[0, 1 + l]$.

The classical Szász-Schurer operators are linear positive operators and approximate the continuous functions over the positive semi axises. Several mathematicians have constructed several generalizations of these sequences of operators to provide flexibility in rate of convergence and order of approximation, e.g., Raza et al. ([5], [6]), Aslan et al. ([7], [8]), Özger et al. ([9], [10]), Cai et al. ([11], [12]), Braha et al. ([13], [14]), Ayman Mursaleen et al. ([15]-[17]), Khursheed et al. ([18], [19]), Mohiuddine et al. ([20], [21]), Mursaleen et al. ([22], [23]), Khan et al. [24], Nasiruzzaman et al. ([25], [26]), Rao et al. ([27], [28]) and Jha et al. [29] provided a number of generalizations for these kinds of sequences to investigate flexibility in approximation properties across several functional spaces.

Recently, Raza et al. [30] provided a class of sequence of operators $G_{\kappa,A}(.;.)$, $\kappa \in \mathbb{N}$, given by the formula

$$\mathcal{G}_{\kappa,A}(\tilde{g}; u) = \frac{e^{-\kappa u}}{\tilde{\Omega}(1)\tilde{\psi}(h, 1)} \sum_{\nu=0}^{\infty} \frac{A_{p,\nu}(\kappa u, h)}{\nu!} \tilde{g}\left(\frac{\nu}{\kappa}\right), \quad u \in \mathbb{R}_0^+, \tag{3}$$

where $A_{p,\nu}$ is the two variable Appell polynomials (see [30]).

The sequence presented by (3) are positive and linear. The basic information about positive linear operators, including their modifications and applications can be found in [31].

As the operators described in (3) are limited for continuous function only, we present a sequence of positive linear operators to provide approximations in the larger class of functions, which is termed as modified Szász-Durrmeyer operators in context of general Appell Polynomials as:

$$\mathcal{R}_{\kappa+l}^{A}(\tilde{g}; u) \quad = \quad \sum_{\nu=0}^{\infty} A_{\nu}^{p}((\kappa+l)u, h) \int_{0}^{\infty} Q_{\kappa+l}^{\nu}(y)\tilde{g}(y)dy, \quad for \quad u \in \mathbb{R}_0^+, \tag{4}$$

where

$$A_{\nu}^{p}((\kappa+l)u, h) = \frac{e^{-(\kappa+l)u}}{\tilde{\Omega}(1)\tilde{\psi}(h, 1)} \frac{A_{p,\nu}((\kappa+l)u, h)}{\nu!}$$
$$and\ Q_{\kappa+l}^{\nu}(y) = \frac{(\kappa+l)^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)} y^{\nu+\lambda} e^{-(\kappa+l)y},$$

with Γ (Gamma) function which is given as:

$$\Gamma m = \int_{0}^{\infty} s^{m-1} e^{-s} ds, \quad \Gamma m = (m-1)\Gamma(m-1) = (m-1)!.$$

Lemma 1.1. *The sequence of operators introduced in (4) are linear.*

Proof. In view of (4) and for $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned}\mathcal{R}^A_{\kappa+l}(\lambda_1\tilde{g}_1 + \lambda_2\tilde{g}_2; u) &= \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u, h) \int_0^\infty Q^\nu_{\kappa+l}(y)(\lambda_1\tilde{g}_1 + \lambda_2\tilde{g}_2)(y)dy \\ &= \lambda_1 \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u, h) \int_0^\infty Q^\nu_{\kappa+l}(y)\tilde{g}_1(y)dy \\ &+ \lambda_2 \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u, h) \int_0^\infty Q^\nu_{\kappa+l}(y)\tilde{g}_2(y)dy \\ &= \lambda_1\mathcal{R}^A_{\kappa+l}(\tilde{g}_1; u) + \lambda_2\mathcal{R}^A_{\kappa+l}(\tilde{g}_2; u).\end{aligned}$$

□

Lemma 1.2. *As discussed by Raza et al. in [30], we can have the following equalities:*

$$\begin{aligned}\sum_{\nu=0}^{\infty} \frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \tilde{\Omega}(1)e^{\kappa u}\tilde{\psi}(h,1); \\ \sum_{\nu=0}^{\infty} \nu\frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \Big[\kappa u\tilde{\Omega}(1)\tilde{\psi}(h,1) + \tilde{\Omega}(1)\tilde{\psi}'(h,1) + \tilde{\Omega}'(1)\tilde{\psi}(h,1)\Big]e^{\kappa u}; \\ \sum_{\nu=0}^{\infty} \nu^2\frac{A_{p,\nu}(\kappa u, h)}{\nu!} &= \Big[((\kappa+l)^2u^2 + \kappa u)\tilde{\Omega}(1)\tilde{\psi}(h,1) + (2\kappa u+1)[\tilde{\Omega}(1)\tilde{\psi}'(h,1) \\ &+ \tilde{\Omega}'(1)\tilde{\psi}(h,1)] + 2\tilde{\psi}'(h,1)\tilde{\Omega}'(1) + \tilde{\psi}''(h,1)\tilde{\Omega}(1) \\ &+ \tilde{\psi}(h,1)\tilde{\Omega}''(1)\Big]e^{\kappa u}.\end{aligned}$$

Lemma 1.3. *Let $\tilde{g}_\theta(y) = y^\theta$, $\theta \in \{0,1,2\}$ be the test functions by (3). Then, we have*

$$\begin{aligned}\mathcal{R}^A_{\kappa+l}(1;u) &= 1; \\ \mathcal{R}^A_{\kappa+l}(y;u) &= u + \frac{1}{\kappa+l}\Big[\lambda + \frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)} + \frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)} + 1\Big]; \\ \mathcal{R}^A_{\kappa+l}(y^2;u) &= u^2 + \frac{1}{(\kappa+l)^2}\Big[(\kappa+l)(2\lambda+4)u + (2(\kappa+l)u + 2\lambda + 4)\Big\{\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)} \\ &+ \frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}\Big\} + 2\frac{\tilde{\psi}'(h,1)\tilde{\Omega}'(1)}{\tilde{\psi}(h,1)\tilde{\Omega}(1)} + \frac{\tilde{\psi}''(h,1)}{\tilde{\psi}(h,1)} + \frac{\tilde{\Omega}''(1)}{\tilde{\Omega}(1)} + \lambda^2 + 3\lambda + 2\Big];\end{aligned}$$

for each $u \in \mathbb{R}^+_0$.

Proof. In the direction of (4), we have

$$\mathcal{R}^A_{\kappa+l}(\tilde{g}_\theta; u) = \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u, h) \int_0^\infty Q^\nu_{\kappa+l}(y)\tilde{g}_\theta(y)dy.$$

Now, for $\theta = 0$,

$$\begin{aligned}
\mathcal{R}^A_{\kappa+l}(1;u) &= \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u,h)\int_0^\infty Q^\nu_{\kappa+l}(y)dy \\
&= \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u,h)\frac{(\kappa+l)^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)}\int_0^\infty y^{\nu+\lambda}e^{-(\kappa+l)y}dy \\
&= \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u,h) = 1.
\end{aligned}$$

For $\theta = 1$,

$$\begin{aligned}
\mathcal{R}^A_{\kappa+l}(y;u) &= \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u,h)\int_0^\infty Q^\nu_{\kappa+l}(y)ydy \\
&= \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u,h)\frac{(\kappa+l)^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)}\int_0^\infty y^{\nu+\lambda+1}e^{-\kappa y}dy \\
&= \frac{1}{\kappa+l}\frac{e^{-(\kappa+l)u}}{\tilde{\Omega}(1)\tilde{\psi}(h,1)}\sum_{\nu=0}^{\infty}\nu\frac{A_{p,\nu}((\kappa+l)u,h)}{\nu!} \\
&\quad+\left(\frac{\lambda+1}{\kappa+l}\right)\frac{e^{-(\kappa+l)u}}{\tilde{\Omega}(1)\tilde{\psi}(h,1)}\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}((\kappa+l)u,h)}{\nu!} \\
&= u+\frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right] := \psi_{\kappa+l}. \qquad (Say)
\end{aligned}$$

.

For $\theta = 2$

$$\begin{aligned}
\mathcal{R}^A_{\kappa+l}(y^2;u) &= \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u,h)\int_0^\infty Q^\nu_{\kappa+l}(y)y^2dy \\
&= \sum_{\nu=0}^{\infty} A^p_\nu((\kappa+l)u,h)\frac{(\kappa+l)^{\nu+\lambda+1}}{\Gamma(\nu+\lambda+1)}\int_0^\infty y^{\nu+\lambda+2}e^{-(\kappa+l)y}dy \\
&= \frac{1}{(\kappa+l)^2}\sum_{\nu=0}^{\infty}\nu^2A^p_\nu((\kappa+l)u,h)+\left(\frac{2\lambda+3}{(\kappa+l)^2}\right)\sum_{\nu=0}^{\infty}\nu A^p_\nu((\kappa+l)u,h) \\
&\quad+\left(\frac{\lambda^2+3\lambda+2}{(\kappa+l)^2}\right)\sum_{\nu=0}^{\infty}A^p_\nu((\kappa+l)u,h) \\
&= \frac{e^{-(\kappa+l)u}}{\tilde{\Omega}(1)\tilde{\psi}(h,1)}\Bigg[\frac{1}{(\kappa+l)^2}\sum_{\nu=0}^{\infty}\nu^2\frac{A_{p,\nu}((\kappa+l)u,h)}{\nu!} \\
&\quad+\left(\frac{2\lambda+3}{(\kappa+l)^2}\right)\sum_{\nu=0}^{\infty}\nu\frac{A_{p,\nu}((\kappa+l)u,h)}{\nu!}
\end{aligned}$$

$$+\left(\frac{\lambda^2+3\lambda+2}{(\kappa+l)^2}\right)\sum_{\nu=0}^{\infty}\frac{A_{p,\nu}((\kappa+l)u,h)}{\nu!}\Bigg]$$

$$= u^2+\frac{1}{(\kappa+l)^2}\Bigg[(\kappa+l)(2\lambda+4)u+(2(\kappa+l)u+2\lambda+4)\left\{\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}\right\}$$

$$+2\frac{\tilde{\psi}'(h,1)\tilde{\Omega}'(1)}{\tilde{\psi}(h,1)\tilde{\Omega}(1)}+\frac{\tilde{\psi}"(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}"(1)}{\tilde{\Omega}(1)}+\lambda^2+3\lambda+2\Bigg].$$

Hence, proof of Lemma 1.3 is completed. □

Lemma 1.4. *Let $\gamma_\theta(y)=(y-u)^\theta$, $\theta=0,1,2$. Then, for the operators (4), we have central moments $\mathcal{R}^A_{\kappa+l}(\gamma_\theta(y),u)$ as:*

$$\mathcal{R}^A_{\kappa+l}(\gamma_0(y),u) = 1;$$

$$\mathcal{R}^A_{\kappa+l}(\gamma_1(y),u) = \frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+\lambda+1\right];$$

$$\mathcal{R}^A_{\kappa+l}(\gamma_2(y),u) = \frac{1}{(\kappa+l)^2}\Bigg[2u(\kappa+l)+(2\lambda+4)\left\{\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}\right\}+$$

$$+2\frac{\tilde{\psi}'(h,1)\tilde{\Omega}'(1)}{\tilde{\psi}(h,1)\tilde{\Omega}(1)}+\frac{\tilde{\psi}"(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}"(1)}{\tilde{\Omega}(1)}+\lambda^2+3\lambda+2\Bigg];$$

for each $u\in\mathbb{R}^+_0$.

Proof. With the aid of the Lemma 1.3 and linearity property, we can easily complete the proof of Lemma 1.4. □

In the following sections, we examine the convergence rate of operators and their approximation order. Specifically, we discuss direct results both globally and locally in several spaces. In the final section, we explore some results of the A-Statistical approximation in various functional spaces.

2. Uniform Rate of Convergence and Order of Approximation

Definition 2.1. [31] The modulus of smoothness for $\tilde{g}\in C[0,\infty)$ is given by

$$\omega(\tilde{g};\delta)=\sup_{|u_1-u_2|\le\delta}|\tilde{g}(u_1)-\tilde{g}(u_2)|, \qquad u_1,u_2\in[0,\infty).$$

Theorem 2.1. *Let $\mathcal{R}^A_{\kappa+l}(.;.)$ be a operators described in Eq. (4). Then, on each bounded and closed interval of $[0,\infty)$, $\mathcal{R}^A_{\kappa+l}(\tilde{g};.)\rightrightarrows\tilde{g}$, for all $\tilde{g}\in C_B[0,\infty)$, where the symbol $\rightrightarrows$ denotes uniform convergence.*

Proof. Considering the classical Korovkin theorem [33], which characterizes the uniform convergence for the sequence of positive linear operators, it is enough to note that

$$\lim_{\kappa\to\infty}\mathcal{R}^A_{\kappa+l}(\tilde{g}_\theta;u)=u^\theta,\ \ \theta=0,1,2,$$

uniformly on all bounded and closed subsets of $[0,\infty)$. We can easily establish this result with the help of Lemma 1.3. □

Now, we show that Voronovskaja-type asymptotic approximation theorem for the $\mathcal{R}^{A}_{\kappa+l}(.;.)$ given in (4).

Theorem 2.2. *Let $\tilde{g} \in C_B[0,\infty)$ and $\tilde{g}', \tilde{g}''$ exist at a fixed point $u \in [0,\infty)$. Then, one has*

$$\lim_{\kappa\to\infty}(\kappa+l)\big(\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)\big)=\tilde{g}'(u)\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right]+u\tilde{g}''(u).$$

Proof. In accordance with Taylor's formula for the function $\tilde{g}$, we get

$$\tilde{g}(y)=\tilde{g}(u)+(y-u)\tilde{g}'(u)+\frac{1}{2}(y-u)^2\tilde{g}''(u)+t(y,u)(y-u)^2, \tag{5}$$

where $t(y,u)$ is Peano remainder and

$$\lim_{y\to u}t(y,u)=0.$$

Applying operators on both the sides in (5), we yield

$$\begin{aligned}(\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)) &= \tilde{g}'(u)\mathcal{R}^{A}_{\kappa+l}((y-u);u)+\frac{1}{2}\tilde{g}''(u)\mathcal{R}^{A}_{\kappa+l}((y-u)^2;u)\\ &+ \mathcal{R}^{A}_{\kappa+l}(t(y,u)(y-u)^2;u).\end{aligned}$$

In view of Lemma 1.4

$$\begin{aligned}(\kappa+l)(\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)) &= g'(u)\left[\lambda+\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1\right]\\ &+ \frac{g''(u)}{2}\frac{1}{\kappa+l}\Bigg[2u(\kappa+l)+(2\lambda+4)\left\{\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}\right\}\\ &+ 2\frac{\tilde{\psi}'(h,1)\tilde{\Omega}'(1)}{\tilde{\psi}(h,1)\tilde{\Omega}(1)}+\frac{\tilde{\psi}''(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}''(1)}{\tilde{\Omega}(1)}+\lambda^2+3\lambda+2\Bigg]\\ &+ (\kappa+l)\mathcal{R}^{A}_{\kappa+l}(t(y,u)(y-u)^2;u).\end{aligned}$$

Operate the limits on both the sides of the above expression, we get

$$\begin{aligned}\lim_{\kappa\to\infty}(\kappa+l)\big(\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)\big) &= \tilde{g}'(u)\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right]+u\tilde{g}''(u)\\ &+ \lim_{\kappa\to\infty}(\kappa+l)\mathcal{R}^{A}_{\kappa+l}(t(y,u)(y-u)^2;u).\end{aligned}$$

Now, we need to show that

$$\lim_{\kappa\to\infty}(\kappa+l)\mathcal{R}^{A}_{\kappa+l}(t(y,u)(y-u)^2;u)=0.$$

In view of Cauchy-Schwarz inequality, we calculate last term of above expression can be written as:

$$(\kappa+l)\mathcal{R}^{A}_{\kappa+l}(t(y,u)(y-u)^2;u)\leq\sqrt{\mathcal{R}^{A}_{\kappa+l}(t^2(y,u);u)}\sqrt{(\kappa+l)^2\mathcal{R}^{A}_{\kappa+l}((y-u)^4;u)}. \tag{6}$$

We see that $t^2(u,u)=0$ and $t^2(y,u)\in C_B[0,\infty)$. Thus we have

$$\lim_{\kappa\to\infty}\mathcal{R}^{A}_{\kappa+l}(t^2(y,u);u)=t^2(u,u)=0. \tag{7}$$

From (6) and (7) it follows that

$$\lim_{\kappa\to\infty}(\kappa+l)\mathcal{R}^{A}_{\kappa+l}(t(y,u)(y-u)^2;u)=0.$$

Hence, the proof is completed. □

According to Shisha et al. [32], order of convergence relative to Ditzian-Totik modulus of continuity can easily be proved.

Theorem 2.3. *Consider $\tilde{g} \in C_B[0,\infty)$ and for the operators $\mathcal{R}^A_{\kappa+l}(.;.)$ presented in Eq. (4), we acquire*

$$|\mathcal{R}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u)| \leq 2\omega(\tilde{g};\delta),$$

where $\delta = \sqrt{\mathcal{R}^A_{\kappa+l}((y-u)^2;u)}$.

Proof. In accordance with Lemma 1.3, 1.4 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|\mathcal{R}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u)| &\leq \mathcal{R}^A_{\kappa+l}\big(|\tilde{g}(y) - \tilde{g}(u)|;u\big) \\
&\leq \mathcal{R}^A_{\kappa+l}\left(\left(1 + \frac{|y-u|}{\delta}\right)\omega(\tilde{g},\delta);u\right) \\
&\leq \left(1 + \frac{1}{\delta}\mathcal{R}^A_{\kappa+l}(|y-u|;u)\right)\omega(\tilde{g},\delta) \\
&\leq \left(1 + \frac{1}{\delta}\sqrt{\mathcal{R}^A_{\kappa+l}((y-u)^2;u)}\right)\omega(\tilde{g},\delta).
\end{aligned}$$

By selecting $\delta = \sqrt{\mathcal{R}^A_{\kappa+l}((y-u)^2;u)}$, we obtained the desired proof. □

3. Locally Approximation Results

We recall a few functional spaces and functional relations in this part as: $C_B[0,\infty)$: Denotes a real valued functional space which acquires bounded and continuous functions. Now, Peetre's K-functional [31] is defined as

$$K_2(\tilde{g},\delta) = \inf_{\tilde{h}\in C^2_B[0,\infty)} \left\{ \|\tilde{g} - \tilde{h}\|_{C_B[0,\infty)} + \delta\|\tilde{h}''\|_{C^2_B[0,\infty)} \right\},$$

where $C^2_B[0,\infty) = \{\tilde{h} \in C_B[0,\infty) : \tilde{h}', \tilde{h}'' \in C_B[0,\infty)\}$ associated with norm $\|\tilde{g}\| = \sup_{0\leq y<\infty} |\tilde{g}(y)|$ and second order Ditzian-Totik modulus of smoothness is presented by

$$\omega_2(\tilde{g};\sqrt{\delta}) = \sup_{0<k\leq\sqrt{\delta}} \sup_{y\in[0,\infty)} |\tilde{g}(y+2k) - 2\tilde{g}(y+k) + \tilde{g}(y)|.$$

As described in ([31] by DeVore and Lorentz on page no. 177, Theorem 2.4) as:

$$K_2(\tilde{g};\delta) \leq \tilde{C}\omega_2(\tilde{g};\sqrt{\delta}), \tag{8}$$

where $\tilde{C}$ is an absolute constant. To establish the next result, we consider the auxiliary operator defined as:

$$\widehat{\mathcal{R}}^A_{\kappa+l}(\tilde{g};u) = R^A_{\kappa+l}(\tilde{g};u) + \tilde{g}(u) - \tilde{g}\left(\frac{1}{\kappa+l}\left[(\kappa+l)u + \frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)} + \frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)} + 1 + \lambda\right]\right) \tag{9}$$

where $\tilde{g} \in C_B[0,\infty)$, $u \geq 0$ and $\kappa > 1$. From Eq. (9), one can yield

$$\widehat{\mathcal{R}}^A_{\kappa+l}(1;u) = 1,\ \widehat{\mathcal{R}}^A_{\kappa+l}(\gamma_1(y);u) = 0 \text{ and } |\widehat{\mathcal{R}}^A_{\kappa+l}(\tilde{g};u)| \leq 3\|\tilde{g}\|. \tag{10}$$

Lemma 3.1. *If $u \geq 0$, one has*

$$|\widehat{\mathcal{R}}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u)| \leq \Theta(u)\|\tilde{g}''\|,$$

where $\tilde{g} \in C^2_B[0,\infty)$ and $\Theta(u) = \widehat{\mathcal{R}}^A_{\kappa+l}(\gamma_1(y);u) + (\widehat{\mathcal{R}}^A_{\kappa+l}(\gamma_1(y);u)^2$.

Proof. For $\tilde{g} \in C^2_B[0,\infty)$ and by Taylor expansion, we get

$$\tilde{g}(y) = \tilde{g}(u) + (y-u)\tilde{g}'(u) + \int_u^y (y-v)\tilde{g}''(v)dv. \tag{11}$$

Implementing the auxiliary operators $\widehat{\mathcal{R}}^A_{\kappa+l}(.;.)$ introduced in Eq.(9) to Eq. (11), we get

$$\widehat{\mathcal{R}}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u) \quad = \quad \tilde{g}'(u)\widehat{\mathcal{R}}^A_{\kappa+l}(\gamma_1(y);u) + \widehat{\mathcal{R}}^A_{\kappa+l}\Big(\int_u^y (y-v)\tilde{g}''(v)dv;u\Big).$$

Using the Eqs. (10) and (11), one yield

$$\widehat{\mathcal{R}}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u) = \widehat{\mathcal{R}}^A_{\kappa+l}\Big(\int_u^y (y-v)\tilde{g}''(v)dv;u\Big)$$

$$= R^A_{\kappa+l}\Big(\int_u^y (y-v)\tilde{g}''(v)dv;u\Big)$$

$$- \int_u^{\psi_{\kappa+l}} \left(\frac{1}{\kappa+l}\left[(\kappa+l)u + \lambda + \frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)} + \frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)} + 1\right] - v\right)\tilde{g}''(v)dv,$$

$$|\widehat{\mathcal{R}}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u)| \leq \left|\widehat{\mathcal{R}}^A_{\kappa+l}\Big(\int_u^y (y-v)\tilde{g}''(v)dv;u\Big)\right|$$

$$+ \left|\int_u^{\psi_{\kappa+l}} \left(\frac{1}{\kappa+l}\left[(\kappa+l)u + \frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)} + \frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)} + 1 + \lambda\right] - v\right)\tilde{g}''(v)dv\right|. \tag{12}$$

Since,

$$\left|\int_u^y (y-v)\tilde{g}''(v)dv\right| \leq (y-u)^2 \parallel \tilde{g}'' \parallel, \tag{13}$$

then

$$\left|\int_u^{\psi_{\kappa+l}} \left(\frac{1}{\kappa+l}\left[(\kappa+l)u + \frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)} + \frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)} + 1 + \lambda\right] - v\right)\tilde{g}''(v)dv\right|$$

$$\leq \left(\frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)} + \frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)} + 1 + \lambda\right]\right)^2 \parallel \tilde{g}'' \parallel. \tag{14}$$

In accordance with (12), (13) and (14), we acquire

$$|\widehat{\mathcal{R}}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)| \leq \left\{\widehat{\mathcal{R}}^{A}_{\kappa+l}(\gamma_2(y);u)+\left(\frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right]\right)^2\right\}\|\tilde{g}''\|$$
$$=\Theta(u)\|\tilde{g}''\|.$$

This proves the required result. □

Theorem 3.2. *For $\tilde{g}\in C^2_B[0,\infty)$, there corresponds a non-negative constant $\tilde{C}>0$ such that*

$$|\ \mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)\ |\leq \tilde{C}\omega_2\big(\tilde{g};\sqrt{\Theta(u)}\big)+\omega(\tilde{g};\mathcal{R}^{A}_{\kappa+l}(\gamma_1(y);u),$$

where $\Theta(u)$ is given by in Lemma 3.1.

Proof. Let $\tilde{h}\in C^2_B[0,\infty)$ and $\tilde{g}\in C_B[0,\infty)$. Then, by definition of $\widehat{\mathcal{R}}^{A}_{\kappa+l}(.;.)$ given in (9), we get

$$|\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\leq|\widehat{\mathcal{R}}^{A}_{\kappa+l}(\tilde{g}-\tilde{h};u)|+|(\tilde{g}-\tilde{h})(u)|+|\widehat{\mathcal{R}}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|$$
$$+\left|\tilde{g}\Big(\frac{1}{\kappa+l}\Big[(\kappa+l)u+\lambda+\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1\Big]\Big)-\tilde{g}(u)\right|.$$

According to Lemma 3.1 and inequalities mentioned in Eq. (10), we acquire

$$|\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\leq 4\|\tilde{g}-\tilde{h}\|+|\widehat{\mathcal{R}}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|$$
$$+\left|\tilde{g}\left(\frac{1}{\kappa+l}\left[(\kappa+l)+\lambda+\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1\right]\right)-\tilde{g}(u)\right|$$
$$\leq 4\|\tilde{g}-\tilde{h}\|+\Theta(u)\|\tilde{h}''\|+\omega\Big(\tilde{g};\mathcal{R}^{A}_{\kappa+l}(y-u);u)\Big).$$

Using Eq. (8), we established the required result. □

Further, we address the next result in Lipschitz type space presented by [34] as:

$$Lip^{\zeta_1,\zeta_2}_{\tilde{M}}(\eta):=\left\{\tilde{g}\in C_B[0,\infty):|\tilde{g}(y)-\tilde{g}(u)|\leq\tilde{M}\frac{|y-u|^{\eta}}{(y+\zeta_1u+\zeta_2u^2)^{\frac{\eta}{2}}}:u,y\in(0,\infty)\right\},$$

where $\tilde{M}>0$, $0<\eta\leq 1$ and $\zeta_1,\zeta_2>0$.

Theorem 3.3. *Consider sequence of linear positive operators in (4) and $\tilde{g}\in Lip^{\zeta_1,\zeta_2}_{\tilde{M}}(\eta)$, one obtain*

$$|\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\leq\tilde{M}\left(\frac{\lambda(u)}{\zeta_1u+\zeta_2u^2}\right)^{\frac{\eta}{2}}, \tag{15}$$

where $0<\eta\leq 1$, $\zeta_1,\zeta_2\in(0,\infty)$ and $\lambda(y)=\mathcal{R}^{A}_{\kappa+l}(\gamma_2(y);u)$.

Proof. For $\eta=1$ and $u\geq 0$, one get

$$|\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\leq\mathcal{R}^{A}_{\kappa+l}(|\tilde{g}(y)-\tilde{g}(u)|;u)$$
$$\leq\tilde{M}\mathcal{R}^{A}_{\kappa+l}\left(\frac{|y-u|}{(y+\zeta_1u+\zeta_2u^2)^{\frac{1}{2}}};u\right).$$

Since $\dfrac{1}{y+\zeta_1 u+\zeta_2 u^2}<\dfrac{1}{\zeta_1 u+\zeta_2 u^2}$, for each $u\in(0,\infty)$, we acquire

$$\begin{aligned}|\mathcal{R}^A_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)| &\le \frac{\tilde{M}}{(\zeta_1 u+\zeta_2 u^2)^{\frac{1}{2}}}(\mathcal{R}^A_{\kappa+l}(\gamma_2(y);u)^{\frac{1}{2}}\\ &\le \tilde{M}\left(\frac{\lambda(u)}{\zeta_1 u+\zeta_2 u^2}\right)^{\frac{1}{2}},\end{aligned}$$

which indicates that the Theorem 3.3 is valid for $\eta=1$. Next, we examine the case where $\eta\in(0,1)$ and in accordance with Hölder's inequality by selecting $p=\frac{2}{\eta}$ and $q=\frac{2}{2-\eta}$, we obtain

$$\begin{aligned}|\mathcal{R}^A_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)| &\le \big(|\mathcal{R}^A_{\kappa+l}(|\tilde{g}(y)-\tilde{g}(u)|^{\frac{2}{\eta}};u)\big)^{\frac{\eta}{2}}\\ &\le \tilde{M}\left(\mathcal{R}^A_{\kappa+l}\left(\frac{|y-u|^2}{(y+\zeta_1 u+\zeta_2 u^2)};u\right)\right)^{\frac{\eta}{2}}.\end{aligned}$$

Since $\dfrac{1}{y+\zeta_1 u+\zeta_2 u^2}<\dfrac{1}{\zeta_1 u+\zeta_2 u^2}$, for all $u\in(0,\infty)$, one get

$$|\mathcal{R}^A_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\le \tilde{M}\left(\frac{|\mathcal{R}^A_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)(|y-u|^2;u)}{\zeta_1 u+\zeta_2 u^2}\right)^{\frac{\eta}{2}}\le \tilde{M}\Big(\frac{\lambda(u)}{\zeta_1 u+\zeta_2 u^2}\Big)^{\frac{\eta}{2}}.$$

Thus, we yield the desired result. □

Further, we address the local approximation in terms of the r^{th} order modulus of smoothness, followed by the Lipschitz-type function introduced by Lenze [34] as:

$$\widetilde{\omega}_r(\tilde{g};u)=\sup_{y\neq u,y\in(0,\infty)}\frac{|\tilde{g}(y)-\tilde{g}(u)|}{|y-u|^r},\quad u\in[0,\infty)\text{ and } r\in(0,1]. \tag{16}$$

Theorem 3.4. *Assume $\tilde{g}\in C_B[0,\infty)$ and $r\in(0,1]$. Then, for every $u\in[0,\infty)$, we have*

$$|\mathcal{R}^A_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\le\widetilde{\omega}_r(\tilde{g};u)\Big(\lambda(u)\Big)^{\frac{r}{2}}.$$

Proof. It can be observed that

$$|\mathcal{R}^A_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\le\mathcal{R}^A_{\kappa+l}|\tilde{g}(y)-\tilde{g}(u)|;u).$$

Using Eq. (16), one get

$$|\mathcal{R}^A_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\le\widetilde{\omega}_s(\tilde{g};u)\mathcal{R}^A_{\kappa+l}(|y-u|^r;u).$$

Then by employing Hölder's inequality with $p=\frac{2}{r}$ and $q=\frac{2}{2-r}$, we obtain

$$|\mathcal{R}^A_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\le\widetilde{\omega}_r(\tilde{g};u)\big(\mathcal{R}^A_{\kappa+l}(|y-u|^2;u)\big)^{\frac{r}{2}}.$$

Thus, we conclude the proof. □

4. Approximation Properties Globally

Consider $\nu(u) = 1 + u^2, 0 \leq u < \infty$ as weight function. Then, $B_\nu[0,\infty) = \{\tilde{g}(u) : |\tilde{g}(u)| \leq \tilde{M}_{\tilde{g}}(1+u^2)$, here the constant $\tilde{M}_{\tilde{g}}$ depends on $\tilde{g}$ and $C_\nu[0,\infty)$ represents the continuous functional space in $B_\nu[0,\infty)$ along with the norm $\|\tilde{g}\|_\nu = \sup\limits_{u\in[0,\infty)} \frac{|\tilde{g}(u)|}{\nu(u)}$ and $C_\nu^{\tilde{k}}[0,\infty) = \{\tilde{g} \in C_\nu[0,\infty) : \lim\limits_{u\to\infty} \frac{\tilde{g}(y)}{\nu(u)} = \tilde{k}$, where constant $\tilde{k}$ depends on $\tilde{g}\}$.

If $\tilde{g}$ is a function defined on $[0,b]$ where $b > 0$. Then, Ditzian-Totik modulus of continuity is given by

$$\omega_b(\tilde{g},\delta) = \sup_{|y-u|\leq\delta} \sup_{u,y\in[0,b]} |\tilde{g}(y) - \tilde{g}(u)|. \tag{17}$$

It is straightforward to observe that for $\tilde{g} \in C_\nu[0,\infty)$, the modulus of continuity defined in Eq. (17) tends to zero.

Theorem 4.1. *Let $\tilde{g} \in C_\nu[0,\infty)$ and $\omega_{b+1}(\tilde{g};\delta)$ denote the modulus of smoothness defined on $[0,b+1] \subset [0,\infty)$. Then, for $y \in [0,b]$, we obtain*

$$\|\mathcal{R}^A_{\kappa+l}(.;.) - \tilde{g}\|_{C[0,b]} \leq 4\tilde{M}_{\tilde{g}}(1+b^2)\delta_s(b) + 2\omega_{b+1}(\tilde{g};\sqrt{\delta_s(b)}),$$

where $\delta_s(b) = \max\limits_{u\in[0,b]} \mathcal{R}^A_{\kappa+l}(\gamma_2;u)$.

Proof. For any $u \in [0,b]$ and $y \in [0,\infty)$, we have

$$|\tilde{g}(y) - \tilde{g}(u)| \leq 4\tilde{M}_{\tilde{g}}(1+b^2)(y-u)^2 + \left(1 + \frac{|y-u|}{\delta}\right)\omega_{b+1}(\tilde{g};\delta).$$

Implementing operator $\mathcal{R}^A_{\kappa+l}(.;.)$ on both the sides, we acquire

$$|\mathcal{R}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u)| \leq 4\tilde{M}_{\tilde{g}}(1+b^2)\mathcal{R}^A_{\kappa+l}(\gamma_2;u) + \left(1 + \frac{\mathcal{R}^A_{\kappa+l}(|y-u|;u)}{\delta}\right)\omega_{b+1}(\tilde{g};\delta).$$

Now, in accordance with Lemma 1.4 and $x \in [0,b]$, one has

$$|\mathcal{R}^A_{\kappa+l}(.;.) - \tilde{g}| \leq 4\tilde{M}_{\tilde{g}}(1+b^2)\delta_s(b) + \left(1 + \frac{\sqrt{\delta_s(b)}}{\delta}\right)\omega_{b+1}(\tilde{g};\delta).$$

By selecting $\delta = \delta_s(b)$, desired result can easily be obtained. □

Remark 4.1. In this article, we employ the test function defined by
$\tilde{g}_\theta(y) = y^\theta$, $\theta \in \{0,1,2\}$.

Theorem 4.2. *([35], [36]) Assume that the sequence of linear positive operators $(L_{\kappa+l})_{\kappa\geq 1}$ mapping from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$ meets the conditions*

$$\lim_{\kappa\to\infty} ||L_{\kappa+l}(\tilde{g}_\theta;.) - \tilde{g}_\theta||_\nu = 0, \text{ where } \theta = 0,1,2,$$

thus, for $\tilde{g} \in C_\nu^{\tilde{k}}[0,\infty)$, we get

$$\lim_{\kappa\to\infty} ||L_(\kappa+l)(\tilde{g};.) - \tilde{g}||_\nu = 0.$$

Theorem 4.3. *Let $\tilde{g} \in C_\nu^{\tilde{k}}[0,\infty)$. Then, we obtain*

$$\lim_{\kappa\to\infty} \|\mathcal{R}^A_{\kappa+l}(\tilde{g};.) - \tilde{g}\|_\nu = 0.$$

Proof. To prove the result of Theorem 4.3, it is enough to verify that

$$\lim_{\kappa\to\infty} \|\mathcal{R}^{A}_{\kappa+l}(\tilde{g}_s;.)-\tilde{g}_s\|_{\nu}=0, for\ \theta=0,1,2.$$

Considering the Lemma 1.3, one can see $\|\mathcal{R}^{A}_{\kappa+l}(\tilde{g}_0;.)-1\|_{\nu}=0$, here $\kappa\to\infty$, also

$$\begin{aligned}\|\mathcal{R}^{A}_{\kappa+l}(\tilde{g}_1;.)-\tilde{g}_1\|_{\nu(u)} &= \sup_{u\in[0,\infty)}\frac{1}{\nu(u)}\left|\frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right]\right| \\ &= \frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right]\sup_{u\in[0,\infty)}\frac{1}{1+u^2}.\end{aligned}$$

For a large value of κ, we get $\|\mathcal{R}^{A}_{\kappa+l}(\tilde{g}_1;.)-\tilde{g}_1\|_{\nu}\to 0$.
Also,

$$\begin{aligned}\|\mathcal{R}^{A}_{\kappa+l}(\tilde{g}_2;.)-\tilde{g}_2\|_{\nu} \le{} & \left(\frac{2}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+\lambda+2\right]\right)\sup_{u\in[0,\infty)}\frac{u}{1+u^2} \\ &+\left(\frac{1}{(\kappa+l)^2}\left[(2\lambda+4)\left\{\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}\right\}+\left\{2\frac{\tilde{\psi}'(h,1)\tilde{\Omega}'(1)}{\tilde{\Omega}(1)\tilde{\psi}(h,1)}\right.\right.\right. \\ &\left.\left.\left.+\frac{\tilde{\psi}''(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}''(1)}{\tilde{\Omega}(1)}\right\}+\lambda^2+3\lambda+2\right]\right)\sup_{y\in[0,\infty)}\frac{1}{1+u^2}.\end{aligned}$$

Which implies $\|\mathcal{R}^{A}_{\kappa+l}(\tilde{g}_2;.)-\tilde{g}_2\|_{\nu}\to 0$ as $\kappa\to\infty$. Thus, we conclude the proof of the Theorem 4.3 □

Theorem 4.4. *Let $\tilde{g}\in C^{\tilde{k}}_{\nu}[0,\infty)$ and $\zeta>0$. Then,*

$$\lim_{\kappa\to\infty}\sup_{u\in[0,\infty)}\frac{\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)}{(1+u^2)^{1+\zeta}}=0.$$

Proof. Since $|\tilde{g}(x)|\le\|\tilde{g}\|_{\nu}(1+u^2)$, for any real fixed number $u_0>0$, we get

$$\begin{aligned}\sup_{u\in[0,\infty)}\frac{\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)}{(1+u^2)^{1+\zeta}} &\le \sup_{u\le u_0}\frac{\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)}{(1+u^2)^{1+\zeta}}+\sup_{u\ge u_0}\frac{\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)}{(1+u^2)^{1+\zeta}} \\ &\le \|\mathcal{R}^{A}_{\kappa+l}(\tilde{g};u)-\tilde{g}(u)|\|_{C[0,u_0]} \\ &+\|\tilde{g}\|_{\nu}\sup_{u\ge u_0}\frac{\mathcal{R}^{A}_{\kappa+l}(1+y^2;u)}{(1+u^2)^{1+\zeta}}+\sup_{u\ge u_0}\frac{|\tilde{g}(u)|}{(1+u^2)^{1+\zeta}} \\ &= \tilde{T}_1+\tilde{T}_2+\tilde{T}_3,\quad say. \qquad (18)\end{aligned}$$

Now,

$$\tilde{T}_3=\sup_{u\ge u_0}\frac{|\tilde{g}(u)|}{(1+u^2)^{1+\zeta}}\le\sup_{u\ge u_0}\frac{\|\tilde{g}\|_{\nu}(1+u^2)}{(1+u^2)^{1+\zeta}}\le\frac{\|\tilde{g}\|_{\nu}}{(1+u_0^2)^{\zeta}}.$$

In view of Lemma 1.3, it gives

$$\lim_{\kappa\to\infty}\sup_{u\in[u_0,\infty)}\frac{\mathcal{R}^{A}_{\kappa+l}(1+y^2;u)}{1+u^2}=1.$$

Therefore, for any arbitrary $\epsilon > 0$, there corresponds $\kappa_1 \in \mathbb{N}$ with

$$\sup_{u\in[u_0,\infty)} \frac{\mathcal{R}^A_{\kappa+l}(1+y^2;u)}{1+u^2} \leq \frac{(1+u_0^2)^\zeta}{\|\tilde{g}\|_\nu}\frac{\epsilon}{3} + 1, \text{ for all } \kappa \geq \kappa_1.$$

Therefore

$$\tilde{T_2} = ||\tilde{g}||_\nu \sup_{u\in[u_0,\infty)} \frac{\mathcal{R}^A_{\kappa+l}(1+y^2;u)}{(1+u^2)^{1+\zeta}} \leq \frac{||\tilde{g}||_\nu}{(1+u_0^2)^\zeta} + \frac{\epsilon}{3}, \text{ for all } \kappa \geq \kappa_1. \tag{19}$$

Hence, we get

$$\tilde{T_2} + \tilde{T_3} < 2\frac{\|\tilde{g}\|_\nu}{(1+u^2)^\zeta} + \frac{\epsilon}{3}.$$

If we take u_0 to be so large that $\frac{\|\tilde{g}\|_\nu}{(1+u^2)^\zeta} < \frac{\epsilon}{6}$, then, we have

$$\tilde{T_2} + \tilde{T_3} < \frac{2\epsilon}{3} \text{ for all } \kappa \geq \kappa_1. \tag{20}$$

Now, from Theorem 4.1, there corresponds $\kappa_2 > \kappa$ with

$$\tilde{T_1} = \|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \tilde{g}\|_{C[0,u_0]} < \frac{\epsilon}{3} \text{ for all } \kappa_2 \geq \kappa. \tag{21}$$

Let $\kappa_3 = \max(\kappa_1, \kappa_2)$. Then, using the Eqs. (18), (20) and (21), we get

$$\sup_{u\in[0,\infty)} \frac{|\mathcal{R}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u)|}{(1+u^2)^{1+\zeta}} < \epsilon,$$

which, completes the proof. □

5. A-Statistical Approximation

We revisit some notation from [37]. Suppose that $B = (b_{(\kappa+l)\mu})$ is an infinite, non-negative summability matrix. A sequence $u := (u_\mu)$ is A-statistically convergent to L, denoted as $st_B - \lim u = L$, if for each $\epsilon > 0$

$$\lim_\kappa \sum_{\mu:|u_\mu - L|\geq\epsilon} b_{(\kappa+l)\mu} = 0.$$

Let $q = (q_{(\kappa+l)})$ be a sequence such that the following assertions are true

$$st_B - \lim_\kappa q_{(\kappa+l)} = 1 \text{ and } st_B - \lim_\kappa q^\kappa_{(\kappa+l)} = b,\ 0 \leq b < 1. \tag{22}$$

Theorem 5.1. *Consider $B = (b_{(\kappa+l)\mu})$ be a non-negative regular summability matrix and sequence $q = (q_{\kappa+l})$ along with condition (22), $q_{\kappa+l} \in (0,1)$, $\kappa \in \mathbb{N}$. Then, for each $\tilde{g} \in C^0_\nu[0,\infty)$, $st_B - \lim_\kappa \|\mathcal{R}^A_{\kappa+l}(\tilde{g};.) - \tilde{g}\|_\nu = 0$.*

Proof. In accordance with Lemma 1.3, one has

$$st_B - \lim_\kappa \|\mathcal{R}^A_{\kappa+l}(\tilde{g}_0;.) - \tilde{g}_0\|_\nu = 0.$$

and

$$\|\mathcal{R}^A_{\kappa+l}(\tilde{g}_1;.)-\tilde{g}_1\|_\nu = \sup_{u\in[0,\infty)} \frac{1}{1+u^2}\left|\frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right]\right|$$
$$= \frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right]\sup_{u\in[0,\infty)}\frac{1}{1+u^2}.$$

Now

$$\tilde{K_1} := \left\{\kappa : \|\mathcal{R}^A_{\kappa+l}(\tilde{g}_1;.)-\tilde{g}_1\| \geq \epsilon\right\},$$
$$\tilde{K_2} := \left\{\kappa : \frac{1}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+1+\lambda\right] \geq \epsilon\right\}.$$

Which implies that $\tilde{K_1} \subseteq \tilde{K_2}$, this shows that $\sum\limits_{\mu\in\tilde{K}_1} b_{(\kappa+l)\mu} \leq \sum\limits_{\mu\in\tilde{K}_2} b_{(\kappa+l)\mu}$. Therefore, we get

$$st_B - \lim_{\kappa} \|\mathcal{R}^A_{\kappa+l}(\tilde{g}_1;.)-\tilde{g}_1\|_\nu = 0. \tag{23}$$

Now by using Lemma 1.3, we have

$$\|\mathcal{R}^A_{\kappa+l}(\tilde{g}_2;.)-\tilde{g}_2\|_{1+u^2} \leq \sup_{u\in[0,\infty)}\frac{1}{\nu(u)}\left|\left(\frac{2}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+\lambda+2\right]u\right)\right.$$
$$+\left(\frac{1}{(\kappa+l)^2}\left[(2\lambda+4)\left\{\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}\right\}+\left\{2\frac{\tilde{\psi}'(h,1)\tilde{\Omega}'(1)}{\tilde{\Omega}(1)\tilde{\psi}(h,1)}\right.\right.\right.$$
$$\left.\left.\left.\left.+\frac{\tilde{\psi}''(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}''(1)}{\tilde{\Omega}(1)}\right\}+\lambda^2+3\lambda+2\right]\right)\right|.$$

For a given $\varepsilon > 0$, we have the following sets

$$\tilde{M_1} := \left\{\kappa : \left\|\mathcal{R}^A_{\kappa+l}(\tilde{g}_2;.)-\tilde{g}_2\right\|_\nu \geq \epsilon\right\},$$
$$\tilde{M_2} := \left\{\kappa : \frac{2}{\kappa+l}\left[\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}+\lambda+2\right] \geq \frac{\epsilon}{2}\right\},$$
$$\tilde{M_3} := \left\{\kappa : \frac{1}{(\kappa+l)^2}\left[(2\lambda+4)\left\{\frac{\tilde{\psi}'(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}'(1)}{\tilde{\Omega}(1)}\right\}+\left\{2\frac{\tilde{\psi}'(h,1)\tilde{\Omega}'(1)}{\tilde{\Omega}(1)\tilde{\psi}(h,1)}\right.\right.\right.$$
$$\left.\left.\left.+\frac{\tilde{\psi}''(h,1)}{\tilde{\psi}(h,1)}+\frac{\tilde{\Omega}''(1)}{\tilde{\Omega}(1)}\right\}+\lambda^2+3\lambda+2\right] \geq \frac{\epsilon}{2}\right\}.$$

It can be observed that $\tilde{M_1} \subseteq \tilde{M_2}\bigcup\tilde{M_3}$. Therefore, we acquire

$$\sum_{\mu\in\tilde{M}_1} b_{(\kappa+l)\mu} \leq \sum_{\mu\in\tilde{M}_2} b_{(\kappa+l)\mu} + \sum_{\mu\in\tilde{M}_3} b_{(\kappa+l)\mu}.$$

As $\kappa\to\infty$, we have

$$st_B - \lim_{\kappa} \|\mathcal{R}^A_{\kappa+l}(\tilde{g}_2;u)-\tilde{g}_2\|_\nu = 0. \tag{24}$$

Thus, we concludes the proof of the Theorem 5.1. □

Next, we will examine the convergence rate of A-Statistical approximation with respect to Peetre's K-functional for the operators $\mathcal{R}^A_{\kappa+l}(.;.)$.

Theorem 5.2. *Let $\tilde{g} \in C^2_B[0,\infty)$. Then,*

$$st_B - \lim_{\kappa} \|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \tilde{g}\|_{C_B[0,\infty)} = 0.$$

Proof. Considering Taylor's result, we have

$$\tilde{g}(y) = \tilde{g}(u) + \tilde{g}'(u)(y-u) + \frac{1}{2}\tilde{g}''(\eta)(y-u)^2,$$

where $v \le \eta \le u$. Operating $\mathcal{R}^A_{\kappa+l}(.;.)$, on both sides in above equation, one get

$$\mathcal{R}^A_{\kappa+l}(\tilde{g};u) - \tilde{g}(u) = \tilde{g}'(u)\mathcal{R}^A_{\kappa+l}(\eta_1;u) + \frac{1}{2}\tilde{g}''(\eta)\mathcal{R}^A_{\kappa+l}(\eta_2;u),$$

which yields that

$$\begin{aligned} \|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \tilde{g}\|_{C_B[0,\infty)} &\le \|\tilde{g}'\|_{C_B[0,\infty)}\|\mathcal{R}^A_{\kappa+l}(\tilde{g}_1-,.)\|_{C_B[0,\infty)} \\ &\quad + \|\tilde{g}''\|_{C_B[0,\infty)}\|\mathcal{R}^A_{\kappa+l}(\tilde{g}_1-,.)^2\|_{C_B[0,\infty)} \\ &= \tilde{W}_1 + \tilde{W}_2, \quad say. \end{aligned} \tag{25}$$

Based on Eqs. (23) and (24), it follows that

$$\begin{aligned} \lim_{\kappa} \sum_{\mu\in\mathbb{N}:\tilde{W}_1\ge\frac{\epsilon}{2}} b_{(\kappa+l)\mu} &= 0, \\ \lim_{\kappa} \sum_{\mu\in\mathbb{N}:\tilde{W}_2\ge\frac{\epsilon}{2}} b_{(\kappa+l)\mu} &= 0. \end{aligned}$$

From Eq. (25), we have

$$\lim_{\kappa} \sum_{\mu\in\mathbb{N}:\|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot)-\tilde{g}\|_{C_B[0,\infty)}\ge\epsilon} b_{(\kappa+l)\mu} \le \lim_{\kappa} \sum_{\mu\in\mathbb{N}:\tilde{W}_1\ge\frac{\epsilon}{2}} b_{(\kappa+l)\mu} + \lim_{\kappa} \sum_{\mu\in N:\tilde{W}_2\ge\frac{\epsilon}{2}} b_{(\kappa+l)\mu}.$$

Thus $st_B - \lim_{\kappa} \|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \tilde{g}\|_{C_B[0,\infty)} \to 0.$ as $\kappa \to \infty$.

Hence, we arrive the proof. □

Theorem 5.3. *For $\tilde{g} \in C^2_B[0,\infty)$,*

$$\|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \tilde{g}\|_{C_B[0,\infty)} \le M\omega_2(\tilde{g};\sqrt{\delta}),$$

where $\delta = \|\mathcal{R}^A_{\kappa+l}(\tilde{g}_1 - \cdot;\cdot)\|_{C_B[0,\infty)} + \|\mathcal{R}^A_{\kappa+l}(\tilde{g}_1 - \cdot)^2;\cdot)\|_{C_B[0,\infty)}$, *and* $\|\tilde{g}\|_{C^2_B[0,\infty)} = \|\tilde{g}\|_{C_B[0,\infty)} + \|\tilde{g}'\|_{C_B[0,\infty)} + \|\tilde{g}''\|_{C_B[0,\infty)}$.

Proof. Let $\tilde{f} \in C^2_B[0,\infty)$. Using Eq. (25), one obtain

$$\begin{aligned} \|\mathcal{R}^A_{\kappa+l}(\tilde{f}) - \tilde{f}\|_{C_B[0,\infty)} &\le \|\tilde{f}'\|_{C_B[0,\infty)}\|\mathcal{R}^A_{\kappa+l}(\tilde{g}_1 - \cdot);\cdot)\|_{C_B[0,\infty)} \\ &\quad + \frac{1}{2}\|\tilde{f}''\|_{C_B[0,\infty)}\mathcal{R}^A_{\kappa+l}(\tilde{g}_1 - \cdot)^2;\cdot)\|_{C_B[0,\infty)} \\ &\le \delta\|\tilde{f}\|_{C^2_B[0,\infty)}. \end{aligned} \tag{26}$$

For each $\tilde{g} \in C_B[0,\infty)$ and $\tilde{f} \in C_B^2[0,\infty)$, Using Eq. (26), we acquire

$$\begin{aligned}
\|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \tilde{g}\|_{C_B[0,\infty)} &\leq \|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \mathcal{R}^A_{\kappa+l}(\tilde{f};\cdot)\|_{C_B[0,\infty)} \\
&+ \|\mathcal{R}^A_{\kappa+l}(\tilde{f};\cdot) - \tilde{f}\|_{C_B[0,\infty)} + \|\tilde{f} - \tilde{g}\|_{C_B[0,\infty)} \\
&\leq 2\|\tilde{f} - \tilde{g}\|_{C_B[0,\infty)} + \|\mathcal{R}^A_{\kappa+l}(\tilde{f};\cdot) - \tilde{f}\|_{C_B[0,\infty)} \\
&\leq 2\|\tilde{f} - \tilde{g}\|_{C_B[0,\infty)} + \delta\|\tilde{f}\|_{C_B^2}.
\end{aligned}$$

Considering Peetre's K-functional, we obtain

$$\|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \tilde{g}\|_{C_B[0,\infty)} \leq 2K_2(\tilde{g};\delta)$$

and

$$\|\mathcal{R}^A_{\kappa+l}(\tilde{g};\cdot) - \tilde{g}\|_{C_B[0,\infty)} \leq \tilde{M}\big\{\omega_2(\tilde{g};\sqrt{\delta}) + \min(1,\delta)\|\tilde{g}\|_{C_B[0,\infty)}\big\}.$$

In view of Eq. (24), we have

$$st_B - \lim_{\kappa} \delta = 0, \text{ thus } st_B - \lim_{\kappa} \omega(\tilde{g};\sqrt{\delta}) = 0,$$

which concludes the proof of desired result. □

6. Conclusion

In this article, we introduce a sequence of positive linear operators using generalized Appell polynomials in integral form. These operators are designed to approximate functions defined on a Lebesgue measurable space and are known as modified Szász-Durrmeyer type operators introduced in (4). Moreover, we derive estimates crucial for establishing the rate of convergence and accuracy of approximation. Further, we explore various aspects of approximation, including local and global results, as well as A-statistical approximation, utilizing these operators to obtain enhanced approximations across different functional spaces.These operators provide the better flexibility in approximations compared to operators defined in (3). In addition, these sequence of operators discuss approximations in wider class in comparison of operators defined in (3).

7. Conflicts of interest

There are no conflicts of interest, according to the authors.

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