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On the lattice of congruence filters of a residuated lattice

Raluca Creţan and Antoaneta Jeflea

Abstract. For a residuated lattice A we denote by $Ds(A)$ the lattice of all congruence filters (deductive systems) of A. The aim of this paper is to put in evidence some new rules of calculus in residuated lattices and some properties of the lattice $(Ds(A), \subseteq)$.

Also, we characterize the residuated lattices for which the lattice of congruence filters is a Boolean lattice.

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1. Introduction

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([22]), Dilworth ([12]), Ward and Dilworth ([29]), Ward ([28]), Balbes and Dwinger ([1]) and Pavelka ([26]).

In [18], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: *BCK- latices* in [17], *full BCK- algebras* in [22], FL_{ew}*- algebras* in [24], and *integral, residuated, commutative l-monoids* in [4].

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [3], [8], [12], [21], [25], [28], [29]).

The paper is organized as follows.

In Section 2 we recall the basic definition and we put in evidence many new rules of calculus in residuated lattices.

Section 3 contains some results relative to the lattice of congruence filters of a residuated lattice. Theorem 3.4 characterize the residuated lattices for which the lattice of congruence filters is a Boolean algebra.

2. Definitions and preliminaries

In this section we review the basic definitions of residuated lattices, with more details and examples. Also we put in evidence connection between residuated lattices and Hilbert algebras and new rules of calculus in residuated lattices.

Definition 2.1. *A residuated lattice (*[3], [27]*) is an algebra*

$$(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$$

of type (2,2,2,2,0,0) equipped with an order $\leq$ *satisfying the following:*

(LR_1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice;*
(LR_2) $(A, \odot, 1)$ *is a commutative ordered monoid;*
(LR_3) $\odot$ *and* $\rightarrow$ *form an adjoint pair, i.e.* $c \leq a \rightarrow b$ *iff* $a \odot c \leq b$ *for all* $a, b, c \in A$.

The relations between the pair of operations $\odot$ and $\to$ expressed by (LR_3), is a particular case of the *law of residuation* ([3]). Namely, let A and B two posets, and $f : A \to B$ a map. Then f is called residuated if there is a map $g : B \to A$, such that for any $a \in A$ and $b \in B$, we have $f(a) \leq b$ iff $b \leq g(a)$ (this is, also expressed by saying that the pair (f, g) is a *residuated pair*).

Now setting A a residuated lattice, $B = A$, and defining, for any $a \in A$, two maps $f_a, g_a : A \to A, f_a(x) = x \odot a$ and $g_a(x) = a \to x$, for any $x \in A$, we see that $x \odot a = f_a(x) \leq y$ iff $x \leq g_a(y) = a \to y$ for every $x, y \in A$, that is, for every $a \in A$, (f_a, g_a) is a pair of residuation.

The symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and logical equivalence.

Proposition 2.1. *([18]) The class $\mathcal{RL}$ of residuated lattices is equational.*

One of the equational axiomatizations of $\mathcal{RL}$ can be:
($\mathcal{L}$) Equations axiomatizing the variety of bounded lattices;
($\mathcal{M}$) Equations axiomatizing the variety of commutative monoids;
($\mathcal{R}_1$) $(x \odot y) \to z = x \to (y \to z)$;
($\mathcal{R}_2$) $[(x \to y) \odot x] \wedge y = (x \to y) \odot x$ (i.e., $(x \to y) \odot x \leq y$);
($\mathcal{R}_3$) $(x \wedge y) \to y = 1$.

Example 2.1. *Let p be a fixed natural number and $I = [0, 1]$ the real unit interval. If for $x, y \in I$, we define $x \odot y = 1 - \min\{1, [(1 - x)^p + (1 - y)^p]^{1/p}\}$ and $x \to y = \sup\{z \in [0, 1] : x \odot z \leq y\}$, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice.*

Example 2.2. *If we preserve the notation from Example 1, and we define for $x, y \in I$, $x \odot y = (\max\{0, x^p + y^p - 1\})^{1/p}$ and $x \to y = \min\{1, (1 - x^p + y^p)^{1/p}\}$, then $(I, \max, \min, \odot, \to, 0, 1)$ become a residuated lattice called generalized Lukasiewicz structure. For $p = 1$ we obtain the notion of Lukasiewicz structure ($x \odot y = \max\{0, x + y - 1\}, x \to y = \min\{1, 1 - x + y\}$).*

Example 2.3. *If on $I = [0, 1]$, for $x, y \in I$ we define $x \odot y = \min\{x, y\}$ and $x \to y = 1$ if $x \leq y$ and y otherwise, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice (called Gődel structure).*

Example 2.4. *If consider on $I = [0, 1]$, $\odot$ to be the usual multiplication of real numbers and for $x, y \in I, x \to y = 1$ if $x \leq y$ and y/x otherwise, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice (called Products structure or Gaines structure).*

Example 2.5. *If $(A, \vee, \wedge, ', 0, 1)$ is a Boolean algebra, then if we define for every $x, y \in A, x \odot y = x \wedge y$ and $x \to y = x' \vee y$, then $(A, \vee, \wedge, \odot, \to, 0, 1)$ become a residuated lattice.*

Examples 2, 3 and 4 have some connections with the notion of *t-norm*.

We call *continuous t-norm* a continuous function $\odot : [0, 1] \times [0, 1] \to [0, 1]$ such that $([0, 1], \odot, 1)$ is an ordered commutative monoid.

So, there are three fundamental t-norms:

Lukasiewicz t-norm: $x \odot_L y = \max\{0, x + y - 1\}$;

Gődel t-norm: $x \odot_G y = \min\{x, y\}$;

*Product (*or *Gaines) t-norm:* $x \odot_P y = x \odot y$.

Since relative to natural ordering on $[0, 1]$, $[0, 1]$ become a complete lattice, every continuous t-norm introduce a natural *residum* (or *implication*) by

$$x \to y = \max\{z \in [0, 1] : x \odot z \leq y\}.$$

So, the implications generated by the three norms mentioned before are

$x \rightarrow_L y = \min\{1, y - x + 1\}$;
$x \rightarrow_G y = 1$ if $x \leq y$ and y otherwise;
$x \rightarrow_P y = 1$ if $x \leq y$ and y/x otherwise.

Definition 2.2. *([27]) A residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called BL-algebra, if the following two identities hold in A :*
(BL_1) $x \odot (x \rightarrow y) = x \wedge y$;
(BL_2) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Remark 2.1. 1. *Łukasiewicz structure, Gődel structure and Product structure are $BL-$ algebras;*
2. *Not every residuated lattice, however, is a BL-algebra (see [27], p.16). Consider, for example a residuated lattice defined on the unit interval I, for all $x, y, z \in I$, such that*
$$x \odot y = 0 \text{ if } x + y \leq \frac{1}{2} \text{ and } x \wedge y \text{ elsewhere}$$
$$x \rightarrow y = 1 \text{ if } x \leq y \text{ and } \max\{\frac{1}{2} - x, y\} \text{ elsewhere.}$$
Let $0 < y < x$, $x + y < \frac{1}{2}$. Then $y < \frac{1}{2} - x$ and $0 \neq y = x \wedge y$, but $x \odot (x \rightarrow y) = x \odot (\frac{1}{2} - x) = 0$. Therefore (BL_1) does not hold.

Remark 2.2. *([27]) If in a $BL-$ algebra A, $x^{**} = x$ for all $x \in A$,(where $x^* = x \rightarrow 0$), and for $x, y \in A$ we denote $x \oplus y = (x^* \odot y^*)^*$, then we obtain an algebra $(A, \oplus, ^*, 0)$ of type $(2, 1, 0)$ satisfying the following:*
$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, x \oplus y = y \oplus x, x \oplus 0 = x,$$
$$x \oplus 0^* = 0^*,$$
$$(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x, \text{ for all } x, y \in A.$$
Then for all $x, y \in A, (y \rightarrow x) \rightarrow x = x \vee y = (x \rightarrow y) \rightarrow y$. $BL-$ algebras of this kind will turn out to be so called $MV-$ algebras (see [27]). Conversely, if $(A, \oplus, ^*, 0)$ is an MV-algebra, then $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra, where for $x, y \in A$:
$$x \odot y = (x^* \oplus y^*)^*,$$
$$x \rightarrow y = x^* \oplus y, 1 = 0^*,$$
$$x \vee y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x \text{ and } x \wedge y = (x^* \vee y^*)^*.$$

Remark 2.3. *([27]) A residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an MV-algebra iff it satisfies the additional condition: $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for any $x, y \in A$.*

Example 2.6. *([19])We give an another example of a finite residuated lattice, which is not a BL-algebra. Let $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but a, b are incomparable. A become a residuated lattice relative to the following operations:*

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

.

The condition $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$, for all $x, y \in A$ is not verified, since $c = a \vee b \neq [(a \rightarrow b) \rightarrow b] \wedge [(b \rightarrow a) \rightarrow a] = (b \rightarrow b) \wedge (a \rightarrow a) = 1$, hence A is not a BL-algebra.

Example 2.7. *([21]) We consider the residuate lattice A with the universe $\{0, a, b, c, d, e, f, 1\}$. Lattice ordering is such that $0 < d < c < b < a < 1$, $0 < d < e < f < a < 1$ and elements $\{b, f\}$ and $\{c, e\}$ are pairwise incomparable. The operations of implication and multiplication are given by the tables below :*

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	1	a	b	c	d	e	f	1

,

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

Clearly, A contains $\{a, b, c, d, e, f\}$ as a sublattice, and that is a copy of the so-called benzene ring, which shows that A is not distributive, and even not modular. But it is easy to see that $a^* = d, b^* = e, c^* = f, d^* = a$, $e^* = b$ and $f^* = c$.

Example 2.8. *([21]) Let A be the residuate lattice with the universe $\{0, a, b, c, d, 1\}$ such that $0 < b < a < 1$, $0 < d < c < a < 1$ and c and d are incomparable with b. The operations of implication and multiplication are given by the tables below :*

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	c	b	b	0	0	b
c	b	d	0	d	d	c
d	b	d	0	d	d	d
1	0	a	b	c	d	1

Then A is obtained from the nonmodular lattice N_5, called the pentagon, by adding the new greatest element 1. Then A is another example of nondistributive residuated lattice.

Example 2.9. *([19]) We give an example of a finite residuate lattice which is an non-linearly MV-algebra. Let $A = \{0, a, b, c, d, 1\}$, with $0 < a, b < c < 1, 0 < b < d < 1$, but a, b and, respective c, d are incomparable. We define*

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

and so A become a $BL-$algebra. We have in A the following operations:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

,

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

It is easy to see that $0^* = 1, a^* = d, b^* = c, c^* = b, d^* = a, 1^* = 0$ *and* $x^{**} = x$, *for all* $x \in A$, *hence* A *is an* $MV-$ *algebra which is not chain.*

Example 2.10. *(*[19]*) We give an another example of a finite residuate lattice* $A = \{0, a, b, c, d, e, f, g, 1\}$, *which is non-linearly* $MV-$ *algebra, with* $0 < a < b < e < 1, 0 < c < f < g < 1, a < d < g, c < d < e$, *but* $\{a, c\}, \{b, d\}, \{d, f\}, \{b, f\}$ *and, respective* $\{e, g\}$ *are incomparable. We define*

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	g	1	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

,

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

and so A *become a residuated lattice.We have* $0^* = 1, a^* = g, b^* = f, c^* = e, d^* = d, e^* = c, f^* = b, g^* = a$.

Example 2.11. *(*[19]*) We give an example of a finite residuate lattice which is an* MV*-algebra. Let* $A = \{0, a, b, c, d, 1\}$, *with* $0 < a < b < 1, 0 < c < d < 1$, *but* a, c *and, respective* b, d *are incomparable. We define*

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

It is easy to see that $0^* = 1, a^* = d, b^* = c, c^* = b, d^* = a$.

In what follows by A we denote a residuated lattice; for $x \in A$ and a natural number n, we define $x^* = x \to 0, (x^*)^* = x^{**}$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

Definition 2.3. *An element* a *in* A *is called idempotent iff* $a^2 = a$, *and it is called nilpotent iff there exists a natural number* n *such that* $a^n = 0$. *The minimum* n *such that* $a^n = 0$ *is called nilpotence order of* a *and will be denoted by* $ord(a)$; *if there is no such* n, *then* $ord(a) = \infty$. *A residuated lattice* A *is called locally finite if every* $a \in A, a \neq 1$, *has finite order. An element* a *in* A *is called dense iff* $a^* = 0$, *and it* *is called a unity iff for all natural numbers n,* $(a^n)^*$ *is nilpotent. The set of dense elements of A will be denoted by* $D(A)$.

Theorem 2.1. *(*[21], [27]*) Let* $x, x_1, x_2, y, y_1, y_2, z \in A$. *Then we have the following rules of calculus:*

$(lr-c_1)$ $1 \to x = x, x \to x = 1, y \le x \to y, x \to 1 = 1, 0 \to x = 1$;
$(lr-c_2)$ $x \odot y \le x, y$, *hence* $x \odot y \le x \wedge y$ *and* $x \odot 0 = 0$;
$(lr-c_3)$ $x \odot y \le x \to y$;
$(lr-c_4)$ $x \le y$ *iff* $x \to y = 1$;
$(lr-c_5)$ $x \to y = y \to x = 1 \Leftrightarrow x = y$;
$(lr-c_6)$ $x \odot (x \to y) \le y, x \le (x \to y) \to y, ((x \to y) \to y) \to y = x \to y$;
$(lr-c_7)$ $x \odot (y \to z) \le y \to (x \odot z) \le (x \odot y) \to (x \odot z)$;
$(lr-c_8)$ $x \to y \le (x \odot z) \to (y \odot z)$;
$(lr-c_9)$ $x \le y$ *implies* $x \odot z \le y \odot z$;
$(lr-c_{10})$ $x \to y \le (z \to x) \to (z \to y)$;
$(lr-c_{11})$ $x \to y \le (y \to z) \to (x \to z)$;
$(lr-c_{12})$ $x \le y$ *implies* $z \to x \le z \to y, y \to z \le x \to z$ *and* $y^* \le x^*$,
$(lr-c_{13})$ $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;
$(lr-c_{14})$ $x_1 \to y_1 \le (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$.

Remark 2.4. *¿From* $lr-c_1$ *and* $lr-c_4$ *we deduce that* 1 *is the greatest element of A.*

Theorem 2.2. *(*[21], [27]*) If* $x, y \in A$, *then :*

$(lr-c_{15})$ $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \le y^*$;
$(lr-c_{16})$ $x \le x^{**}, x^{**} \le x^* \to x$;
$(lr-c_{17})$ $1^* = 0\ ,\ 0^* = 1$;
$(lr-c_{18})$ $x \to y \le y^* \to x^*$;
$(lr-c_{19})$ $x^{***} = x^*, (x \odot y)^* = x \to y^* = y \to x^* = x^{**} \to y^*$.

Theorem 2.3. *(*[21], [27]*) If A is a complete residuated lattice,* $x \in A$ *and* $(y_i)_{i \in I}$ *a family of elements of A, then :*

$(lr-c_{20})$ $x \odot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \odot y_i)$;
$(lr-c_{21})$ $x \odot (\bigwedge_{i \in I} y_i) \le \bigwedge_{i \in I} (x \odot y_i)$;
$(lr-c_{22})$ $x \to (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \to y_i)$;
$(lr-c_{23})$ $(\bigvee_{i \in I} y_i) \to x = \bigwedge_{i \in I} (y_i \to x)$;
$(lr-c_{24})$ $\bigvee_{i \in I} (y_i \to x) \le (\bigwedge_{i \in I} y_i) \to x$;
$(lr-c_{25})$ $\bigvee_{i \in I} (x \to y_i) \le x \to (\bigvee_{i \in I} y_i)$;
$(lr-c_{26})$ $(\bigvee_{i \in I} y_i)^* = \bigwedge_{i \in I} y_i^*$;
$(lr-c_{27})$ $(\bigwedge_{i \in I} y_i)^* \ge \bigvee_{i \in I} y_i^*$.

Corollary 2.1. *(*[8]*) If* $x, x', y, y', z \in A$ *then:*

$(lr-c_{28})$ $x \vee y = 1$ *implies* $x \odot y = x \wedge y$;
$(lr-c_{29})$ $x \to (y \to z) \ge (x \to y) \to (x \to z)$;
$(lr-c_{30})$ $x \vee (y \odot z) \ge (x \vee y) \odot (x \vee z)$, *hence* $x \vee y^n \ge (x \vee y)^n$ *and* $x^m \vee y^n \ge (x \vee y)^{mn}$, *for any* m, n *natural numbers;*
$(lr-c_{31})$ $(x \to y) \odot (x' \to y') \le (x \vee x') \to (y \vee y')$;
$(lr-c_{32})$ $(x \to y) \odot (x' \to y') \le (x \wedge x') \to (y \wedge y')$.

If $B = \{a_1, a_2, ..., a_n\}$ is a finite subset of A we denote $\Pi B = a_1 \odot ... \odot a_n$.

Proposition 2.2. *([2], [5]) Let $A_1, ..., A_n$ finite subsets of A.*
$(lr - c_{33})$ *If $a_1 \vee ... \vee a_n = 1$, for all $a_i \in A_i, i \in \{1, ..., n\}$, then*

$$(\Pi A_1) \vee ... \vee (\Pi A_n) = 1.$$

Proof. For $n = 2$ it is proved in [5] and for $n = 2$, A_1 a singleton and A_2 a doubleton in [2] (Lemma 6.4). The proof for arbitrary n is a simple mathematical induction argument. ■

Corollary 2.2. *Let $a_1, ..., a_n \in A$.*
$(lr - c_{34})$ *If $a_1 \vee ... \vee a_n = 1$, then $a_1^k \vee ... \vee a_n^k = 1$, for every natural number k.*

Proposition 2.3. *Suppose A is a locally finite residuated lattice. Then for all $a, b \in A, a \vee b = 1$ iff $a = 1$ or $b = 1$.*

Proof. Assume $a \vee b = 1$. Then, since $a \vee b \leq [(a \to b) \to b] \wedge [(b \to a) \to a]$ we deduce that $(a \to b) \to b = (b \to a) \to a = 1$, hence $a \to b = b$ and $b \to a = a$. Let now $a \neq 1$. Since the residuated lattice A is locally finite (under consideration) there is a natural number m such that $a^m = 0$. Now $b = a \to b = a \to (a \to b) = a^2 \to b = ... = a^m \to b = 0 \to b = 1$. ■

Proposition 2.4. *In any locally finite residuated lattice A, for all $x \in A$*
(i) $0 < x < 1$ *iff* $0 < x^* < 1$;
(ii) $x^* = 0$ *iff* $x = 1$;
(iii) $x^* = 1$ *iff* $x = 0$.

Proof. (i). Assume $0 < x < 1, ord(x) = m \geq 2$. Then, $x^{m-1} \odot x = 0, x^{m-2} \odot x \neq 0$, so by the definition of $x^*, 0 < x^{m-1} \leq x^* < x^{m-2} \leq 1$. Conversely, let $0 < x^* < 1$, $ord(x^*) = n \geq 2$. Then by similar argument, $0 < (x^*)^{n-1} \leq x^{**} < (x^*)^{n-2} \leq 1$.
If now $x = 0$, then $x^* = 1$, a contradiction. Therefore $0 < x \leq x^{**} < 1$.
(ii). If $x^* = 0$ but $x \neq 1$, then $0 < x < 1$, which leads to a contradiction $x^* \neq 0$. Thus $x = 1$.
(iii). Analogously as (ii). ■
Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Recall (see [15]) that an element $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such element b exists it is called a *complement* of a. We will denote $b = a'$ and the set of all complemented elements in L by $B(L)$. Complements are generally not unique, unless the lattice is distributive.
In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.

Lemma 2.1. *([21]) Suppose that $a \in A$ have a complement $b \in A$. Then, the following hold:*
(i) *If c is another complement of a in A, then $c = b$;*
(ii) $a' = b$ *and* $b' = a$;
(iii) $a^2 = a$.

Let $B(A)$ the set of all complemented elements of the lattice $L(A) = (A, \wedge, \vee, 0, 1)$.

Proposition 2.5. *([6], [21]) A nontrivial residuated lattice A is directly indecomposable iff $B(A) = \{0, 1\}$.*

Corollary 2.3. *([6], [21]) If A is subdirectly irreducible, then $B(A) = \{0, 1\}$.*

Lemma 2.2. *([8]) If $e \in B(A)$, then $e' = e^*$ and $e^{**} = e$.*

Remark 2.5. *([21]) If $e, f \in B(A)$, then $e \wedge f, e \vee f \in B(A)$. Moreover, $(e \vee f)' = e' \wedge f'$ and $(e \wedge f)' = e' \vee f'$. So, $e \rightarrow f = e' \vee f \in B(A)$.*

Lemma 2.3. *([21]) If $e \in B(A)$, then*

$(lr - c_{35})$ $e \odot x = e \wedge x$, *for every* $x \in A$.

Corollary 2.4. *([21]) The set $B(A)$ is the universe of a Boolean subalgebra of A (called the Boolean center of A).*

Proposition 2.6. *([8]) For $e \in A$ the following are equivalent:*

(i) $e \in B(A)$;

(ii) $e \vee e^* = 1$.

Definition 2.4. *A totally ordered (linearly ordered) residuated lattice will be called chain.*

Remark 2.6. *If A is a chain, then $B(A) = \{0, 1\}$.*

Proposition 2.7. *([8]) For $e \in A$ we consider the following assertions:*

(1) $e \in B(A)$;

(2) $e^2 = e$ *and* $e = e^{**}$;

(3) $e^2 = e$ *and* $e^* \rightarrow e = e$;

(4) $(e \rightarrow x) \rightarrow e = e$, *for every* $x \in A$;

(5) $e \wedge e^* = 0$.

Then:

(i) $(1) \Rightarrow (2), (3), (4)$ *and* (5),

(ii) $(2) \not\Rightarrow (1), (3) \not\Rightarrow (1), (4) \not\Rightarrow (1), (5) \not\Rightarrow (1)$,

(iii) *If A is a $BL-$algebra then the conditios $(1) - (5)$ are equivalent.*

Remark 2.7. 1. *If $A = \{0, a, b, c, 1\}$, is the residuated lattice from Example 2.6, then $B(A) = \{0, 1\}$;*

2. *If $A = \{0, a, b, c, d, e, f, 1\}$, is the residuated lattice from Example 2.7, then $B(A) = \{0, 1\}$; also $B(A) = \{0, 1\}$, where A is the residuated lattice from Example 2.8;*
3. *If $A = \{0, a, b, c, d, 1\}$, is the residuated lattice from Example 2.9, then $B(A) = \{0, a, d, 1\}$;*
4. *If $A = \{0, a, b, c, d, e, f, g, 1\}$, is the residuated lattice from Example 2.10, then $B(A) = \{0, b, f, 1\}$;*
5. *If $A = \{0, a, b, c, d, 1\}$, is the residuated lattice from Example 2.11, then $B(A) = \{0, b, c, 1\}$.*

Lemma 2.4. *([8]) If $e, f \in B(A)$ and $x, y \in A$, then:*

$(lr - c_{36})$ $x \odot (x \rightarrow e) = e \wedge x, e \odot (e \rightarrow x) = e \wedge x$;

$(lr - c_{37})$ $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$;

$(lr - c_{38})$ $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$;

$(lr - c_{39})$ $e \odot (x \rightarrow y) = e \odot [(e \odot x) \rightarrow (e \odot y)]$;

$(lr - c_{40})$ $x \odot (e \rightarrow f) = x \odot [(x \odot e) \rightarrow (x \odot f)]$;

$(lr - c_{41})$ $e \rightarrow (x \rightarrow y) = (e \rightarrow x) \rightarrow (e \rightarrow y)$.

Corollary 2.5. *If $e \in B(A)$ and $x, y \in A$, then:*

$(lr - c_{42})$ $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$.

Definition 2.5. *A Heyting algebra is a lattice $(L, \vee, \wedge)$ with 0 such that for every $a, b \in L$, there exists an element $a \rightarrow b \in L$ (called the pseudocomplement of a with respect to b) such that for every $x \in L$, $a \wedge x \leq b$ iff $x \leq a \rightarrow b$ (that is, $a \rightarrow b = \sup\{x \in L : a \wedge x \leq b\}$).*

Definition 2.6. *([11]) Following Diego, by Hilbert algebra we mean an algebra* $(A, \rightarrow, 1)$ *of type* $(2, 0)$ *satisfying the following identities*:
(H_1) $x \rightarrow (y \rightarrow x) = 1$;
(H_2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$;
(H_3) *If* $x \rightarrow y = y \rightarrow x = 1$, *then* $x = y$.

Remark 2.8. *([11]) If* $(L, \vee, \wedge, \rightarrow, 0)$ *is a Heyting algebra, then* $(L, \rightarrow, 1)$ *is a Hilbert algebra, where* $1 = a \rightarrow a$ *for an element* $a \in L$.

Taking as a guide -line the case of $BL-$ algebras ([7]), a residuated lattice A will be called *G- algebra* if $x^2 = x$, for every $x \in A$.

Remark 2.9. *In a G-algebra* A, $x \odot y = x \wedge y$ *for every* $x, y \in A$.

Proposition 2.8. *In a residuated lattice* A *the following assertions are equivalent :*
(i) $x^2 = x$ *for every* $x \in A$;
(ii) $x \odot (x \rightarrow y) = x \odot y = x \wedge y$ *for every* $x, y \in A$.

Proof. $(i) \Rightarrow (ii)$. Let $x, y \in A$. By $(lr - c_7)$ we have

$$x \odot (x \rightarrow y) \leq (x \odot x) \rightarrow (x \odot y) \Leftrightarrow x \odot (x \rightarrow y) \leq x \rightarrow (x \odot y) \Leftrightarrow$$
$$x \rightarrow y \leq x \rightarrow (x \rightarrow (x \odot y)) = x^2 \rightarrow (x \odot y) = x \rightarrow (x \odot y) \Rightarrow$$
$$x \odot (x \rightarrow y) \leq x \odot y.$$

Since $y \leq x \rightarrow y$, then $x \odot y \leq x \odot (x \rightarrow y)$, so $x \odot (x \rightarrow y) \leq x \odot y$.

Clearly, $x \odot y \leq x, y$. To prove $x \odot y = x \wedge y$, let $t \in A$ such that $t \leq x$ and $t \leq y$. Then $t = t^2 \leq x \odot y$, that is, $x \odot y = x \wedge y$.

$(ii) \Rightarrow (i)$. In particular for $x = y$ we obtain $x \odot x = x \wedge x = x \Leftrightarrow x^2 = x$. ■

Proposition 2.9. *For a residuated lattice* $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *the following are equivalent:*
(i) $(A, \rightarrow, 1)$ *is a Hilbert algebra;*
(ii) $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a G-algebra.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $(A, \rightarrow, 1)$ is a Hilbert algebra, then for every $x, y, z \in A$ we have

$$x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z).$$

¿From $lr - c_{13}$ we have

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z \text{ and } (x \rightarrow y) \rightarrow (x \rightarrow z) = (x \odot (x \rightarrow y)) \rightarrow z,$$

so we obtain

$$(x \odot y) \rightarrow z = (x \odot (x \rightarrow y)) \rightarrow z$$

hence $x \odot y = x \odot (x \rightarrow y)$; for $x = y$ we obtain $x^2 = x$, that is, A is a G- algebra.

$(ii) \Rightarrow (i)$. Follows from Proposition 2.8. ■

3. The lattice of congruence filters of a residuated lattice

In this section we present new results relative to lattice of congruence filters of a residuated lattice. We characterize the residuated lattices for which the lattice of congruence filters is a Boolean algebra.

Definition 3.1. *([21], [27]) A non empty subset* $D \subseteq A$ *is called a congruence filters of* A *if the following conditions are satisfied:*
(Ds_1) $1 \in D$;
(Ds_2) *If* $x, x \rightarrow y \in D$, *then* $y \in D$.

Clearly $\{1\}$ and A are congruence filters ; a congruence filter D of A is called *proper* if $D \neq A$.

Remark 3.1. 1. *A congruence filter D is proper iff $0 \notin D$ iff no element $a \in A$ holds $a, a^* \in D$;*
2. *$a \in D$ iff $a^n \in D$ for every $n \geq 1$.*

Remark 3.2. *(*[21], [27]*) A nonempty subset $D \subseteq A$ is a congruence filters of A iff for all $x, y \in A$:*
(Ds'_1) *If $x,\ y \in D$, then $x \odot y \in D$;*
(Ds'_2) *If $x \in D, y \in A, x \leq y$, then $y \in D$.*

Remark 3.3. *Congruence filters are called also deductive systems in literature. To avoid confusion we reserve, however in this paper, the name filter to lattice filters and deductive system (**ds**) for congruence filters. From $(lr - c_2)$ and Remark 3.2 we deduce that every ds of A is a filter for $L(A)$, but filters of $L(A)$ are not, in general, congruence filters for A (see* [27]*).*

We denote by $Ds(A)$ the set of all congruence filters (deductive systems, **ds** for short) of A.

Whith any **ds** D of A we can (see [21], [27]) associate a congruence θ_D on A by defining : $(a, b) \in \theta_D$ iff $a \to b, b \to a \in D$ iff $(a \to b) \odot (b \to a) \in D$. Conversely, for $\theta \in Con(A)$, the subset D_θ of A defined by $a \in D_\theta$ iff $(a, 1) \in \theta$ is a **ds** of A. Moreover the natural maps associated whith the above are mutually inverse and establish an isomorphism between the lattices $Ds(A)$ and $Con(A)$.

For $a \in A$, let a/D be the equivalence class of a modulo θ_D. If we denote by A/D the quotient set A/θ_D, then A/D becomes a residuated lattice with the natural operations induced from those of A. Clearly, in A/D, $\mathbf{0} = 0/D$ and $\mathbf{1} = 1/D$.

Proposition 3.1. *Let $D \in Ds(A)$, and $a, b \in A$, then*
(i) *$a/D = 1/D$ iff $a \in D$, hence $a/D \neq \mathbf{1}$ iff $a \notin D$;*
(ii) *$a/D = 0/D$ iff $a^* \in D$;*
(iii) *If D is proper and $a/D = 0/D$, then $a \notin D$;*
(iv) *$a/D \leq b/D$ iff $a \to b \in D$.*

Proof. (i). We have $a/D = 1/D$ iff $(a \to 1) \odot (1 \to a) \in D$ iff $1 \odot a = a \in D$.
(ii). We have $a/D = 0/D$ iff $(a \to 0) \odot (0 \to a) \in D$ iff $a^* \odot 1 = a^* \in D$.
(iii). Follow from Remark 3.1.
(iv). By $lr - c_4$ we have $a/D \leq b/D$ iff $a/D \to b/D = \mathbf{1}$ iff $(a \to b)/D = 1/D$ iff $a \to b \in D$ (by (i)). ■

It follows immediately from the above that a residuated lattice A (see and [6]) is subdirectly irreducible iff it has the second smallest **ds**, i.e. the smallest **ds** among all ds except $\{1\}$. The next theorem characterises internally subdirectly irreducible and simple residuated lattices.

Theorem 3.1. *(*[21]*) A residuated lattice A is:*
(i) *subdirectly irreducible (**si** for short) iff there exists an element $a < 1$ such that for any $x < 1$ there exists a natural number $n \geq 1$ such that $x^n \leq a$;*
(ii) *simple iff a can be taken to be 0.*

Proposition 3.2. *(*[21]*) In any **si** residuated lattice, if $x \vee y = 1$, then either $x = 1$ or $y = 1$ holds.*

Therefore, every **si** residuated lattice has at most one *coatom* (recall that are element a of a lattice L with the greatest element 1 is a coatom if it is maximal among elements in $L \backslash \{1\}$).

The next result characterises these **si** residuated lattices which have the coatom:

Theorem 3.2. *([20]) A residuated lattice A has the unique coatom iff there exists an element $a < 1$ and a natural number n such that $x^n \leq a$ holds for any $x < 1$.*

Directly indecomposable residuated lattices also have quite a handly description. It was obtained for a subvariety of residuated latticers, called product algebras, by Cignoli and Torrens in [10].

For arbitrary residuated lattices we have:

Theorem 3.3. *([21]) A nontrivial residuated lattice A is directly indecomposable iff $B(A) = \{0, 1\}$.*

Remark 3.4. *The lattices from Examples 2.6, 2.7 and 2.8 are directly indecomposable.*

For a nonempty subset $S \subseteq A$, the smallest **ds** of A which contains S, i.e. $\cap\{D \in Ds(A) : S \subseteq D\}$, is said to be *the **ds** of A generated by S* and will be denoted by $[S)$.

If $S = \{a\}$, with $a \in A$, we denote by $[a)$ the **ds** *generated by* $\{a\}$ ($[a)$ is called *principal*).

For $D \in Ds(A)$ and $a \in A$, we denote by $D(a) = [D \cup \{a\})$ (clearly, if $a \in D$, then $D(a) = D$).

Proposition 3.3. *([21], [27]) Let $S \subseteq A$ a nonempty subset of A, $a \in A$, $D, D_1, D_2 \in Ds(A)$. Then*

(*i*) *If S is a **ds**, then $[S) = S$;*
(*ii*) *$[S) = \{x \in A : s_1 \odot ... \odot s_n \leq x$, for some $n \geq 1$ and $s_1, ..., s_n \in S\}$. In particular, $[a) = \{x \in A : x \geq a^n$, for some $n \geq 1\}$;*
(*iii*) *$D(a) = \{x \in A : x \geq d \odot a^n$, whith $d \in D$ and $n \geq 1\}$;*
(*iv*) *$[D_1 \cup D_2) = \{x \in A : x \geq d_1 \odot d_2$ for some $d_1 \in D_1$ and $d_2 \in D_2\}$.*

Lemma 3.1. *Let $D \in Ds(A)$ and $a \in A$. Then $D(a) = \{x \in A : a^n \to x \in D$, for some $n \geq 1\}$.*

Proof. If $x \in D(a)$, then $x \geq d \odot a^n$, for some $n \geq 1$ and $d \in D$. Thus, $d \leq a^n \to x$, so $a^n \to x \in D$.

Conversely, assume that $d = a^n \to x \in D$ for some $n \geq 1$. We also have $(a^n \odot d) \to x = d \to (a^n \to x) = d \to d = 1$, hence $a^n \odot d \leq x$. Therefore, $x \in D(a)$. ■

Proposition 3.4. *For any element x of a residuated lattice A, there is a proper **ds** D of A such that $x \in D$ iff $ord(x) = \infty$.*

Proof. Let D be a proper ds and $x \in D$. Then $x^n \in D$, for some natural number $n \geq 1$, whence $x^n \neq 0$ for any natural number n. Therefore $ord(x) = \infty$. Conversely, if $ord(x) = \infty$, then $D = [x) = \{y \in A : x^n \leq y$ for some natural number $n\}$ is a proper ds of A and $x \in D$. ■

For D_1, $D_2 \in Ds(A)$ we put

$$D_1 \wedge D_2 = D_1 \cap D_2 \text{ and } D_1 \vee D_2 = [D_1 \cup D_2).$$

Proposition 3.5. *If $a, b \in A$, then*

(*i*) *$[a) = \{x \in A : a \leq x\}$ iff $a \odot a = a$;*
(*ii*) *$a \leq b$ implies $[b) \subseteq [a)$;*
(*iii*) *$[a) \cap [b) = [a \vee b)$;*
(*iv*) *$[a) \vee [b) = [a \wedge b) = [a \odot b)$;*

(v) $[a) = 1$ *iff* $a = 1$.

Proof. $(i), (ii)$. Obviously.

(iii). Since $a, b \leq a \vee b$, by (ii), $[a \vee b) \subseteq [a), [b)$, hence $[a \vee b) \subseteq [a) \cap [b)$. Let now $x \in [a) \cap [b)$; then $x \geq a^m, x \geq b^n$ for some natural numbers $m, n \geq 1$, hence $x \geq a^m \vee b^n \geq (a \vee b)^{mn}$, (by $lr - c_{30}$), so $x \in [a \vee b)$, that is, $[a) \cap [b) \subseteq [a \vee b)$. Hence $[a) \cap [b) = [a \vee b)$.

(iv). Since $a \odot b \leq a \wedge b \leq a, b$, by (ii), we deduce that $[a), [b) \subseteq [a \wedge b) \subseteq [a \odot b)$, hence $[a) \vee [b) \subseteq [a \wedge b) \subseteq [a \odot b)$.

For the converse inclusions, let $x \in [a \odot b)$. Then for some natural number $n \geq 1$, $x \geq (a \odot b)^n = a^n \odot b^n \in [a) \vee [b)$ (since $a^n \in [a), b^m \in [b)$), (by Proposition 3.3, (ii)), hence $x \in [a) \vee [b)$, that is, $[a \odot b) \subseteq [a) \vee [b)$, so $[a) \vee [b) = [a \wedge b) = [a \odot b)$.

(v). Obviously. ■

Corollary 3.1. *If we denote by $Ds_p(A)$ the family of all principal* **ds** *of A, then $Ds_p(A)$ is a bounded sublattice of $Ds(A)$.*

Proof. Apply Proposition 3.5, (iii), (iv) and the fact that $\{1\} = [1) \in Ds_p(A)$ and $A = [0) \in Ds_p(A)$. ■

Definition 3.2. *We recall (*[15]*, p.93) that a lattice $(L, \vee, \wedge)$ is called Brouwerian if it satisfies the identity $a \wedge (\bigvee\limits_i b_i) = \bigvee\limits_i (a \wedge b_i)$) (whenever the arbitrary unions exists). Let L be a complete lattice and let a be an element of L. Then a is called compact if $a \leq \vee X$ for some $X \subseteq L$ implies that $a \leq \vee X_1$ for some finite $X_1 \subseteq X$. A complete lattice is called algebraic if every element is the join of compact elements (in the literature, algebraic lattices are also called compactly generated lattices).*

Proposition 3.6. *The lattice $(Ds(A), \subseteq)$ is a complete Brouwerian lattice (hence distributive), the compacts elements being exactly the principal* **ds** *of A.*

Proof. Clearly, if $(D_i)_{i\in I}$ is a family of ds from A, then the infimum of this family is $\bigwedge\limits_{i\in I} D_i = \bigcap\limits_{i\in I} D_i$ and the supremum is $\bigvee\limits_{i\in I} D_i = [\bigcup\limits_{i\in I} D_i) = \{x \in A : x \geq x_{i_1} \odot ... \odot x_{i_m}$, where $i_1, ..., i_m \in I, x_{i_j} \in D_{i_j}$, $1 \leq j \leq m\}$, that is, $Ds(A)$ is complete.

We will to prove that the compacts elements of $Ds(A)$ are exactly the principal **ds** of A. Let D be a compact element of $Ds(A)$. Since $D = \bigvee\limits_{a\in D} [a)$, there are $m \geq 1$ and $a_1, ..., a_m \in A$ such that $D = [a_1) \vee ... \vee [a_m) = [a_1 \odot .... \odot a_m)$, (by Proposition 3.5, (iv)). Hence D is a principal **ds** of A.

Conversely, let $a \in A$ and $(D_i)_{i\in I}$ be a family of **ds** of A such that $[a) \subseteq \bigvee\limits_{i\in I} D_i$. Then $a \in \bigvee\limits_{i\in I} D_i = [\bigcup\limits_{i\in I} D_i)$, so we deduce that there are $m \geq 1$, $i_1, ..., i_m \in I, x_{i_j} \in D_{i_j}$ $(1 \leq j \leq m)$ such that $a \geq x_{i_1} \odot ... \odot x_{i_m}$.

It follows that $a \in [D_{i_1} \cup ... \cup D_{i_m})$, so $[a) \subseteq [D_{i_1} \cup ... \cup D_{i_m}) = D_{i_1} \vee ... \vee D_{i_m}$.

For any ds D we have $D = \bigvee\limits_{a\in D} [a)$, so the lattice $Ds(A)$ is algebraic.

In order to prove that $Ds(A)$ is Brouwerian we must show that for every ds D and every family $(D_i)_{i\in I}$ of **ds**, $D \wedge (\bigvee\limits_{i\in I} D_i) = \bigvee\limits_{i\in I} (D \wedge D_i) \Leftrightarrow D \cap (\bigvee\limits_{i\in I} D_i) = [\bigcup\limits_{i\in I} (D \cap D_i))$. Clearly, $[\bigcup\limits_{i\in I} (D \cap D_i)) \subseteq D \cap (\bigvee\limits_{i\in I} D_i)$.

Let now $x \in D \cap (\bigvee\limits_{i\in I} D_i)$. Then $x \in D$ and there exist $i_1, ..., i_m \in I, x_{i_j} \in D_{i_j}$ $(1 \leq j \leq m)$ such that $x \geq x_{i_1} \odot ... \odot x_{i_m}$. Then $x = x \vee (x_{i_1} \odot ... \odot x_{i_m}) \geq (x \vee x_{i_1}) \odot .... \odot (x \vee x_{i_m})$ (by $lr - c_{30}$). Since $x \vee x_{i_j} \in D \cap D_{i_j}$, for every $1 \leq j \leq m$ we deduce that $x \in \bigvee\limits_{i\in I} (D \cap D_i)$, hence $D \cap (\bigvee\limits_{i\in I} D_i) \subseteq \bigvee\limits_{i\in I} (D \cap D_i)$, that is, $D \cap (\bigvee\limits_{i\in I} D_i) = \bigvee\limits_{i\in I} (D \cap D_i)$.■

For D_1, $D_2 \in Ds(A)$ we put

$$D_1 \to D_2 = \{a \in A : D_1 \cap [a) \subseteq D_2\}.$$

Lemma 3.2. *If D_1, $D_2 \in Ds(A)$ then*
(*i*) $D_1 \to D_2 \in Ds(A)$;
(*ii*) *If $D \in Ds(A)$, then $D_1 \cap D \subseteq D_2$ iff $D \subseteq D_1 \to D_2$, that is,*

$$D_1 \to D_2 = \sup\{D \in Ds(A) : D_1 \cap D \subseteq D_2\}.$$

Proof. (*i*). Since $[1) = \{1\}$ and $[1) \cap D_1 = \{1\} \subseteq D_2$ we deduce that $1 \in D_1 \to D_2$.
Let x, $y \in A$ such that $x \leq y$ and $x \in D_1 \to D_2$, that is, $[x) \cap D_1 \subseteq D_2$. Then $[y) \subseteq [x)$, so $[y) \cap D_1 \subseteq [x) \cap D_1 \subseteq D_2$, hence $[y) \cap D_1 \subseteq D_2$, that is, $y \in D_1 \to D_2$.
To proof that (Ds'_1) is verified, let x, $y \in A$ such that $x, y \in D_1 \to D_2$, hence

$$[x) \cap D_1 \subseteq D_2 \text{ and } [y) \cap D_1 \subseteq D_2.$$

We deduce $([x) \cap D_1) \vee ([y) \cap D_1) \subseteq D_2$, hence by Proposition 3.6, $([x) \vee [y)) \cap D_1 \subseteq D_2$. By Proposition 3.5 we deduce that $[x \odot y) \cap D_1 \subseteq D_2$, hence, $x \odot y \in D_1 \to D_2$, that is, $D_1 \to D_2 \in Ds(A)$.
(*ii*). Suppose $D_1 \cap D \subseteq D_2$ and let $x \in D$. Then $[x) \subseteq D$, hence $[x) \cap D_1 \subseteq D \cap D_1 \subseteq D_2$, so $x \in D_1 \to D_2$, that is, $D \subseteq D_1 \to D_2$.
Suppose $D \subseteq D_1 \to D_2$ and let $x \in D_1 \cap D$. Then $x \in D$, hence $x \in D_1 \to D_2$, that is, $[x) \cap D_1 \subseteq D_2$. Since $x \in [x) \cap D_1 \subseteq D_2$ we obtain $x \in D_2$, that is, $D_1 \cap D \subseteq D_2$. ■

For D_1, $D_2 \in Ds(A)$, we denote

$$D_1 * D_2 = \{x \in A : x \vee y \in D_2, \text{ for all } y \in D_1\}.$$

Proposition 3.7. *For all D_1, $D_2 \in Ds(A), D_1 * D_2 = D_1 \to D_2$.*

Proof. Let $x \in D_1 * D_2$ and $z \in [x) \cap D_1$, that is, $z \in D_1$ and $z \geq x^n$ for some $n \geq 1$. Then $x \vee z \in D_2$. Since $z = z \vee x^n \geq (z \vee x)^n$ (by $lr - c_{30}$) we deduce that $z \in D_2$, hence $x \in D_1 \to D_2$, so $D_1 * D_2 \subseteq D_1 \to D_2$.
For converse inclusion, let $x \in D_1 \to D_2$. Thus $[x) \cap D_1 \subseteq D_2$, so, if $y \in D_1$ then $x \vee y \in [x) \cap D_1$, hence $x \vee y \in D_2$. We deduce that $x \in D_1 * D_2$, so $D_1 \to D_2 \subseteq D_1 * D_2$. Since $D_1 * D_2 \subseteq D_1 \to D_2$ we deduce that $D_1 * D_2 = D_1 \to D_2$. ■

Corollary 3.2. *$(Ds(A), \vee, \wedge, \to, \{1\}, A)$ is a Heyting algebra, where for $D \in Ds(A)$,*

$$D^* = D \to \mathbf{0} = D \to \{1\} = \{x \in A : x \vee y = 1, \text{ for every } y \in D\},$$

hence for every $x \in D$ and $y \in D^*, x \vee y = 1$. In particular, for every $a \in A$,

$$[a)^* = \{x \in A : x \vee a = 1\}.$$

Proposition 3.8. *If $x, y \in A$, then $[x \odot y)^* = [x)^* \cap [y)^*$.*

Proof. If $a \in [x \odot y)^*$, then $a \vee (x \odot y) = 1$. Since $x \odot y \leq x, y$ then $a \vee x = a \vee y = 1$, hence $a \in [x)^* \cap [y)^*$, that is, $[x \odot y)^* \subseteq [x)^* \cap [y)^*$.
Let now $a \in [x)^* \cap [y)^*$, that is, $a \vee x = a \vee y = 1$.
By $lr - c_{30}$ we deduce $a \vee (x \odot y) \geq (a \vee x) \odot (a \vee y) = 1$, hence $a \vee (x \odot y) = 1$, that is, $a \in [x \odot y)^*$.
It follows that $[x)^* \cap [y)^* \subseteq [x \odot y)^*$, hence $[x \odot y)^* = [x)^* \cap [y)^*$. ■

Theorem 3.4. *If A is a residuated lattice, then the following assertions are equivalent:*
(*i*) *$(Ds(A), \vee, \wedge, ^*, \{1\}, A)$ is a Boolean algebra;*

(*ii*) *Every* **ds** *of* A *is principal and for every* $a \in A$ *there exists* $n \geq 1$ *such that* $a \vee (a^n)^* = 1$.

Proof. $(i) \Rightarrow (ii)$. Let $D \in Ds(A)$; since $Ds(A)$ is supposed Boolean algebra, then $D \vee D^* = A$. So, since $0 \in A$, there exist $a \in D$, $b \in D^*$ such that $a \odot b = 0$.

Since $b \in D^*$, by Corollary 3.2, it follow that $a \vee b = 1$. By $(lr - c_{28})$ we deduce that $a \wedge b = a \odot b = 0$, that is, b is the complement of a in $L(A)$. Hence $a, b \in B(A) = B(L(A))$.

If $x \in D$, since $b \in D^*$, we have $b \vee x = 1$. Since $a = a \wedge (b \vee x) \overset{lr-c_{42}}{=} (a \wedge b) \vee (a \wedge x) = a \wedge x$ we deduce that $a \leq x$, that is, $D = [a)$. Hence every **ds** of A is principal.

Let now $x \in A$; since $Ds(A)$ is a Boolean algebra, then $[x) \vee [x)^* = A \Leftrightarrow [x)^*(x) = A \Leftrightarrow \{a \in A : a \geq c \odot x^n$, with $c \in [x)^*$ and $n \geq 1\} = A$ (see Proposition 3.3, (ii)).

So, since $0 \in A$, there exist $c \in [x)^*$ and $n \in \omega$ such that $c \odot x^n = 0$. Since $c \in [x)^*$, then $x \vee c = 1$. By $(lr - c_{15})$, from $c \odot x^n = 0$ we deduce $c \leq (x^n)^*$. So, $1 = x \vee c \leq x \vee (x^n)^*$, hence $x \vee (x^n)^* = 1$.

$(ii) \Rightarrow (i)$. By Corollary 3.2, $Ds(A)$ is a Heyting algebra. To prove $Ds(A)$ is a Boolean algebra, we must show that for $D \in Ds(A)$, $D^* = \{1\}$ only for $D = A$ ([1], p. 175). By hypothesis every **ds** of A is principal, so we have $a \in A$ such that $D = [a)$.

Also, by hypothesis, for $a \in A$, there is $n \in \omega$ such that $a \vee (a^n)^* = 1$. By Corollary 3.2, $(a^n)^* \in [a)^* = \{1\}$, hence $(a^n)^* = 1$, that is, $a^n = 0$. By Remark 3.1, we deduce that $0 \in D$, hence $D = A$. ■

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(Raluca Creţan) Technological Secondary School Ion Mincu,
3, Locotenent Dumitru Petrescu st., Tg. Jiu
E-mail address: ralucacretan11@yahoo.com

(Antoaneta Jeflea) Faculty of Bookkeeping Financial Management, University Spiru Haret,
32-34, Unirii st., Constantza, Romania
E-mail address: antojeflea@yahoo.com