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General Decay of the Euler-Bernoulli Beam Fixed Into a Moving Base

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Abstract. In this paper, we aim to discuss the stability of a viscoelastic Euler Bernoulli beam that is fixed to a base in a translational motion at one end and to a tip mass at its free end. The beam is supposed to be influenced by an external unknown disturbance which may destabilize the system. Under some assumptions related to the unknown disturbance together with a suitable selection of the external force applied to the base in motion, we should establish the uniform stability of the system for a large class of relaxation functions using the method of the multiplier technique. In fact, such procedure and other simulations have ameliorated earlier work where the disturbance on the beam was ignored.

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1. Introduction

Robots are widely used through the manufacturing and logistic sectors.They enable companies to carry out work at larger scales and standardize many processes of the new global economy. Due to speediness and preciseness in achieving work at the lowest cost, Robots have replaced human beings in many tasks. Nowadays, and since their adoption in the 1960s, robots are reaching full potential due to digital, artificial intelligence and automation technologies.

Robotic arms, programmable machines, whose links are connected by joints allowing either rotational motion or translational displacement, have become essential in various industrial fields. They can be used in numerous tasks involved in manufacturing industry as welding, drilling and spraying. Their importance increases when it comes to productivity, efficiency and ability to reduce the complexity. Most robotic arms have different characteristics that make them more suitable for certain applications compared to others types of robotic arms. Learning about types and functions of robotic arms has received a great attention by many scientific researchers to work on such machines and develop their performance.

As any machine, robotic arms, which in many times referred as cantilevered Euler-Bernouli beams [10, 26, 44, 12, 18, 27], may suffer from numerous deficiencies which reduce their effectiveness. Those deficiencies are usually due to the manifestation of the mechanical system vibrations when working. One efficient method to eliminate or to reduce the undesirable vibrations is to consider the beam made of vescoelestic materials. In fact, using such materials in various engineering fields has proved them

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to be the most suitable, for more details, see ([9, 15]). In particular, they play a prominent role in the stabilization of the wave equations.

Rivera and Salvatierra [39] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau) \operatorname{div}[a(x) \nabla u(\tau)] \, d\tau = 0, \quad \text{in } \Omega \times (0, \infty),$$

with

$$\begin{cases} u = 0 \quad \text{on } \Gamma \times (0, \infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$, with a smooth boundary $\Gamma, u_0(x)$ and $u_1(x)$ are the initial data, $a(x)$ is a nonnegative C^2 function defined over Ω such that $a(x) = 0$ on $\omega \backslash \omega_\varepsilon$ and $a(x) = 1$ on $\omega_{\varepsilon/2}$ for $\varepsilon > 0$, where ω_ε is a subset of Ω satisfying some assumptions. Assuming some conditions on the kernels g, they demonstrated the well posedness and the stability result of the system (1).

In fact, numerous results related to the stabilization of the system have been established, for further study, see [3, 6, 7, 28–30, 40]. In the pervious system, Mustafa and Messaoudi [31] considered $a(x) = 1$ on Ω, for a larger class of relaxation functions g that fulfills

$$g'(t) \leq -H(g(t)), \quad t \geq 0,$$

where H is a nonnegative function, strictly increasing and convex on $(0, r]$ for some $0 < r < 1$, verifying $H(0) = H'(0) = 0$. Using arguments related to convexity [2, 8, 24], the authors proved a general decay result, where exponential and polynomial decay represent particular cases. Related to similar problems, we recall works of Mustafa [32] Tatar [40]- [42].

In the few last decades, the stabilization of Euler-Bernoulli beam appeared in mechanical engineering, mathematical control system theory and other areas. The reason why many researchers devoted their works on this level. Among those, we mention the works of Park and Kim [34] on the stabilization of a viscoelastic Euler-Bernoulli beam controlled by a nonlinear force in one end and the other end considered to be fixed. Namely they considered the system

$$u_{tt}(x,t) + u_{xxxx} - \int_0^t k(t-s) \, u_{xxxx}(x,s) \, ds + g(u_t(x,t)) = 0, \quad (x,t) \in [0, L] \times \mathbb{R}^+, \tag{2}$$

with the boundary conditions

$$\begin{cases} u(0,t) = u_x(0,t) = u_{xx}(0,t) = u_{xx}(L,t) = u_{xxx}(0,t) = 0, \quad t > 0, \\ u_{xxx}(L,t) - \int\limits_0^t k(t-s) \, u_{xxx}(L,s) \, ds = f(u(L,t)), \quad t > 0, \end{cases}$$

where k is the kernel of finite memory term.

The authors, at first, used the Faedo-Galerkin method to establish the well posedness of the system, then they elaborated the uniform decay of the energy exploring the multiplier technique. We indicate that the authors imposed certain conditions on the functions g and f along with the kernel k. The same problem was tackled by Park et al [35], where they considered other kinds of boundary conditions, and resulted in the exponenential energy decay. In what concerns theoretical stability of Euler-Bernoulli beam with different categories of dissipation, analogue problems were studied by many other authors like Guo and Huang [20] Morgül [33], Park and Kang [36], Park and Park [37], de Querioz et al [14] and Yan et al [45].

The goal of this paper is to investigate the stability result of a viscoelastic cantilevered Euler-Bernoulli beam, subject to an unknown disturbance . We consider the beam to be fixed to a tip mass at the free end whereas the other end is fixed to a base in a translational motion. Namely, we consider

$$\begin{cases} mS_{tt}\left(t\right)+\displaystyle\int_0^L \rho\left(S_{tt}\left(t\right)+u_{tt}\left(x,t\right)\right)dx+m_E\left(S_{tt}\left(t\right)+u_{tt}\left(L,t\right)\right)=f\left(t\right),\ t\geq 0, \\ \rho\left(S_{tt}\left(t\right)+u_{tt}\left(x,t\right)\right)+EIu_{xxxx}\left(x,t\right)-EI\displaystyle\int_0^t g\left(t-s\right)u_{xxxx}\left(s\right)ds=h\left(x,t\right), \\ \qquad\qquad\left(x,t\right)\in\left(0,L\right)\times\mathbb{R}_+, \end{cases} \tag{3}$$

along with following boundary and the initial conditions

$$\begin{cases} u\left(0,t\right)=u_x\left(0,t\right)=0, \quad t\geq 0, \\ EIu_{xxx}\left(L,t\right)-EI\displaystyle\int_0^t g\left(t-s\right)u_{xxx}\left(L,s\right)ds=m_E\left(u_{tt}\left(L,t\right)+S_{tt}\left(t\right)\right),\ t\geq 0, \\ EIu_{xx}\left(L,t\right)-EI\displaystyle\int_0^t g\left(t-s\right)u_{xx}\left(L,s\right)ds=-Ju_{xtt}\left(L,t\right), \quad t\geq 0, \\ u\left(x,0\right)=u_0\left(x\right),u_t\left(x,0\right)=u_1\left(x\right),S\left(0\right)=S^0,S_t\left(0\right)=S^1,x\in\left(0,L\right), \end{cases} \tag{4}$$

where

$u\left(x,t\right)$ stands for transversal displacement of the beam in the position x at the time t.

$S\left(t\right)$ stands for the base motion.

$f(t)$ is the external force applied to the base.

L and m are the length and the mass of the beam respectively.

EI and ρ are the bending stiffness and the linear density respectively.

m_E and J stands for the rigid body at the free end of the beam and the rotary inertia respectively.

g is called the relaxation function to be determined later.

To be more precise, the previous model (3) − (4) is usually applied in mechanical engineering, such as the flexible robot manipulators. Here, the function $h\left(x,t\right)$ is the consequence of certain hydrodynamics effects which are in most time external, and this is considered advantageous if the robotic arms are placed in the external effects.

Historically, analogue problem to (3)–(4), but with three dissipations, was studied by Dadfarnia et al [13], namely

$$\begin{cases} mS_{tt}\left(t\right)+\displaystyle\int_0^L \rho\left(S_{tt}\left(t\right)+u_{tt}\left(x,t\right)\right)dx+m_E\left(S''\left(t\right)+u_{tt}\left(L,t\right)\right)=f\left(t\right),t\geq 0, \\ \rho\left(S_{tt}\left(t\right)+u_{tt}\left(x,t\right)\right)+EIu_{xxxx}\left(x,t\right)-EI\displaystyle\int_0^t g\left(t-s\right)u_{xxxx}\left(s\right)ds \\ +Bu_t\left(x,t\right)+Cu_{xt}\left(x,t\right)+Du_{xxxxt}\left(x,t\right)=h(x,t), \quad \left(x,t\right)\in\left(0,L\right)\times\mathbb{R}_+, \end{cases} \tag{5}$$

with the boundary conditions

$$\begin{cases} u(0,t) = u_x(0,t) = 0, \quad t \geq 0, \\ Du_{xxxt}(L,t) + EIu_{xxx}(L,t) - EI\displaystyle\int_0^t g(t-s)\, u_{xxx}(L,s)\, ds \\ \quad = m_E\left(u_{tt}(L,t) + S_{tt}(t)\right), \quad t \geq 0, \\ Du_{xxt}(L,t) + EIu_{xx}(L,t) - EI\displaystyle\int_0^t g(t-s)\, u_{xx}(L,s)\, ds = -Ju_{xtt}(L,t), \quad t \geq 0, \end{cases} \tag{6}$$

where B, C and D are the viscous damping, the structural damping and the Kelvin-Voigt damping coefficients respectively. In the case $g \equiv h \equiv 0$, the authors proved the exponential stability of the system under a suitable control force.

Very lately, Berkani et al [4] studied the case $B = 0$, $h \equiv 0$ and $g \neq 0$, and used a viscoelastic damping instead of viscous one, taking the relaxation function g such that

$$0 \leq g'(t) + \mu g(t) \leq \xi(t),$$

where μ is a positive constant, $\xi(t)$ is a nonnegative function (i.e. that is the standard assumption $g'(t) \leq -\mu g(t)$ is slightly perturbed). The authors guaranteed the exponential stability if $\overline{\xi} = \|\xi\|_{L^1(0,L)}$ is sufficiently small.

Later, an arbitrary decay of the energy, for a large class of relaxation functions, has been proved in [5]. We refer the reader to see [**?**,25,16,17,19,46,11,12], for other kinds of dissipations concerning the translational Euler-Bernoulli beam.

In this paper, we resume the main contributions related to the stability of the translational Euler -Bernouli beam:

(i) In the absence of any damping, if the beam is made of a viscoelastic material, then the uniform stability of our problem is guaranteed under an unknown distributed disturbance.

(ii) The uniform stability of the system (8) and (9) under the unknown spatiotemporally varying disturbance $h(x,t)$ is established for a large class of kernels. Namely, there exists a nondecreasing function $\gamma(t) > 0$ such that $\gamma'(t)/\gamma(t) = \delta(t)$ is a decreasing function and

$$\gamma(t)\, g(t) \in L^1(0,\infty)$$

where the kernels g verifies: $g'(t) \leq 0$.

Remark 1.1. We recall that our goal in this paper is to investigate the stabilization of the system (5)–(6) in the case $h \neq 0$ without taking in the account the viscous, structural and Kalvin voigt dampings , that is $B \equiv C \equiv D \equiv 0$. This should ameliorate the earlier results [4,5], where the boundary control of the translational Euler -Bernouli beam was supposed to reduce its vibration in the case of $h \equiv 0$.

Remark 1.2. We indicate that vibration abatement of the translational beam with other kinds of of dissipations, in which the unknown distributed disturbance was ignored, has been widely studied in [23,11–13]. In this article, we will improve the later results, by considering the unknown distributed disturbance with viscoelastic damping for a large class of kernels.

The contents of this paper are organized as follows. In section 2, we transform the system (3) – (4) into a simple form in order to be able exploit some functionals analysis tools, we state the well-posedness result without proof, in what concern the stability, we introduced some useful lemmas. In Section 3, we state our main result of stability. Simulations are presented to verify the efficiency of the proposed control in Section 4.

2. Preliminary results

In this section, we mention some lemmas to be used in the proof of our result. To begin with, we consider new functions for the reason to simplify the study of our system (3) – (4). As in [4], by $S^1(t)$ and $w(x,t)$ we denote

$$S^1(t) = S(t) - S_d, \quad S_d > 0,$$

and

$$w(x,t) = S^1(t) + u(x,t), \quad t \geq 0$$

where S_d represents the point at which the beam is regulated. As a consequence problem (3) – (4) is reformulated to

$$m w_{tt}(0,t) + \rho \int_0^L w_{tt}(x,t)\,dx + m_E w_{tt}(L,t) = f(t), \quad t \geq 0, \tag{7}$$

$$\rho w_{tt}(x,t) + EI w_{xxxx}(x,t) - EI \int_0^t g(t-s)\, w_{xxxx}(s)\,ds = h(x,t), \ \ (x,t) \in (0,L)\times\mathbb{R}_+, \tag{8}$$

with the boundary conditions and the initial data

$$\begin{cases} w_x(0,t) = 0, \quad t \geq 0, \\ EI w_{xxx}(L,t) - EI \displaystyle\int_0^t g(t-s)\, w_{xxx}(L,s)\,ds = m_E w_{tt}(L,t), \quad t \geq 0, \\ EI w_{xx}(L,t) - EI \displaystyle\int_0^t g(t-s)\, w_{xx}(L,s)\,ds = -J w_{xtt}(L,t), \quad t \geq 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in (0,L). \end{cases} \tag{9}$$

By replacing Eq. (8) in the Eq. (7) and using the boundary conditions (9), the ordinary differential equation (7) becomes

$$m w_{tt}(0,t) + EI w_{xxx}(0,t) - EI \int_0^t g(t-s)\, w_{xxx}(0,s)\,ds + \int_0^L h(x,t)dx = f(t), \quad t \geq 0, \tag{10}$$

To get the uniform stability of system (8) – (10) we consider the control force

$$f(t) = -K w_t(0,t) - w(0,t), \quad t \geq 0, \tag{11}$$

where K is a positive constant to be determined later.

The well-posedness result of our problem is stated without proof, in fact, we can use the standard Faedo-Galerkin method (see for example: Hrusa [22] and Park et al [35, 38]). For this end, we denote

$$\mathbb{V} = \left\{ y \in H^2(0,L), y_x(0) = 0 \right\}.$$

where $H^2(0,L)$ is the usual Sobolev space.

Proposition 2.1. *Suppose that* $(w_0, w_1) \in \mathbb{V} \times L^2(0,L)$ *and* $h(t)$ *be a nonnegative summable kernel. Then, there exists a unique solution* w *of the system* $(8) - (10)$ *under the external force* $f(t)$ *defined in* (11)*, satisfying*

$$w \in C\big([0,T];\mathbb{V}\big),\ w_t \in C\big([0,T];L^2(0,L)\big),\ w_{tt} \in C\big([0,T];\mathbb{V}_*\big)$$

where $T > 0$ *and* $\mathbb{V}_*$ *is the dual of* $\mathbb{V}$*.*

Throughout this paper, for all $t \geq 0$, we denote by $*, \circ, \circledcirc, \Box$ and $\diamond$ the binary operators, respectively, defined by

$$\begin{aligned}
(g * v)(t) &= \int_0^t g(t-s)\left(v(x,t) - v(x,s)\right) ds,\\
(g \circ v)(t) &= \int_0^t g(t-s)\left(v(0,t) - v(0,s)\right) ds\\
(g \circledcirc v)(t) &= \int_0^t g(t-s)\left(v(L,t) - v(L,s)\right) ds,\\
(g \Box v)(t) &= \int_0^t g(t-s)\left(v(x,t) - v(x,s)\right)^2 ds
\end{aligned}$$

and

$$(g \diamond v)(t) = \int_0^t g(t-s)\left(v(0,t) - v(0,s)\right)^2 ds,$$

Further, we will use the Young inequality

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \quad \forall a, b \in \mathbb{R}, \epsilon > 0.$$

We define the modified energy functional of problem $(8) - (10)$ as

$$\begin{aligned}
E(t) =& \frac{m}{2} w_t^2(0,t) + \frac{\rho}{2}\|w_t\|_2^2 + \frac{EI}{2}\left(1 - \int_0^t g(s)\,ds\right)\|w_{xx}\|_2^2 + \frac{EI}{2}\int_0^L (g \Box w_{xx})\,dx\\
&+ \frac{m_E}{2} w_t^2(L,t) + \frac{J}{2} w_{xt}^2(L,t)
\end{aligned} \tag{12}$$

where $\|\cdot\|_2$ denotes the norm in $L^2(0,L)$. If we differentiate $E(t)$ with respect to time t, then using $(8) - (10)$ together with the identity

$$\begin{aligned}
2\int_0^L v_t \int_0^t g(t-s)\,v(s)\,ds\,dx =& \int_0^L (g' \Box v)\,dx - g(t)\|v\|_2^2\\
&- \frac{d}{dt}\left[\int_0^L (g \Box v)\,dx - \left(\int_0^t g(s)\,ds\right)\|v\|_2^2\right],
\end{aligned}$$

we obtain

$$\begin{aligned}
E'(t) =& \frac{EI}{2}\int_0^L (g' \Box w_{xx})\,dx - \frac{EI}{2} g(t)\|w_{xx}\|_2^2 + w_t(0,t)\,f(t)\\
&- w_t(0,t)\int_0^L h(x,t)\,dx + \int_0^L w_t h(x,t)\,dx.
\end{aligned}$$

Consequently, using (11) together with Young's inequality, we find

$$\begin{aligned} E'(t) \leq & \frac{EI}{2}\int_0^L (g'\Box w_{xx})\,dx - \left(K - \frac{1}{4\epsilon} - \frac{L}{4\epsilon}\right) w_t^2(0,t) + \epsilon w^2(0,t) \\ & + \epsilon_0 \|w_t\|_2^2 + \left(\epsilon + \frac{1}{4\epsilon_0}\right)\|h\|_2^2, \end{aligned} \tag{13}$$

for any $\epsilon, \epsilon_0 > 0$. Now, Consider

$$1 - \int_0^{+\infty} g(s)ds = 1 - \zeta > 0,$$

and recall

Lemma 2.2 (see [21]). *We infer from the boundary conditions* (4) *and* (9), *that*

$$u^2(x,t) \leq L\|u_x\|_2^2, \quad u^2(x,t) \leq L^3\|u_{xx}\|_2^2, \quad u_x^2(x,t) \leq L\|u_{xx}\|_2^2, \quad \forall x \in [0,L] \tag{14}$$

$$\|u\|^2 \leq L^2\|u_x\|_2^2 \leq L^4\|u_{xx}\|_2^2 \tag{15}$$

and

$$w_x^2(x,t) \leq L\|w_{xx}\|_2^2, \quad \|w_x\|_2^2 \leq L^2\|w_{xx}\|_2^2, \quad \forall x \in [0,L]. \tag{16}$$

Lemma 2.3 (see [40]). *Let* $g \in C(0,\infty)$ *and* $v \in C\left(0,\infty; L^2(0,L)\right)$, *then*

$$\begin{aligned} 2\int_0^L v\int_0^t g(t-s)v(s)\,dsdx = & \left(\int_0^t g(s)\,ds\right)\|v\|_2^2 + \int_0^t g(t-s)\|v(s)\|_2^2\,ds \\ & - \int_0^L (g\Box v)\,dx, \quad t \geq 0. \end{aligned} \tag{17}$$

Lemma 2.4 (see [1]). *Let* $\chi(t), \sigma(t), \beta(t) \in C^1[0,\infty)$. *We assume that, there exists a positive function* $\mu(t) \in C^1[0,\infty)$ *such that*

$$0 \leq \sigma(t) \leq \frac{\mu(t)}{2}\left(\chi(t) - \frac{\mu'(t)}{\mu(t)}\right) \quad \text{and} \quad \beta(t) \leq \frac{1}{2\mu(t)}\left(\chi(t) - \frac{\mu'(t)}{\mu(t)}\right), \quad t \geq 0.$$

Then a non-negative solution v *of the following inequality*

$$v'(t) \leq -\chi(t)v(t) + \sigma(t)v^2(t) + \beta(t), \quad t \geq 0,$$

such that $\mu(0)v(0) < 1$, *satisfies the estimate*

$$v(t) < \frac{1}{\mu(t)}, \quad t \geq 0.$$

Now, let us consider the following Lyapunov functional

$$F(t) = E(t) + \sum_{i=1}^{5}\eta_i\Phi_i(t), \quad t \geq 0, \tag{18}$$

where $\eta_3 = \eta_4 = \frac{1}{2}$, and $\eta_i > 0, i = 1,2$ are positive constants to be chosen later. We define also

$$\Phi_1(t) = \rho\int_0^L ww_t dx + mw(0,t)w_t(0,t) + m_E w(L,t)w_t(L,t) + Jw_x(L,t)w_{xt}(L,t),$$

$$\begin{aligned}\Phi_2(t) = & -\rho\int_0^L w_t\,(g*w)\,dx - m_E w_t(L,t)\,(g\circledcirc w) - m w_t(0,t)\,(g\circ w)\\ & - J w_{xt}(L,t)\,(g\circledcirc w_x)\,,\end{aligned}$$

$$\Phi_3(t) = \frac{1}{2}(g\diamond w) + \frac{1}{2}w^2(0,t)\,,\quad \Phi_4(t) = \int_0^t G_\beta(t-s)\,(w(0,t)-w(0,s))^2\,ds$$

and

$$\Phi_5(t) = \int_0^t G_\gamma(t-s)\,\|w_{xx}(s)\|_2^2\,ds$$

where

$$G_\beta(t) = e^{-\beta t}\int_t^\infty g(s)\,e^{\beta s}ds \quad \text{and} \quad G_\gamma(t) = \frac{1}{\gamma(t)}\int_t^\infty g(s)\,\gamma(s)\,ds,\quad t\geq 0.$$

Proposition 2.5. *There exist two positive constants* α_1 *and* α_2*, such that*

$$\alpha_1\left(E(t)+\sum_{i=3}^5\Phi_i(t)\right) \leq F(t) \leq \alpha_2\left(E(t)+\sum_{i=3}^5\Phi_i(t)\right),\quad \forall t\geq 0. \tag{19}$$

Proof. By Young's inequality, we have

$$\begin{aligned}\Phi_1(t) \leq & \frac{\rho}{2}\|w_t\|_2^2 + \frac{\rho}{2}\|w\|_2^2 + \frac{m}{2}w^2(0,t) + \frac{m}{2}w_t^2(0,t) + \frac{m_E}{2}w^2(L,t) + \frac{m_E}{2}w_t^2(L,t)\\ & + \frac{J}{2}w_x^2(L,t) + \frac{J}{2}w_{xt}^2(L,t)\,.\end{aligned}$$

Noting that $w(x,t) = S^1(t) + u(x,t) = w(0,t) + u(x,t)$ and $w_{xx}(x,t) = u_{xx}(x,t)\,.$ Therefore, by exploiting (14) and (15), we arrive at

$$w^2(L,t) \leq 2w^2(0,t) + 2u^2(L,t) \leq 2w^2(0,t) + 2L^3\|w_{xx}\|_2^2 \tag{20}$$

and

$$\|w\|_2^2 \leq 2Lw^2(0,t) + 2\|u\|_2^2 \leq 2Lw^2(0,t) + 2L^4\|w_{xx}\|_2^2\,. \tag{21}$$

Using $(16)\,,(20)$ and $(21)\,,$ we obtain

$$\begin{aligned}\Phi_1(t) \quad\leq\quad & \frac{\rho}{2}\|w_t\|_2^2 + \left(\rho L^4 + m_E L^3 + \frac{JL}{2}\right)\|w_{xx}\|_2^2 + \frac{m}{2}w_t^2(0,t)\\ & + \left(\rho L + \frac{m}{2} + m_E\right)w^2(0,t) + \frac{m_E}{2}w_t^2(L,t) + \frac{J}{2}w_{xt}^2(L,t)\,.\end{aligned} \tag{22}$$

For the functional $\Phi_2(t)\,,$ we have

$$\begin{aligned}\Phi_2(t) \quad\leq\quad & \frac{\rho}{2}\|w_t\|_2^2 + \frac{\rho}{2}\int_0^t g(s)\,ds\int_0^L (g\square w)\,dx + \frac{m}{2}w_t^2(0,t) + \frac{m}{2}\int_0^t g(s)\,ds\,(g\diamond w)\\ & + \frac{m_E}{2}w_t^2(L,t) + \frac{m_E}{2}\int_0^t g(s)\,ds\int_0^t g(t-s)\,(w(L,t)-w(L,s))^2\,ds\\ & + \frac{J}{2}w_{xt}^2(L,t) + \frac{J}{2}\int_0^t g(s)\,ds\int_0^t g(t-s)\,(w_x(L,t)-w_x(L,s))^2\,ds\end{aligned} \tag{23}$$

By observing that

$$\begin{aligned}(w(x,t)-w(x,s))^2 \quad &=\quad [(w(0,t)-w(0,s)) + (u(x,t)-u(x,s))]^2\\ &\leq\quad 2(w(0,t)-w(0,s))^2 + 2(u(x,t)-u(x,s))^2\end{aligned} \tag{24}$$

and

$$\begin{aligned}(w(L,t)-w(L,s))^2 &= [(w(0,t)-w(0,s))+(u(L,t)-u(L,s))]^2\\ &\leq 2(w(0,t)-w(0,s))^2+2(u(L,t)-u(L,s))^2, \quad (25)\end{aligned}$$

then, using the fact that $u_{xx}(x,t)=w_{xx}(x,t)$, (24), (25) and lemma (2.2), the inequality (23) becomes

$$\begin{aligned}\Phi_2(t) \leq& \frac{\rho}{2}\|w_t\|_2^2+\frac{m}{2}w_t^2(0,t)+\frac{m_E}{2}w_t^2(L,t)+\frac{J}{2}w_{xt}^2(L,t)+\left(\rho L+\frac{m}{2}+m_E\right)\zeta(g\diamond w)\\ &+\left(\rho L^4+m_EL^3+\frac{JL}{2}\right)\zeta\int_0^L(g\square w_{xx})\,dx \qquad (26)\end{aligned}$$

Combining (12), (18), (22) and (26), we show that

$$\begin{aligned}F(t) \leq& (1+\eta_1+\eta_2)\frac{m}{2}w_t^2(0,t)+(1+\eta_1+\eta_2)\frac{m_E}{2}w_t^2(L,t)+(1+\eta_1+\eta_2)\frac{J}{2}w_{xt}^2(L,t)\\ &+(1+\eta_1+\eta_2)\frac{\rho}{2}\|w_t\|_2^2+\left[\frac{1}{4}+\eta_1\left(\rho L+\frac{m}{2}+m_E\right)\right]w^2(0,t)\\ &+\left[\frac{EI}{2}\left(1-\int_0^t g(s)\,ds\right)+\eta_1\left(\rho L^4+m_EL^3+\frac{JL}{2}\right)\right]\|w_{xx}\|_2^2+\eta_4\Phi_4(t)\\ &+\left[\frac{EI}{2}+\eta_2\left(\rho L^4+m_EL^3+\frac{JL}{2}\right)\zeta\right]\int_0^L(g\square w_{xx})\,dx\\ &+\left[\frac{1}{4}+\eta_2\left(\rho L+\frac{m}{2}+m_E\right)\zeta\right](g\diamond w)+\eta_5\Phi_5(t).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}2F(t) \geq& (1-\eta_1-\eta_2)mw_t^2(0,t)+(1-\eta_1-\eta_2)Jw_{xt}^2(L,t)+(1-\eta_1-\eta_2)m_Ew_t^2(L,t)\\ &+(1-\eta_1-\eta_2)\rho\|w_t\|_2^2+\left[\frac{1}{2}-\eta_1(2\rho L+m+2m_E)\right]w^2(0,t)\\ &+\left[EI(1-\zeta)-\eta_1\left(2\rho L^4+2m_EL^3+JL\right)\right]\|w_{xx}\|_2^2+2\eta_4\Phi_4(t)\\ &+\left[EI-\eta_2\left(2\rho L^4+2m_EL^3+JL\right)\zeta\right]\int_0^L(g\square w_{xx})\,dx\\ &+\left[\frac{1}{2}-\eta_2(2\rho L+m+2m_E)\zeta\right](g\diamond w)+2\eta_5\Phi_5(t).\end{aligned}$$

Therefore,

$$\alpha_1\left(E(t)+\sum_{i=3}^{5}\Phi_i(t)\right)\leq F(t)\leq\alpha_2\left(E(t)+\sum_{i=3}^{5}\Phi_i(t)\right), \quad \forall t\geq 0$$

for some α_1, $\alpha_2>0$, provided that

$$\eta_1<\min\left\{1,\frac{1}{(4\rho L+2m+4m_E)},\frac{EI(1-\zeta)}{2\rho L^4+2m_EL^3+JL}\right\}$$

and

$$\eta_2<\min\left\{1-\eta_1,\frac{EI}{(2\rho L^4+2m_EL^3+JL)\zeta},\frac{1}{(4\rho L+2m+4m_E)\zeta}\right\}.$$

This completes the proof. □

3. Asymptotic behavior

In this section, we state and prove the energy decay result under the control force $f(t)$. To begin with, let $t_* > 0$ be a number such that $\int_0^{t_*} g(s)\, ds = g_* > 0$. As in [43], for every measurable set $\mathcal{A} \subset \mathbb{R}^+$, we define the probability measure $\widehat{g}$ by

$$\widehat{g}(\mathcal{A}) = \frac{1}{\zeta} \int_{\mathcal{A}} g(s)\, ds. \tag{27}$$

The non-decreasingness set of g is defined by

$$Q_g = \left\{ s \in \mathbb{R}^+ : g'(s) \geq 0 \right\},$$

and the non-decreasingness rate of g by

$$\mathcal{R}_g = \widehat{g}(Q_g).$$

We also define

$$\widetilde{Q}_{gt} = \left\{ s \in \mathbb{R}^+ : 0 \leq s \leq t, g(t-s) > 0 \text{ and } g'(t-s) = 0 \right\}.$$

Now, our assumptions on the kernel g are

(**H1**) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable non-negative function satisfying

$$\int_0^\infty g(s)\, ds = \zeta < 1.$$

(**H2**) $g'(t) \leq 0, \quad \text{a.e. } t \geq 0.$

(**H3**) There exists a nondecreasing function $\gamma(t) > 0$ such that $\gamma'(t)/\gamma(t) = \delta(t)$ is a decreasing function and

$$\gamma(t)\, g(t) \in L^1(0, \infty).$$

The unknown distributed disturbance is assumed to fulfill the condition

(**H4**) $h \in C\left(\mathbb{R}^+; L^2(0, L)\right).$

Now, we are ready to state and prove the uniform stability under the control force.

Theorem 3.1. *Assume that the hypotheses* $(H1) - (H4)$ *hold and* $\mathcal{R}_g < 3/16$. *If* $G_\gamma(0) < [4g_*(4-\zeta) - 3\zeta]/8$ *such that* $g_* > 3\zeta/(8-2\zeta)$ *and there exists a positive function* $\mu(t) \in C^1[0, \infty)$ *where*

$$0 \leq \chi(t) - \frac{\mu'(t)}{\mu(t)} \quad \text{and} \quad \|h\|_2^2 \leq \frac{1}{2c\mu(t)} \left(\chi(t) - \frac{\mu'(t)}{\mu(t)} \right), \quad t \geq 0$$

where c *is given later in* (35). *Suppose further that* $\mu(0) F(0) < 1$. *We deduce that, there exist some positive constant* C *such that*

$$E(t) \leq \frac{C}{\mu(t)}, \quad \forall t \geq 0 \tag{28}$$

in the cases:

(a) $\lim_{t \to \infty} \delta(t) = 0$ *and* $\chi(t) = c_1 \delta(t)$ *(*c_1 *will be chosen in* (36)*), or*

(b) $\lim_{t \to \infty} \delta(t) = \overline{\delta} \neq 0$ *and* $\chi(t) = c_2$ *(*c_2 *as in* (37)*).*

Proof. Differentiating the expression (18), using Lemma 5.1 to Lemma 5.4 (see Appendix for more detailed derivations), making use of (13), we find that for any $t \geq t_* > 0$

$$\begin{aligned}
F'(t) \leq & \left(\frac{EI}{2} - \eta_2 C_1(\epsilon_2)\right) \int_0^L (g' \Box w_{xx})\, dx + \left(\frac{\eta_3}{2} - \eta_2 C_2(\epsilon_2)\right)(g' \Diamond w) + \left[\frac{1+L}{4\epsilon} + \frac{\eta_4}{2\epsilon_5}\right. \\
& \left. + \eta_1 \left(m + \frac{1}{4\epsilon_1}\right) + \eta_3 \left(\frac{1}{4\epsilon_3} + \frac{1}{4\epsilon_4}\right) - K - \eta_2 (g_* m - \epsilon_2 m - \epsilon_2)\right] w_t^2(0,t) \\
& + [\eta_1 - \eta_2 (g_* - \epsilon_2)]\, m_E w_t^2(L,t) + \left[\eta_2 \epsilon_2 + \epsilon + \eta_3 \epsilon_4 - \eta_1 \left(1 - \epsilon_1 K^2 - 3\epsilon_1 L\right)\right] w^2(0,t) \\
& + [\eta_1 - \eta_2 (g_* - \epsilon_2)]\, J w_{xt}^2(L,t) + \left[\eta_1 + \frac{\epsilon_0}{\rho} - \eta_2 (g_* - \epsilon_2)\right] \rho \left\|w_t\right\|_2^2 \\
& + \left[\eta_2 (1 - g_*) \left(\epsilon_2 + \frac{3}{2} \zeta \widehat{g}(Q)\right) + \eta_5 \frac{G_\gamma(0)}{EI} - \eta_1 \left(1 - \frac{\zeta}{2} - 2\epsilon_1 \frac{L^4}{EI}\right)\right] EI \left\|w_{xx}\right\|_2^2 \\
& + \eta_2 C_3(\epsilon_2) \int_0^L \int_{\mathcal{A}_t} g(t-s) \left(w_{xx}(t) - w_{xx}(s)\right)^2 ds dx + \left(\eta_2 C_4(\epsilon_2) + 2\eta_3 \epsilon_3 \zeta\right) \\
& \times \int_{\mathcal{A}_t} g(t-s) \left(w(0,t) - w(0,s)\right)^2 ds + \left[\left(2\eta_3 \epsilon_3 + \eta_2 C_5(\epsilon_2)\right) \zeta \widehat{g}(Q) - \eta_4\right] (g \Diamond w) \\
& + \left(2\eta_2 \zeta \widehat{g}(Q) - \frac{\eta_1}{2}\right) EI \int_0^L (g \Box w_{xx})\, dx + \left(\frac{\eta_1}{2\epsilon_1} + \eta_2 \left(\epsilon_2 + \frac{L^4}{EI}\right) + \epsilon + \frac{1}{4\epsilon_0}\right) \left\|h\right\|_2^2 \\
& - \eta_5 \delta(t) \Phi_5(t) + \eta_4 \left(2\epsilon_5 \overline{G}_\beta - \beta\right) \Phi_4(t) \\
& + \left[\eta_2 \frac{EI}{2}(1 - g_*) + \eta_1 \frac{EI}{2} - \eta_5\right] \int_0^t g(t-s) \left\|w_{xx}(s)\right\|_2^2 ds,
\end{aligned} \tag{29}$$

As in Tatar [43], we introduce the following sets

$$\mathcal{A}_n = \left\{s \in \mathbb{R}^+, n g'(s) + g(s) \leq 0\right\}, \quad n \in \mathbb{N}$$

and

$$\widetilde{\mathcal{A}}_{nt} = \left\{s \in \mathbb{R}^+, 0 \leq s \leq t, n g'(t-s) + g(t-s) \leq 0\right\}.$$

We notice that

$$\bigcup_n \mathcal{A}_n = \mathbb{R}^+ \backslash \left\{Q_g \cup \mathcal{N}_g\right\},$$

where $\mathcal{N}_g$ is the null set where g' is not defined. Furthermore, if denoting $Q_n = \mathbb{R}^+ \backslash \mathcal{A}_n$, then $\lim_{n\to\infty} \widehat{g}(Q_n) = \widehat{g}(Q_g)$ because $Q_{n+1} \subset Q_n$ for all n and

$$\bigcap_n Q_n = Q_g \cup \mathcal{N}_g.$$

We take $\mathcal{A}_t = \widetilde{\mathcal{A}}_{nt}$ and $Q_t = \widetilde{Q}_{nt}$ in (29) , then, we have

$$\int_0^L (g' \Box w_{xx})\, dx \leq -\frac{1}{n} \int_0^L \int_{\widetilde{\mathcal{A}}_{nt}} g(t-s) \left(w_{xx}(t) - w_{xx}(s)\right)^2 ds dx \tag{30}$$

and

$$(g' \Diamond w) \leq -\frac{1}{n} \int_{\widetilde{\mathcal{A}}_{nt}} g(t-s) \left(w(0,t) - w(0,s)\right)^2 ds. \tag{31}$$

Now, we start selecting the deferent parameters. First, choosing $\epsilon_2 < g_*$, $\eta_1 = \frac{3(g_* - \epsilon_2)}{4} \eta_2$, $\eta_5 = \frac{3EI}{4} \eta_2, \epsilon = \frac{1}{8}(g_* - \epsilon_2)\eta_2$, $\epsilon_0 = \frac{1}{8} \rho (g_* - \epsilon_2) \eta_2$, $\epsilon_1 < \min\left\{\frac{1}{2\left(K^2 + 3L\right)}, \frac{\zeta EI}{16 L^4}\right\}$, $\epsilon_3 = \frac{1}{16\zeta n}$, $\epsilon_4 = \frac{1}{4}\eta_2 (g_* - \epsilon_2)$, $\epsilon_5 = \frac{\beta}{4\overline{G}_\beta}$ and taking into account the inequalities (30) and (31) , we infer from (29) that

$$
\begin{aligned}
F'(t) \leq{} & \left(\frac{EI}{4} - \eta_2 C_1(\epsilon_2)\right) \int_0^L (g' \Box w_{xx})\, dx + \left(\frac{1}{8} - \eta_2 C_2(\epsilon_2)\right)(g' \Diamond w) - \frac{(g_* - \epsilon_2)}{4} \eta_2 m_E w_t^2(L,t) \\
& + \left[\epsilon_2 - \frac{1}{8}(g_* - \epsilon_2)\right] \eta_2 w^2(0,t) - \frac{(g_* - \epsilon_2)}{4} \eta_2 J w_{xt}^2(L,t) - \frac{(g_* - \epsilon_2)}{8} \eta_2 \rho \|w_t\|_2^2 \\
& + \left[\frac{3}{4} G_\gamma(0) + (1 - g_*)\left(\epsilon_2 + \frac{3}{2}\zeta \widehat{g}(Q)\right) - \frac{3}{4}[(1-\sigma)+\sigma](g_* - \epsilon_2)\left(1 - \frac{3\zeta}{4}\right)\right] \eta_2 EI \|w_{xx}\|_2^2 \\
& + \left[\frac{1+L}{4\epsilon} + \eta_1\left(m + \frac{1}{4\epsilon_1}\right) + \eta_3\left(\frac{1}{4\epsilon_3} + \frac{1}{4\epsilon_4}\right) + \frac{\eta_4}{2\epsilon_5} - K - \eta_2(g_* m - \epsilon_2 m - \epsilon_2)\right] w_t^2(0,t) \qquad (32) \\
& + \left[\left(\frac{1}{16\zeta n} + \eta_2 C_5(\epsilon_2)\right)\zeta \widehat{g}(Q_n) - \frac{1}{2}\right](g \Diamond w) + \left[2\zeta \widehat{g}(Q) - \frac{3(g_* - \epsilon_2)}{8}\right] \eta_2 EI \int_0^L (g \Box w_{xx})\, dx \\
& - \frac{(g_* + 2)}{8} \eta_2 EI \int_0^t g(t-s) \|w_{xx}(s)\|_2^2\, ds + \left(\eta_2 C_4(\epsilon_2) - \frac{1}{16n}\right) \int_{\widetilde{\mathcal{A}}_{nt}} g(t-s)(w(0,t) - w(0,s))^2\, ds \\
& + \left(\eta_2 C_3(\epsilon_2) - \frac{EI}{4n}\right) \int_0^L \int_{\widetilde{\mathcal{A}}_{nt}} g(t-s)(w_{xx}(t) - w_{xx}(s))^2\, ds dx - \frac{\beta}{4}\Phi_4(t) - \eta_5 \delta(t) \Phi_5(t) \\
& + \left(\frac{1}{2\epsilon_1} \frac{3(g_* - \epsilon_2)}{4} \eta_2 + \eta_2\left(\epsilon_2 + \frac{L^4}{EI}\right) + \frac{1}{8}(g_* - \epsilon_2)\eta_2 + \frac{1}{4\epsilon_0}\right) \|h\|_2^2,
\end{aligned}
$$

for some $0 < \sigma < 1$. For small ϵ_2 and large values of n and t_*, if $\widehat{g}(Q_n) < 3/16$, we have

$$
2\zeta \widehat{g}(Q_n) - \frac{3(g_* - \epsilon_2)}{8} < 0
$$

and

$$
(1 - g_*) \frac{3}{2} \zeta \widehat{g}(Q_n) < \sigma \frac{3}{4}(g_* - \epsilon_2)\left(1 - \frac{5\zeta}{8}\right), \qquad (33)
$$

with $\sigma = \frac{3\zeta(1-g_*)}{g_*(8-5\zeta)}$. Note that for t_* large enough, we have $0 < \sigma < 1$. On other hand, we require that $G_\gamma(0)$ satisfy

$$
\frac{3}{4} G_\gamma(0) < \frac{3}{4}(1-\sigma)(g_* - \epsilon_2)\left(1 - \frac{5\zeta}{8}\right), \qquad (34)
$$

then (34) is satisfied if

$$
G_\gamma(0) < \frac{1}{8}[4g_*(4-\zeta) - 3\zeta]
$$

and $g_* > \frac{3\zeta}{8-2\zeta}$. By taking into account (33), (34) and selecting ϵ_2 small enough if needed yield

$$
\begin{cases}
(1 - g_*)\left(\epsilon_2 + \frac{3}{2}\zeta \widehat{g}(Q_n)\right) + \frac{3G_\gamma(0)}{4} - \frac{3}{4}[(1-\sigma)+\sigma](g_* - \epsilon_2)\left(1 - \frac{5\zeta}{8}\right) < 0, \\
\epsilon_2 - \frac{1}{8}(g_* - \epsilon_2) < 0.
\end{cases}
$$

Once ϵ_2, n and t_* are fixed, we take η_2 small enough so that

$$
\begin{cases}
\frac{EI}{4} - \eta_2 C_1(\epsilon_2) > 0, \\
\frac{1}{8} - \eta_2 C_2(\epsilon_2) > 0, \\
\eta_2 C_3(\epsilon_2) - \frac{EI}{4n} < 0, \\
\eta_2 C_4(\epsilon_2) - \frac{1}{16n} < 0, \\
\frac{\widehat{g}(Q_n)}{16n} + \eta_2 C_5(\epsilon_2) \zeta \widehat{g}(Q_n) - \frac{1}{2} < 0.
\end{cases}
$$

Finally, we choose K large enough, such that

$$
\frac{1+L}{4\epsilon} + \eta_1\left(m + \frac{1}{4\epsilon_1}\right) + \eta_3\left(\frac{1}{4\epsilon_3} + \frac{1}{4\epsilon_4}\right) + \frac{\eta_4}{2\epsilon_5} - K - \eta_2[(g_* - \epsilon_2)m - \epsilon_2] < 0.
$$

Consequently, we deduce from (32) that

$$
F'(t) \leq -c_0(\Phi_3(t) + \Phi_4(t)) - \eta_5 \delta(t) \Phi_5(t) + c\|h\|_2^2, \qquad (35)
$$

for some positive constants c_0 and c. Now,

(a) If $\lim\limits_{t \to \infty} \delta(t) = 0$, then, there exists a $\widehat{t}(c_0) \geq t_*$ such that $\delta(t) \leq c_0$ for $t \geq \widehat{t}(c_0)$. Hence, by exploiting the right-hand side of inequality (19), we infer from (35) that

$$
F'(t) \leq -c_1 \delta(t) F(t) + c\|h\|_2^2, \quad t \geq \widehat{t}, \qquad (36)
$$

for some positive constants c_1. Now, applying Lemma (2.4) with

$$\chi(t) = c_1\delta(t), \quad \sigma(t) = 0, \quad \beta(t) = c\|h\|_2^2,$$

we conclude from (36) and the left-hand side of (19) that

$$E(t) \leq \frac{C}{\mu(t)}, \quad t \geq \widehat{t},$$

for some positive constant C, provided that $\mu(0)F(0) < 1$.

(b) If $\lim\limits_{t\to\infty} \delta(t) = \overline{\delta} \neq 0$, then, there exists a $\widehat{t}(c_0) \geq t_*$ such that $\delta(t) \geq c_0$ for $t \geq \widehat{t}$. Therefore, (35) we get

$$F'(t) \leq -c_2F(t) + c\|h\|_2^2, \quad t \geq \widehat{t}, \tag{37}$$

for some $c_2 > 0$. In view of Lemma (2.4) for

$$\chi(t) = c_2, \quad \sigma(t) = 0, \quad \beta(t) = c\|h\|_2^2,$$

we infer from (37) that

$$E(t) \leq \frac{C}{\mu(t)}, \quad t \geq \widehat{t}$$

for a constant $C > 0$, provided that $\mu(0)F(0) < 1$. Furthermore, using the continuity and boundedness of $E(t)$ in $[0, \widehat{t}]$, we get (28). □

4. Simulation analysis

This section is devoted to the numerical simulations, to clarify the efficiency ,of the given control. The parameters associated to our system are: $E = 69 \times 10^9 Nm^{-2}$, $I = 8.93 \times 10^{-13}m^4$, $L = 0.4m$, $\rho = 0.065Kgm^{-1}$, $m = 0.50Kg$, $m_E = 0.01Kg$, $J = 0.0001Kgm^2$, $K = 110$ and the initial displacement and velocity are:

$$w_0(x) = \sin\left(\frac{\pi}{L}x\right) \quad \text{and} \quad w_1(x) = 0.$$

We indicate that we selected the kernel function as $g(t) = e^{-\tau t}$ for some $\tau \in \mathbb{R}^+$. The external disturbance $h(x,t)$ is given as:

$$h(x,t) = 0.01\sin(0.001\pi xt).$$

We have exploited the method of finite difference for numerical approximation of the problem (7)–(9). We have adopted, for both spatial and temporal derivatives in the differential equation (8) and the boundary condition (9), Central difference approximations.

In what concerns the convolution term, the Trapezoid method has been used. We observe that, the convolution term requires to store the solutions from $t = 0$ and use them at every time step. This, together with large values of the model parameters, require very small values of time steps, which increases calculated time. Hence, it is possible to scale the parameters for computer simulations to overcome time performance.

The solution behavior associated to the problem (7)–(9), with the control (11), is illustrated in Figures 1 at $x = 0.1$ and 2 at $x = 0.2$.

Figure 2 displays the discrepancy of solutions in the presence or absence of the control affect at the points $x = 0.2$ and the boundary $x = L$. The affect of the control (11) may be observed, after a short time thanks to the convolution term, in especially, when getting closer to the boundary point $x = L$.

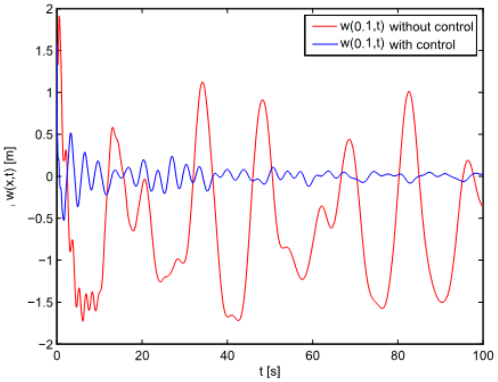

FIGURE 1. Beam displacement at $x = 0.1$.

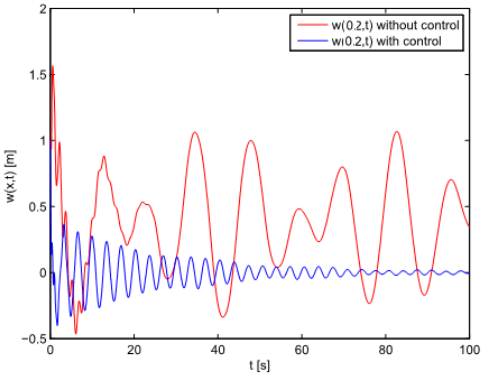

FIGURE 2. Beam displacement at $x = 0.2$.

5. Appendix

In this section, we state some useful which were helpful in the proof of the Theorem 3.1.

Lemma 5.1. *Under the assumptions* $(H1)-(H4)$*, along the solution of* (8) − (10)*, the derivative of* $\Phi_1(t)$ *satisfies the estimate*

$$\begin{aligned}\Phi_1'(t) \leq & \rho \left\| w_t \right\|_2^2 + \left(m + \frac{1}{4\epsilon_1} \right) w_t^2(0,t) + m_E w_t^2(L,t) + J w_{xt}^2(L,t) \\ & - \left(1 - \epsilon_1 K^2 - 3\epsilon_1 L\right) w^2(0,t) - \left(1 - \frac{\zeta}{2} - 2\epsilon_1 \frac{L^4}{EI}\right) EI \left\| w_{xx} \right\|_2^2 - \frac{EI}{2} \int_0^L (g \Box w_{xx})\, dx \\ & + \frac{EI}{2} \int_0^t g(t-s) \left\| w_{xx}(s) \right\|_2^2 ds + \frac{1}{2\epsilon_1} \left\| h \right\|_2^2, \end{aligned} \tag{38}$$

for all $t \geq 0$ *and* $\epsilon_1 > 0$.

Proof. Differentiating $\Phi_1(t)$ with respect to t and using (8) − (10), we get

$$\begin{aligned}
\Phi_1'(t) &= \rho\|w_t\|_2^2 + \int_0^L w\left(-EIw_{xxxx}(x,t) + EI\int_0^t g(t-s)\,w_{xxxx}(s)\,ds + h(x,t)\right)dx \\
&+ mw_t^2(0,t) + w(0,t)\left(-EIw_{xxx}(0,t) + EI\int_0^t g(t-s)\,w_{xxx}(0,s)\,ds - \int_0^L h(x,t)\,dx + f(t)\right) \\
&+ m_E w_t^2(L,t) + w(L,t)\left(EIw_{xxx}(L,t) - EI\int_0^t g(t-s)\,w_{xxx}(L,s)\,ds\right) \\
&+ Jw_{xt}^2(L,t) + w_x(L,t)\left(-EIw_{xx}(L,t) + EI\int_0^t g(t-s)\,w_{xx}(L,s)\,ds\right). \qquad (39)
\end{aligned}$$

Integrating by parts, we have

$$-\int_0^L ww_{xxxx}dx = -w(L,t)\,w_{xxx}(L,t) + w(0,t)\,w_{xxx}(0,t) + w_x(L,t)\,w_{xx}(L,t) - \|w_{xx}\|_2^2, \qquad (40)$$

and

$$\begin{aligned}
&\int_0^L w\int_0^t g(t-s)\,w_{xxxx}(s)\,dsdx \\
&= w(L,t)\int_0^t g(t-s)\,w_{xxx}(L,s)\,ds - w(0,t)\int_0^t g(t-s)\,w_{xxx}(0,s)\,ds \\
&\quad - w_x(L,t)\int_0^t g(t-s)\,w_{xx}(L,s)\,ds + \int_0^L w_{xx}(t)\int_0^t g(t-s)\,w_{xx}(s)\,dsdx. \qquad (41)
\end{aligned}$$

Substituting the relations (40) and (41) into (39), and using the fact that (11), we obtain

$$\begin{aligned}
\Phi_1'(t) =& \rho\|w_t\|_2^2 + mw_t^2(0,t) + m_E w_t^2(L,t) + Jw_{xt}^2(L,t) - w^2(0,t) - Kw(0,t)\,w_t(0,t) \\
&- w(0,t)\int_0^L h(x,t)\,dx + \int_0^L w(x,t)h(x,t)\,dx - EI\|w_{xx}\|_2^2 \\
&+ EI\int_0^L w_{xx}(t)\int_0^t g(t-s)\,w_{xx}(s)\,dsdx. \qquad (42)
\end{aligned}$$

By using Young's inequality, we estimated the terms in the right-hand side of (42) as follows

$$-Kw(0,t)\,w_t(0,t) \le \epsilon_1 K^2 w^2(0,t) + \frac{1}{4\epsilon_1}w_t^2(0,t), \qquad (43)$$

and

$$-w(0,t)\int_0^L h(x,t)\,dx \le \epsilon_1 L w^2(0,t) + \frac{1}{4\epsilon_1}\|h\|_2^2, \quad \epsilon_1 > 0. \qquad (44)$$

Moreover, from (21), we have

$$\begin{aligned}
\int_0^L w(x,t)h(x,t)\,dx &\le \epsilon_1\|w\|_2^2 + \frac{1}{4\epsilon_1}\|h\|_2^2 \\
&\le 2\epsilon_1 L w^2(0,t) + 2\epsilon_1 L^4\|w_{xx}\|_2^2 + \frac{1}{4\epsilon_1}\|h\|_2^2, \quad \epsilon_1 > 0. \qquad (45)
\end{aligned}$$

Thank's to the identity (17) and by inserting the above estimates (43)-(45) into (42), we get (38). ☐

Lemma 5.2. *The derivative of the functional $\Phi_2(t)$ satisfies along solutions of the problem* (8) – (10)

$$\begin{aligned}
&\Phi_2'(t) \leq -(g_* - \epsilon_2)\,\rho\,\|\omega_t\|_2^2 - (g_* m - \epsilon_2 m - \epsilon_2 K)\,\omega_t^2(0,t) - (g_* - \epsilon_2)\,m_E\omega_t^2(L,t) \\
&- (g_* - \epsilon_2)\,J\omega_{xt}^2(L,t) + \epsilon_2\omega^2(0,t) + (1-g_*)\left(\epsilon_2 + \frac{3}{2}\zeta\widehat{g}(Q)\right) EI\,\|\omega_{xx}\|_2^2 \\
&+ \frac{EI}{2}(1-g_*)\int_{Q_t} g(t-s)\,\|\omega_{xx}(s)\|_2^2\,ds - C_1(\epsilon_2)\int_0^L (g'\Box\omega_{xx})\,dx - C_2(\epsilon_2)\,(g'\Diamond\omega) + \\
&+ C_3(\epsilon_2)\int_0^L\int_{\mathcal{A}_t} g(t-s)\,(\omega_{xx}(t)-\omega_{xx}(s))^2\,dsdx + C_4(\epsilon_2)\int_{\mathcal{A}_t} g(t-s)\,(\omega(0,t)-\omega(0,s))^2\,ds \\
&+ 2EI\zeta\widehat{g}(Q)\int_0^L\int_{Q_t} g(t-s)\,(\omega_{xx}(t)-\omega_{xx}(s))^2\,dsdx \qquad (46)\\
&+ C_5(\epsilon_2)\,\zeta\widehat{g}(Q)\int_{Q_t} g(t-s)\,(\omega(0,t)-\omega(0,s))^2\,ds + \left(\epsilon_2 + \frac{L^4}{EI}\right)\|h\|_2^2, \quad t \geq t_* > 0,
\end{aligned}$$

for any $\epsilon_2 > 0$, *where*

$$\begin{aligned}
C_1(\epsilon_2) &= \left(\frac{\rho L^4}{2\epsilon_2} + \frac{m_E L^3}{2\epsilon_2} + \frac{JL}{4\epsilon_2}\right) g(0), \quad C_2(\epsilon_2) = \left(\frac{\rho}{2\epsilon_2} + \frac{m}{4\epsilon_2} + \frac{m_E}{2\epsilon_2}\right) g(0), \\
C_3(\epsilon_2) &= \frac{(1-g_*)\,\zeta EI}{4\epsilon_2} + 4\zeta EI, \\
C_4(\epsilon_2) &= \frac{\zeta EI}{L^3} + \frac{\zeta K^2}{2\epsilon_2} + \frac{\zeta + L}{2\epsilon_2}, \ \textit{and } C_5(\epsilon_2) = \frac{K^2}{2\epsilon_2} + \frac{1+L}{2\epsilon_2} + \frac{EI}{L^3}.
\end{aligned}$$

Proof. Differentiating $\Phi_2(t)$ and using (8) – (10), we arrive at

$$\begin{aligned}
&\Phi_2'(t) = -\left(\int_0^t g(s)\,ds\right)\left(\rho\,\|w_t\|_2^2 + m w_t^2(0,t) + m_E w_t^2(L,t) + J w_{xt}^2(L,t)\right) \\
&- EI\int_0^L (g*w)\int_0^t g(t-s)\,w_{xxxx}(s)\,dsdx + EI\int_0^L w_{xxxx}\,(g*w)\,dx \\
&- \int_0^L h(x,t)\,(g*w)\,dx - \rho\int_0^L w_t\,(g'*w)\,dx - EIw_{xxx}(L,t)\,(g\circledcirc w) \\
&+ EI\,(g\circledcirc w)\int_0^t g(t-s)\,w_{xxx}(L,s)\,ds - m_E w_t(L,t)\,(g'\circledcirc w) + EIw_{xxx}(0,t)\,(g\circ w) \\
&- EI\,(g\circ w)\int_0^t g(t-s)\,w_{xxx}(0,s)\,ds + (g\circ w)\int_0^L h(x,t)\,dx - (g\circ w)\,f(t) - m w_t(0,t)\,(g'\circ w) \\
&- Jw_{xt}(L,t)\,(g'\circledcirc w_x) + EIw_{xx}(L,t)\,(g\circledcirc w_x) - EI\,(g\circledcirc w_x)\int_0^t g(t-s)\,w_{xx}(L,s)\,ds.
\end{aligned} \tag{47}$$

Integrating by parts in (47), we have

$$\begin{aligned}
\int_0^L w_{xxxx}\,(g*w)\,dx =& w_{xxx}(L,t)\,(g\circledcirc w) - w_{xxx}(0,t)\,(g\circ w) - w_{xx}(L,t)\,(g\circledcirc w_x) \\
&+ \int_0^L w_{xx}\,(g*w_{xx})\,dx,
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
&-\int_0^L (g*w)\int_0^t g(t-s)\,w_{xxxx}(s)\,dsdx\\
=&-(g\circledcirc w)\int_0^t g(t-s)\,w_{xxx}(L,s)\,ds+(g\circ w)\int_0^t g(t-s)\,w_{xxx}(0,s)\,ds\\
&+(g\circledcirc w_x)\int_0^t g(t-s)\,w_{xx}(L,s)\,ds-\int_0^L (g*w_{xx})\int_0^t g(t-s)\,w_{xx}(s)\,dsdx.
\end{aligned}\tag{49}$$

Using the relations (11), (48) and (49), the identity (47) takes the following form

$$\begin{aligned}
&\Phi_2'(t)=-\left(\int_0^t g(s)\,ds\right)\left(\rho\|w_t\|_2^2+mw_t^2(0,t)+m_Ew_t^2(L,t)+Jw_{xt}^2(L,t)\right)\\
&+EI\left(1-\int_0^t g(s)\,ds\right)\int_0^L w_{xx}(g*w_{xx})\,dx+EI\int_0^L (g*w_{xx})^2\,dx\\
&-\int_0^L h(x,t)(g*w)\,dx-\rho\int_0^L w_t\left(g'*w\right)dx-m_Ew_t(L,t)\left(g'\circledcirc w\right)+Kw_t(0,t)(g\circ w)\\
&+w(0,t)(g\circ w)+(g\circ w)\int_0^L h(x,t)\,dx-mw_t(0,t)\left(g'\circ w\right)-Jw_{xt}(L,t)\left(g'\circledcirc w_x\right).
\end{aligned}\tag{50}$$

Now, we estimate the terms on the right-hand side of expression (50). For all measurable sets $\mathcal{A}$ and Q such that $\mathcal{A}=\mathbb{R}^+\backslash Q$. For the fifth term, we have

$$\begin{aligned}
&\int_0^L w_{xx}(g*w_{xx})\,dx=c\int_0^L w_{xx}\int_{\mathcal{A}\cap[0,t]} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx\\
&\quad+\int_0^L w_{xx}\int_{Q\cap[0,t]} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx\\
&\leq\int_0^L w_{xx}\int_{\mathcal{A}\cap[0,t]} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx+\left(\int_{Q\cap[0,t]} g(t-s)\,ds\right)\|w_{xx}\|_2^2\\
&\quad-\int_0^L w_{xx}\int_{Q\cap[0,t]} g(t-s)\,w_{xx}(s)\,dsdx.
\end{aligned}\tag{51}$$

For simplicity, we use the notation $\mathcal{B}_t=\mathcal{B}\cap[0,t]$. By Young's inequality, we have for $\epsilon_2>0$

$$\begin{aligned}
&\int_0^L w_{xx}\int_{\mathcal{A}_t} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)dsdx\\
&\qquad\leq\epsilon_2\|w_{xx}\|_2^2+\frac{\varsigma}{4\epsilon_2}\int_0^L\int_{\mathcal{A}_t} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2dsdx
\end{aligned}$$

and

$$-\int_0^L w_{xx}\int_{Q_t} g(t-s)\,w_{xx}(s)\,dsdx\leq\frac{1}{2}\left(\int_{Q_t} g(t-s)\,ds\right)\|w_{xx}\|_2^2+\frac{1}{2}\int_{Q_t} g(t-s)\|w_{xx}(s)\|_2^2\,ds$$

Now, (51) becomes

$$\begin{aligned}
\int_0^L w_{xx}(g*w_{xx})\,dx\quad\leq\quad&\frac{\varsigma}{4\epsilon_2}\int_0^L\int_{\mathcal{A}_t} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2dsdx\\
&+\left(\epsilon_2+\frac{3}{2}\varsigma\widehat{g}(Q)\right)\|w_{xx}\|_2^2+\frac{1}{2}\int_{Q_t} g(t-s)\|w_{xx}(s)\|_2^2\,ds
\end{aligned}\tag{52}$$

where $\widehat{g}(Q)$ is defined in (27). Next,

$$\begin{aligned}\int_0^L (g * w_{xx})^2\,dx \leq\, & 3\zeta \int_0^L \int_{\mathcal{A}_t} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx \\ & + \frac{3}{2}\zeta \widehat{g}(Q) \int_0^L \int_{Q_t} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx. \qquad (53)\end{aligned}$$

For 8th, as $u_{xx}(x,t) = w_{xx}(x,t)$, using Young's inequality, (15) and (24), we obtain

$$\begin{aligned}-\int_0^L w_t\left(g' * w\right) dx & \leq \epsilon_2 \left\|w_t\right\|_2^2 + \frac{1}{4\epsilon_2}\left(\int_0^t -g'(s)\,ds\right)\int_0^L \left(-g' \Box w\right) dx \\ & \leq \epsilon_2 \left\|w_t\right\|_2^2 - \frac{g(0)}{2\epsilon_2}\left(g' \Diamond w\right) - \frac{g(0)L^4}{2\epsilon_2}\int_0^L \left(g' \Box w_{xx}\right) dx, \qquad (54)\end{aligned}$$

Using the relations (14), (16) and (25), The last fifth terms on the right-hand side of (50) can be estimated for $\epsilon_2 > 0$ as follows

$$\begin{aligned}-w_t(L,t)\left(g' \circledcirc w\right) & \leq \epsilon_2 w_t^2(L,t) - \frac{1}{4\epsilon_2} g(0) \int_0^t g'(t-s)\left(w(L,t)-w(L,s)\right)^2 ds \\ & \leq \epsilon_2 w_t^2(L,t) - \frac{g(0)}{2\epsilon_2}\left(g' \Diamond w\right) - \frac{g(0)L^3}{2\epsilon_2}\int_0^L \left(g' \Box w_{xx}\right) dx, \qquad (55)\end{aligned}$$

$$\begin{aligned}K w_t(0,t)(g \circ w) & \leq \epsilon_2 w_t^2(0,t) + \frac{\zeta K^2}{2\epsilon_2}\int_{\mathcal{A}_t} g(t-s)\left(w(0,t)-w(0,s)\right)^2 ds \\ & \quad + \frac{K^2}{2\epsilon_2}\zeta \widehat{g}(Q)\int_{Q_t} g(t-s)\left(w(0,t)-w(0,s)\right)^2 ds, \qquad (56)\end{aligned}$$

$$\begin{aligned}w(0,t)(g \circ w) & \leq \epsilon_2 w^2(0,t) + \frac{\zeta}{2\epsilon_2}\int_{\mathcal{A}_t} g(t-s)\left(w(0,t)-w(0,s)\right)^2 ds \\ & \quad + \frac{1}{2\epsilon_2}\zeta \widehat{g}(Q)\int_{Q_t} g(t-s)\left(w(0,t)-w(0,s)\right)^2 ds, \qquad (57)\end{aligned}$$

$$\begin{aligned}(g \circ w)\int_0^L h(x,t)\,dx & \leq \epsilon_2 \left\|h\right\|_2^2 + \frac{\zeta L}{2\epsilon_2}\int_{\mathcal{A}_t} g(t-s)\left(w(0,t)-w(0,s)\right)^2 ds \\ & \quad + \frac{L}{2\epsilon_2}\zeta \widehat{g}(Q)\int_{Q_t} g(t-s)\left(w(0,t)-w(0,s)\right)^2 ds, \qquad (58)\end{aligned}$$

$$-\,w_t(0,t)\left(g' \circ w\right) \leq \epsilon_2 w_t^2(0,t) - \frac{g(0)}{4\epsilon_2}\left(g' \Diamond w\right) \qquad (59)$$

and

$$-\,w_{xt}(L,t)\left(g' \circledcirc w_x\right) \leq \epsilon_2 w_{xt}^2(L,t) - \frac{g(0)L}{4\epsilon_2}\int_0^L \left(g' \Box w_{xx}\right) dx \qquad (60)$$

Finally, by exploiting the relations (24) and for $\epsilon_2 = \frac{EI}{4L^4}$, we obtain the estimate

$$\begin{aligned}-\int_0^L h(x,t)(g * w)\,dx & \leq \frac{EI}{2L^4}\int_0^L \left(\int_{\mathcal{A}_t} g(t-s)\left(w(t)-w(s)\right) ds\right)^2 dx \\ & \quad + \frac{EI}{2L^4}\int_0^L \left(\int_{Q_t} g(t-s)\left(w(t)-w(s)\right) ds\right)^2 dx + \frac{L^4}{EI}\left\|h\right\|_2^2 \\ & \leq \zeta EI \int_0^L \int_{\mathcal{A}_t} g(t-s)\left(w_{xx}(t)-w_{xx}(s)\right)^2 dsdx \\ & \quad + \frac{\zeta EI}{L^3}\int_{\mathcal{A}_t} g(t-s)\left(w(0,t)-w(0,s)\right)^2 ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{EI}{2}\zeta\widehat{g}\left(Q\right)\int_0^L\int_{Q_t}g\left(t-s\right)\left(w_{xx}\left(t\right)-w_{xx}\left(s\right)\right)^2dsdx+\frac{L^4}{EI}\left\|h\right\|_2^2\\
&+\frac{EI}{L^3}\zeta\widehat{g}\left(Q\right)\int_{Q_t}g\left(t-s\right)\left(w\left(0,t\right)-w\left(0,s\right)\right)^2ds. \qquad (61)
\end{aligned}
$$

Therefore, collecting the previous estimates (52) to (61) into (50) , the Lemma 5.2 is established. □

Lemma 5.3. *The functional* $\Phi_3\left(t\right)$ *satisfies, for all* $\epsilon_3,\epsilon_4>0$

$$
\begin{aligned}
\Phi_3'\left(t\right) \leq\ & \frac{1}{2}\left(g'\diamond w\right)+2\epsilon_3\zeta\int_{\mathcal{A}_t}g\left(t-s\right)\left(w\left(0,t\right)-w\left(0,s\right)\right)^2ds+\left(\frac{1}{4\epsilon_3}+\frac{1}{4\epsilon_4}\right)w_t^2\left(0,t\right)\\
&+\epsilon_4w^2\left(0,t\right)+2\epsilon_3\zeta\widehat{g}\left(Q\right)\int_{Q_t}g\left(t-s\right)\left(w\left(0,t\right)-w\left(0,s\right)\right)^2ds, \quad t\geq0. \qquad (62)
\end{aligned}
$$

Proof. By differentiating $\Phi_3\left(t\right)$, we obtain

$$
\Phi_3'\left(t\right)=\frac{1}{2}\left(g'\diamond w\right)+w_t\left(0,t\right)\left(g\circ w\right)+w_t\left(0,t\right)w\left(0,t\right). \qquad (63)
$$

Clearly, it suffices to observe that

$$
\begin{aligned}
w_t\left(0,t\right)\left(g\circ w\right) \leq\ & 2\epsilon_3\zeta\widehat{g}\left(Q\right)\int_{Q_t}g\left(t-s\right)\left(w\left(0,t\right)-w\left(0,s\right)\right)^2ds\\
&+2\epsilon_3\zeta\int_{\mathcal{A}_t}g\left(t-s\right)\left(w\left(0,t\right)-w\left(0,s\right)\right)^2ds+\frac{1}{4\epsilon_3}w_t^2\left(0,t\right) \qquad (64)
\end{aligned}
$$

and

$$
w_t\left(0,t\right)w\left(0,t\right)\leq\epsilon_4w^2\left(0,t\right)+\frac{1}{4\epsilon_4}w_t^2\left(0,t\right), \qquad (65)
$$

to get (62). □

Lemma 5.4. *For any* $\epsilon_5>0$, *the functionals* $\Phi_4\left(t\right)$ *and* $\Phi_5\left(t\right)$ *satisfies,*

$$
\Phi_4'\left(t\right)\leq-\left(\beta-2\epsilon_5\overline{G}_\beta\right)\Phi_4\left(t\right)+\frac{1}{2\epsilon_5}w_t^2\left(0,t\right)-\left(g\diamond w\right), \quad t\geq0 \qquad (66)
$$

and

$$
\Phi_5'\left(t\right)\leq G_\gamma\left(0\right)\left\|w_{xx}\right\|_2^2-\delta\left(t\right)\Phi_5\left(t\right)-\int_0^tg\left(t-s\right)\left\|w_{xx}\left(s\right)\right\|_2^2ds, \quad t\geq0. \qquad (67)
$$

Proof. A differentiation of $\Phi_4\left(t\right)$ gives

$$
\begin{aligned}
\Phi_4'\left(t\right) =\ & -\beta\Phi_4\left(t\right)-\int_0^tg\left(t-s\right)\left(w\left(0,t\right)-w\left(0,s\right)\right)^2ds\\
&+2w_t\left(0,t\right)\int_0^tG_\beta\left(t-s\right)\left(w\left(0,t\right)-w\left(0,s\right)\right)ds. \qquad (68)
\end{aligned}
$$

Noting that

$$
w_t\left(0,t\right)\int_0^tG_\beta\left(t-s\right)\left(w\left(0,t\right)-w\left(0,s\right)\right)ds\leq\frac{1}{4\epsilon_5}w_t^2\left(0,t\right)+\epsilon_5\overline{G}_\beta\Phi_4\left(t\right), \qquad (69)
$$

where

$$
\overline{G}_\beta=\int_0^\infty G_\beta\left(s\right)ds.
$$

Then, making use (68) and (69), leads to (66). For the proof of (67) see [43]. □

References

[1] R.G. Airapetyan, A.G. Ramm, A.B. Smirnova, Continuous methods for solving nonlinear Ill-posed problems, In: *Operator Theory and Applications* **25** (2000), 111-137. DOI: 10.1090/fic/025/05

[2] F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, *Appl Math Opt.* **51** (2005), no. 1, 61–105.

[3] S. Berrimi, S.A. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, *Nonlinear Anal.* **64** (2006), 2314–2331.

[4] A. Berkani, N-e. Tatar, A. Khemmoudj, Control of a viscoelastic translational Euler-Bernoulli beam, *Math Methods Appl Sci.* **40** (2017), 237–254.

[5] A. Berkani, N-e. Tatar, A. Kelleche, Vibration control of a viscoelastic translational Euler-Bernoulli beam, *J Dyn Control Syst.* **24** (2018), 167–199.

[6] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.S. Prates Filho, J.A. Soriano, Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping, *Differential Integr. Equations* **14** (2001), no. 1, 85–116.

[7] M.M. Cavalcanti, H.P. Oquendo, Frictional versus viscoelastic damping in a semilinear wave equation, *SIAM J. Control Optim.* **42** (2003), no. 4, 1310–1324.

[8] M.M. Cavalcanti, V.N.D. Cavalcanti, I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping–source interaction, *Int. J. Differ. Equ.* **236** (2007), no. 2, 407–459.

[9] R.M. Christensen, *Theory of Viscoelasticity: An Introduction*, New York/London, Academic Press, 1982.

[10] H. Diken, Vibration control of a rotating Euler-Bernoulli beam, *J. Sound Vib.* **232** (2000), no. 3, 541–551.

[11] M. Dadfarnia, N. Jalili, B. Xian, D.M. Dawson, Lyapunov-based piezoelectric control of flexible cartesian robot manipulators, *Proceedings of the American Control Conference Denver*, Colorado June 4-6. **9** (2003), 5227–5232.

[12] M. Dadfarnia, N. Jalili, Z. Liu, D.M. Dawson, An observer-based piezoelectric control of flexible cartesian robot arms: theory and experiment, *Control Eng. Pract.* **12** (2004), 1041–1053.

[13] M. Dadfarnia, N. Jalili, B. Xian, D.M. Dawson, Lyapunouv-based vibration control of translational Euler-Bernoulli beams using the stabilizing effect of beam damping mechanisms, *J Vib Control.* **10** (2004), 933–961.

[14] M.S. de Querioz, D.M. Dawson, M. Agrawal, F. Zhang, Adaptive nonlinear boundary control of a flexible link robot arm, *IEEE Trans Robot Autom.* **15** (1999), 779–787.

[15] M. Fabrizio, A. Morro, *Mathematical problems in linear viscoelasticity*, SIAM Studies in Applied Mathematics, Philadelphia, 1992.

[16] S.S. Ge, T.H. Lee, G. Zhu, Energy-based robust controller design for multi-link flexible robots, Mechatronics **6** (1996), 779–798.

[17] S.S. Ge, T.H. Lee,G. Zhu, A nonlinear feedback controller for a single-link flexible manipulator based on a finite element model, *J. Robot. Syst.* **14** (1997),165–178.

[18] S.S. Ge, T.H. Lee, G. Zhu, Asymptotically stable end-point regulation of a flexible SCARA/Cartesian robot, *IEEE/ASME Trans. Mechatronics* **3** (1998), no. 2, 138–144.

[19] S.S. Ge, T.H. Lee, J.Q. Gong, A robust distributed controller of a single-link SCARA/Cartesian smart materials robot, *Mechatronics* **9** (1999), 65–93.

[20] F. Guo, F. Huang, Boundary feedback stabilization of the undamped Euler-Bernoulli beam with both ends free, *SIAM J Control Optim.* **43** (2004), no. 1, 341–356.

[21] G.H. Hardy, J.E. Littlewood, *G. Polya Inequalities*, Cambridge, U.K., Cambridge Univ. Press, 1959.

[22] W.J. Hrusa, Global existence and asymptotic stability for semilinear hyperbolic Volterra equation with large initial data, *SIAM J Math Anal.* **16** (1985), no. 1, 110–134.

[23] N. Jalili, M. Dadfarnia, F. Hong, S.S. Ge, Adaptive non model-based piezoelectric control of flexible beams with translational base, *Proceedings of the American Control Conference Anchorage*, AK May 8-1, **5** (2002), 3802–3807.

[24] I. Lasiecka, D. Doundykov, Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms, *Nonlinear Anal.* **64** (2006), no. 8, 1757-1797.

[25] Z.H. Luo, N. Kitamura, B.Z. Guo, Shear force feedback control of flexible robot arms, *IEEE Trans. Robot. Autom.* **11** (1995), 760–765.

[26] Z.H. Luo, B.Z. Guo, Shear force feedback control of a single-link flexible robot with a revolute joint, *IEEE Trans. Automat. Contr.* **42** (1997), no. 1, 53–65.

[27] Z.H. Luo, N. Kitamura, B.Z. Guo, Shear force feedback control of flexible robot arms, *IEEE Transactions on Robotics and Automation* **11** (1995), no. 5, 760-765.

[28] S.A. Messaoudi, General decay of the solution energy in a viscoelastic equation with a nonlinear source, *Nonlinear Anal.* **69** (2008), no. 8, 2589-2598.

[29] S.A. Messaoudi, N-e. Tatar, Exponential and polynomial decay for a quasilinear viscoelastic equation, *Nonlinear Ana-Theor.* **68** (2008), no. 4, 785–793.

[30] M.I. Mustafa, Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations, *Nonlinear Anal. Real World Appl.* **13** (2012), 452–463.

[31] M.I. Mustafa, S.A. Messaoudi, General stability result for viscoelastic wave equations, *J Math Phys.* **53** (2012), no. 5, 053702.

[32] M.I. Mustafa, Optimal decay rates for the viscoelastic wave equation, *Math Methods Appl Sci.* **41** (2018), 192–204.

[33] Ö. Morgül, Dynamic boundary control of a Euler-Bernoulli beam, *IEEE Trans Autom Control* **37** (1992), no. 5, 639–642.

[34] J.Y. Park, J.A. Kim, Existence and uniform decay for Euler-Bernoulli beam equation with memory term, *Math Meth Appl Sci.* **27** (2004), 1629–1640.

[35] J.Y. Park, Y.H. Kang, J.A. Kim, Existence and exponential stability for a Euler-Bernoulli beam equation with memory and boundary output feedback control term, *Acta Appl. Math.* **104** (2008), 287–301.

[36] J.Y. Park, J.R. Kang, Global existence and uniform decay for a nonlinear viscoelastic equation with damping, *Acta Appl Math.* **110** (2010), no. 3, 1393–1406.

[37] J.Y. Park, S.H. Park, General decay for quasilinear viscoelastic equations with nonlinear weak damping, *J Math Phys.* **50** (2009), no. 8, 083505.

[38] J.Y. Park, J.A. Kim, Global existence and stability for Euler-Bernoulli beam equation with memory condition at the boundary, *J Korean Math Soc.* **42** (2005), no. 6, 1137–1152.

[39] M.J.E. Rivera, P.A. Salvatierra, Asymptotic behaviour of the energy in partially viscoelastic materials, *Q Appl Math.* **59** (2001), no. 3, 557–578.

[40] N-e. Tatar, A new class of kernels leading to an arbitrary decay in viscoelasticity, *Mediterr. J. Math.* **10** (2013), 213–226.

[41] N-e. Tatar, On a perturbed kernel in viscoelasticity, *Appl Math Letters* **24** (2011), 766–770.

[42] N-e. Tatar, On a large class of kernels yielding exponential stability in viscoelasticity, *Appl Math Comput.* **215** (2009), no. 6, 2298–2306.

[43] N-e. Tatar, Arbitrary decays in linear viscoelasticity, *J Math Phys.* **52** (2011), no. 1, 013502.

[44] J.B. Yang, L.J. Jiang, D.C. Chen, Dynamic modelling and control of a rotating Euler–Bernoulli beam, *J Sound Vib.* **274** (2004), no. 3-5, 863–875.

[45] Q. Yan, S. Hou, L. Zhang, Stabilization of Euler–Bernoulli beam with a nonlinear locally distributed feedback control, *J Syst Sci Complex.* **24** (2011), 1100–1109.

[46] G. Zhu, S.S. Ge, T.H. Lee, Variable structure regulation of a flexible arm with a translational base, *Proceedings of the 36th Conference on Decision and Control San Diego*, California USA, **2** (1997), 1361–1366.

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