Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 53(1), 2026, Pages 202–219, DOI: 10.52846/ami.v53i1.2132
ISSN: 1223-6934

Generalized Results on Levinson-Type Inequalities via Montgomery Identity and Green Functions

Awais Rasheed, Khuram Ali Khan, Josip Pečarić, and Đilda Pečarić

Abstract. The aim of this study is to obtain new generalizations of Levinson type inequalities for the class of n-convex($n \geq 3$) functions using new Green functions along with Montgomery identity. Some new estimations for novel functionals are derived via Bullen-type inequalities. Furthermore, generalized Levinson-type inequalities are established for positive real weights involving Montgomery identity and Green function.

2020 *Mathematics Subject Classification.* 26D10, 26D15, 26D20, 26B25.
Key words and phrases. Levinson's inequality; Montgomery identity; Convex functions; Green functions.

1. Introduction and Preliminaries

Convex functions are useful in many areas of mathematics. Convex functions are crucial to modern analysis as well as the study of optimization issues. There is a strong relationship between convex functions and the theory of inequalities. Many mathematicians and physicists have utilising higher order convexity to take advantage of the inequalities and tackle issues requiring more dimensions. In [1, p. 16], criteria for convex function of higher order is provided as; "f is n-convex if and only if nth-order derivative of f is non-negative, provided that $f^{(n)}$ exists."
Divided difference [1, p. 14], technique is defined as follows:

Definition 1.1. For a function $\Lambda : [\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2] \to \mathbb{R}$, the nth order divided difference, at mutually exclusive points $u_0, ..., u_n \in [\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2]$ is defined recursively by

$$[u_\varepsilon; \Lambda] = \Lambda(u_\varepsilon), \quad \varepsilon = 0, \ldots, n,$$

$$[u_0, \ldots, u_n; \Lambda] = \frac{[u_1, \ldots, u_n; \Lambda] - [u_0, \ldots, u_{n-1}; \Lambda]}{u_n - u_0}. \tag{1}$$

It is known that (1) is equivalent to

$$[u_0, \ldots, u_n; \Lambda] = \sum_{\varepsilon=0}^{n} \frac{\Lambda(u_\varepsilon)}{\mathfrak{c}'(\Lambda_\varepsilon)}, \quad where\ \mathfrak{c}(u) = \prod_{e=0}^{n}(u - u_e).$$

Using divided difference, n-convex function [1, p. 15], is defined as follows:

Definition 1.2. A function $\Lambda : [\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2] \to \mathbb{R}$ is called n-convex $(0 \leq n)$, for $u_0, \ldots, u_n \in [\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2]$, if and only if

$$[u_0, \ldots, u_n; \Lambda] \geq 0$$

Received March 23, 2025. Accepted August 9, 2025.

holds.

If $[u_0, \ldots, u_n; \Lambda] \leq 0$,then Λ is n-concave.

The Ky Fan's inequality, for 3-convex functions was expanded by Levinson [2], as:

Theorem A. Consider $\Lambda : \mathbb{I}_2 = (0, 2\delta) \to \mathbb{R}$ with $\frac{d^3}{dz^3}\Lambda(z) \geq 0$. Consider $x_\varepsilon \in (0, \delta)$ and $p_\varepsilon > 0$ with $\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon = Q$. Then

$$\frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \Lambda(x_\varepsilon) - \Lambda\left(\frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\right) \leq \frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \Lambda(2\delta - x_\varepsilon) - \Lambda\left(\frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon (2\delta - x_\varepsilon)\right). \quad (2)$$

Popoviciu [3], pointed out that Levinson Inequality (2) significantly affects on $(0, 2\delta)$, and Bullen [6], offered a unique form of Popoviciu's conclusions as well as the opposite of (2).

Theorem B. (i) Consider a 3-convex function $\Lambda : \mathfrak{t} = [\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2] \to \mathbb{R}$ and $x_\varepsilon, y_\varepsilon \in \mathfrak{t}$, $p_\varepsilon > 0(\varepsilon = 1, 2, \ldots, \hat{\varrho})$ and $\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon = Q_{\hat{\varrho}}$ so that

$$\max\{x_1 \ldots x_{\hat{\varrho}}\} \leq \min\{y_1 \ldots y_{\hat{\varrho}}\}, \; y_1 + x_1 = \ldots = y_{\hat{\varrho}} + x_{\hat{\varrho}} \quad (3)$$

then

$$\frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \Lambda(x_\varepsilon) - \Lambda\left(\frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\right) \leq \frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \Lambda(y_\varepsilon) - \Lambda\left(\frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon y_\varepsilon\right). \quad (4)$$

(ii) If Λ is continuous, (3) and (4) hold then Λ is 3-convex, for $p_\varepsilon > 0$.

From (4), we have following functional:

$$\mathbb{D}(\Lambda(.)) = \frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \Lambda(y_\varepsilon) - \Lambda\left(\frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon y_\varepsilon\right) - \frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \Lambda(x_\varepsilon) + \Lambda\left(\frac{1}{Q_{\hat{\varrho}}}\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\right) \geq 0. \quad (5)$$

Pečarić [7], explained inequality (4) in the following result by employing weaker criteria (3).

Theorem C. Suppose $\Lambda : \mathfrak{t} \to \mathbb{R}$, p_ε and $\Lambda^{(3)}(t)$ are non-negative. Assume $x_\varepsilon, y_\varepsilon \in \mathfrak{t}$ be such that $x_\varepsilon + y_\varepsilon = 2\check{c}$, for $\varepsilon = 1, \ldots, \hat{\varrho}$, $x_\varepsilon + x_{\hat{\varrho}-\varepsilon+1} \leq 2\check{c}$ and $\check{c} \geq \frac{p_\varepsilon x_\varepsilon + p_{\hat{\varrho}-\varepsilon+1} x_{\hat{\varrho}-\varepsilon+1}}{p_\varepsilon + p_{\hat{\varrho}-\varepsilon+1}}$, then (4) holds.

Mercer demonstrated in [8] that the following theorem's symmetric distribution of the points makes (4) correct.

Theorem D. Let Λ be a 3-convex function, defined on $\mathfrak{t}$ and p_ε be such that $\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon = 1$. If $\min\{y_1 \ldots y_{\hat{\varrho}}\} \geq \max\{x_1 \ldots x_{\hat{\varrho}}\}$ and

$$\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \left(x_\varepsilon - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\right)^2 = \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \left(y_\varepsilon - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon y_\varepsilon\right)^2, \quad (6)$$

then (4), holds.

Let $\mathfrak{t} = [\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2] \subset (-\infty, \infty)$. In [11], Awais *et al.*, defined extended version of 3-convex

Green functions and proved following identities:

$$\Lambda(\Psi) = \Lambda(\hat{\mathbb{b}}_1) + (\Psi - \hat{\mathbb{b}}_1)\Lambda'(\hat{\mathbb{b}}_2) + (\Psi - \hat{\mathbb{b}}_1)(\Psi - \hat{\mathbb{b}}_2)\Lambda''(\hat{\mathbb{b}}_1) - \frac{(\Psi - \hat{\mathbb{b}}_1)^2}{2}\Lambda''(\hat{\mathbb{b}}_2) + \int_{\mathfrak{t}} G_1(\Psi, \Theta)\Lambda'''(\Theta)d\Theta, \quad (7)$$

$$\Lambda(\Psi) = \Lambda(\hat{\mathbb{b}}_2) - (\hat{\mathbb{b}}_2 - \Psi)\Lambda'(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_1)\frac{(\hat{\mathbb{b}}_2 - \Psi)^2}{2} + (\Psi - \hat{\mathbb{b}}_1)(\Psi - \hat{\mathbb{b}}_2)\Lambda''(\hat{\mathbb{b}}_2) - \int_{\mathfrak{t}} G_2(\Psi, \Theta)\Lambda'''(\Theta)d\Theta, \quad (8)$$

$$\begin{aligned}\Lambda(\Psi) = {} & \Lambda(\hat{\mathbb{b}}_2) + (\Psi - \hat{\mathbb{b}}_1)\Lambda'(\hat{\mathbb{b}}_1) - (\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1)\Lambda'(\hat{\mathbb{b}}_2) \\ & - \Lambda''(\hat{\mathbb{b}}_1)\left[\frac{(\Psi - \hat{\mathbb{b}}_1)^2}{2} + (\Psi - \hat{\mathbb{b}}_1)(\hat{\mathbb{b}}_1 - \hat{\mathbb{b}}_2)\right] \\ & + \Lambda''(\hat{\mathbb{b}}_2)\left[\frac{(\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1)^2}{2} + (\Psi - \hat{\mathbb{b}}_1)(\Psi - \hat{\mathbb{b}}_2)\right] - \int_{\mathfrak{t}} G_3(\Psi, \Theta)\Lambda'''(\Theta)d\Theta, \quad (9)\end{aligned}$$

$$\begin{aligned}\Lambda(\Psi) = {} & \Lambda(\hat{\mathbb{b}}_1) + (\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1)\Lambda'(\hat{\mathbb{b}}_1) - (\hat{\mathbb{b}}_2 - \Psi)\Lambda'(\hat{\mathbb{b}}_2) \\ & + \Lambda''(\hat{\mathbb{b}}_1)\left[(\Psi - \hat{\mathbb{b}}_2)(\Psi - \hat{\mathbb{b}}_1) + \frac{(\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1)^2}{2}\right] \\ & - \left[(\Psi - \hat{\mathbb{b}}_2)(\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1) + \frac{(\Psi - \hat{\mathbb{b}}_2)^2}{2}\right]\Lambda''(\hat{\mathbb{b}}_2) + \int_{\mathfrak{t}} \Lambda'''(\Theta)G_4(\Psi, \Theta)d\Theta. \quad (10)\end{aligned}$$

Where $G_k : \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$, and for $k \in \{1, 2, 3, 4\}$, G_k is given as:

$$G_1(\Psi, \Theta) = \begin{cases} \frac{1}{2}(\Theta - \hat{\mathbb{b}}_1)^2 + (\Psi - \hat{\mathbb{b}}_1)(\Psi - \hat{\mathbb{b}}_2), & \hat{\mathbb{b}}_1 \le \Theta \le \Psi, \\ (\Psi - \hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_2) + \frac{(\Psi - \hat{\mathbb{b}}_1)^2}{2}, & \Psi \le \Theta \le \hat{\mathbb{b}}_2. \end{cases} \quad (11)$$

$$G_2(\Psi, \Theta) = \begin{cases} (\Psi - \hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_1) + \frac{1}{2}(\Psi - \hat{\mathbb{b}}_2)^2, & \hat{\mathbb{b}}_1 \le \Theta \le \Psi, \\ \frac{(\Theta - \hat{\mathbb{b}}_2)^2}{2} + (\Psi - \hat{\mathbb{b}}_1)(\Psi - \hat{\mathbb{b}}_2), & \Psi \le \Theta \le \hat{\mathbb{b}}_2. \end{cases} \quad (12)$$

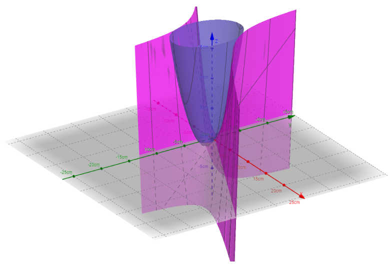
$$G_3(\Psi, \Theta) = \begin{cases} (\Psi - \hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_2) + \frac{(\Psi - \hat{\mathbb{b}}_1)^2}{2}, & \hat{\mathbb{b}}_1 \le \Theta \le \Psi, \\ \frac{1}{2}(\Theta - \hat{\mathbb{b}}_1)^2 + (\Psi - \hat{\mathbb{b}}_1)(\Psi - \hat{\mathbb{b}}_2), & \Psi \le \Theta \le \hat{\mathbb{b}}_2. \end{cases} \quad (13)$$

$$G_4(\Psi, \Theta) = \begin{cases} \frac{(\Theta - \hat{\mathbb{b}}_2)^2}{2} + (\Psi - \hat{\mathbb{b}}_2)(\Psi - \hat{\mathbb{b}}_1), & \hat{\mathbb{b}}_1 \le \Theta \le \Psi, \\ (\Psi - \hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_1) + \frac{1}{2}(\Psi - \hat{\mathbb{b}}_2)^2, & \Psi \le \Theta \le \hat{\mathbb{b}}_2. \end{cases} \quad (14)$$

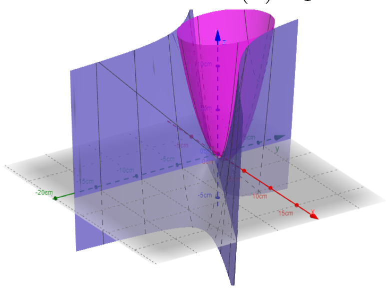
To obtain our primary results, we use Montgomery identity with Taylor formula [4], stated as:

Theorem I. Let $\mathbb{I} \subset \mathbb{R}$ be the open interval and $\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2 \in \mathbb{I}$ such that $\hat{\mathbb{b}}_1 < \hat{\mathbb{b}}_2$. Let $\Lambda^{(n-1)}$ be absolutely continuous defined on $\mathbb{I}$. Then for $n \in \mathbb{N}$

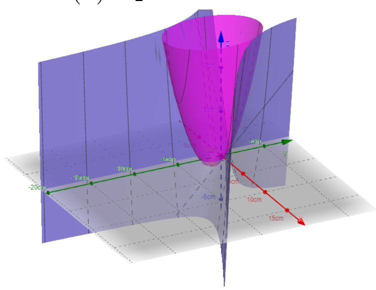
$$\Lambda(\Theta) \quad = \quad \frac{1}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\int_{\mathfrak{t}} \Lambda(v)dv + \sum_{\mathfrak{c}=0}^{n-2} \frac{\Lambda^{(\mathfrak{c}+1)}(\hat{\mathbb{b}}_1)}{\mathfrak{c}!(\mathfrak{c}+2)} \frac{(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}+2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}$$



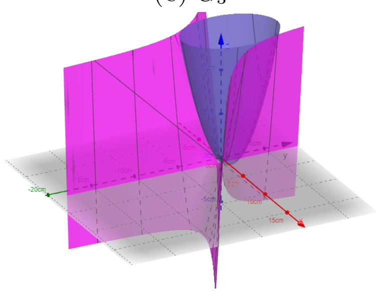
(A) G_1



(B) G_2



(C) G_3



(D) G_4

FIGURE 1. Graph of $G_k (k = 1, \dots, 4)$ for different values of Ψ and Θ.

$$-\sum_{\mathfrak{c}=0}^{n-2}\frac{\Lambda^{(\mathfrak{c}+1)}(\hat{\mathbb{b}}_2)}{\mathfrak{c}!(\mathfrak{c}+2)}\frac{(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}+2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}+\frac{1}{(n-1)!}\int_{\mathfrak{t}}R_n(\Theta,v)\Lambda^{(n)}(v)dv \tag{15}$$

holds, where

$$R_n(\Theta,v)=\begin{cases}-\frac{(\Theta-v)^n}{n(\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1)}+\frac{\Theta-\hat{\mathbb{b}}_1}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}((\Theta-v)^{n-1}), & \hat{\mathbb{b}}_1\le v\le\Theta;\\ -\frac{(\Theta-v)^n}{n(\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1)}+\frac{\Theta-\hat{\mathbb{b}}_2}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}((\Theta-v)^{n-1}), & \Theta<v\le\hat{\mathbb{b}}_2.\end{cases} \tag{16}$$

For $n=1$ in (15), we have well-known Montgomery identity given in [5].

$$\Lambda(\Theta) \;=\; \frac{1}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\int_{\mathfrak{t}}\Lambda(v)dv+\int_{\mathfrak{t}}P(\Theta,v)\Lambda^{'}(v)dv,$$

where the Peano kernel, $P(\Theta,v)$ is defined as

$$P(\Theta,v)=\begin{cases}\frac{v-\hat{\mathbb{b}}_1}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}, & \hat{\mathbb{b}}_1\le v\le\Theta;\\ \frac{v-\hat{\mathbb{b}}_2}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}, & \Theta<v\le\hat{\mathbb{b}}_2.\end{cases} \tag{17}$$

Through Montgomery inequality, Pečarić *et al.* provided a new generalisation of Popoviciu-type inequality in [9]. Additionally, they used the newly created four Green functions to state new identities. These generalised identities were employed by numerous writers to calculate the limits of various inequalities. Adeel *et al.* employed Montgomery identity and Green functions in [10] to generalise Levinson type inequalities.

In recent years many authors presented a fruitful derivations in the field of higher order convex functions using Levinson-type inequlities and gave their fruitful applications. In [11], Rasheed *et al.* gave novel 3-convex four Green functions with graphical representation. They also obtained new identities using these Green functions and derived generalizations of Levinson type inequalities. They also calculated fruitful results of their main findings in information theory. In [12], authors derived new bounds for Levinson-type and Bullen-type inequalities by using Hermite interpolation and new Green's functions. In 2024, bounds for Levinson-type inequalities are derived in the form of Taylor's formula involving Green's functions by Rasheed [20] *et al.*. They also established various estimations for Bullen-type inequalities for positive real weights.

In the domain of analysis, Adeel *et al.* constructed the Levinson inequality for 3-convex functions [13] and achieved successful outcomes. Abel-Gontscharoff interpolation was used by Khan [14] *et al.* to establish generalisations of Levinson-type inequalities for convex functions of higher order. Adeel *et al.* computed Shannon entropy and provided results for f-divergence in [15] using new Green functions and the Lidstone polynomial in conjunction with Levinson type inequalities. By using the Hermite interpolating polynomial, Adeel [16] *et al.* successfully obtained the Levinson-type inequality for convex functions of higher order. They also provided estimates for the Shannon entropy and f-divergence. Adeel *et al.* presented numerous results linked to Levinson-type inequalities for higher order convex functions associated to different interpolations including: Hermite interpolating polynomial, Fink's identity and Lidstone interpolation (see [17–19]). Bilal *et al.* gave extended version of Shannon type inequalities and inequalities for Csiszár's f-divergence using diamond integrals in [21,22]. The obtained results are more comprehensive and generalized in Time scales calculus.

2. Main Results

The results associated to Bullen-type and Levinson-type inequalities using new Green functions are provided.

$\mathfrak{F}$: Suppose a function $\Lambda : \mathfrak{t} = [\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2] \to \mathbb{R}$. Let $(p_1, \ldots, p_{\hat{\varrho}}) \in \mathbb{R}^{\hat{\varrho}}$ and $(q_1, \ldots, q_\omega) \in \mathbb{R}^\omega$ be such that $\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon = 1$, $\sum_{\epsilon=1}^{\omega} q_\epsilon = 1$ and $x_\varepsilon, y_\epsilon, \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon, \sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon \in \mathfrak{t}$. Then

$$\mathbb{M}(\Lambda(\cdot)) = \sum_{\epsilon=1}^{\omega} q_\epsilon \Lambda(y_\epsilon) - \Lambda\bigg(\sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon\bigg) - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \Lambda(x_\varepsilon) + \Lambda\bigg(\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\bigg). \tag{18}$$

Also, choose $x_1 \ldots x_{\hat{\varrho}}$ and $y_1 \ldots y_\omega \in \mathfrak{t}$ such that $\max\{x_1 \ldots x_{\hat{\varrho}}\} \le \min\{y_1 \ldots y_\omega\}$ and

$$\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \left(x_\varepsilon - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\right)^2 = \sum_{\epsilon=1}^{\omega} q_\epsilon \left(y_\epsilon - \sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon\right)^2. \tag{19}$$

$\mathfrak{G}$: Suppose $\Lambda \in C^n[\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_1]$ and $\Lambda^{(n-1)}$ is absolutely continuous. Let $(p_1, \ldots, p_{\hat{\varrho}})$ be positive real numbers such that $\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon = 1$.

2.1. Extension of Bullen-type inequalities for higher order convex functions. Identity (5), gives us motivation for the construction of following identities:

Theorem 2.1. *Suppose $\mathfrak{F}$. Let $\Lambda^{(n-1)}$ be absolutely continuous and $\Lambda \in C^n[\hat{\mathbb{b}}_1, \hat{\mathbb{b}}_2]$. Then, for $k = 1, 4$, we have*

$$\begin{aligned}(i)\ \mathbb{M}(\Lambda(\cdot)) = {} & \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}](2\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)) \\ & + 2\bigg(\frac{\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\bigg) \int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot, \Theta)) d\Theta + \int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot, \Theta)) \\ & \times \sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!} \left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right] d\Theta \\ & + \frac{1}{(n-4)!} \int_{\mathfrak{t}} \Lambda^{(n)}(v) \bigg(\int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot, \Theta)) \tilde{R}_{n-3}(\Theta, v) d\Theta \bigg) dv, \end{aligned} \tag{20}$$

and

$$\begin{aligned}(ii)\ \mathbb{M}(\Lambda(\cdot)) = {} & \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}](2\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)) \\ & + 2\bigg(\frac{\Lambda''(\hat{\mathbb{b}}_2) - \Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\bigg) \int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot, \Theta)) d\Theta + \int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot, \Theta)) \\ & \times \sum_{\mathfrak{c}=4}^{n-1} \frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!} \left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right] d\Theta \\ & + \frac{1}{(n-4)!} \int_{\mathfrak{t}} \Lambda^{(n)}(v) \bigg(\int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot, \Theta)) R_{n-3}(\Theta, v) d\Theta \bigg) dv, \end{aligned} \tag{21}$$

For $k = 2, 3,$

$$(iii)\ \mathbb{M}(\Lambda(\cdot)) = \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}](2\Lambda^{''}(\hat{\mathbb{b}}_2) - \Lambda^{''}(\hat{\mathbb{b}}_1))$$
$$- 2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_1) - \Lambda^{''}(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))d\Theta - \int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))$$
$$\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right]d\Theta$$
$$- \frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta,v)d\Theta\right)dv, \quad (22)$$

and

$$(iv)\ \mathbb{M}(\Lambda(\cdot)) = \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}](2\Lambda^{''}(\hat{\mathbb{b}}_2) - \Lambda^{''}(\hat{\mathbb{b}}_1))$$
$$- 2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_2) - \Lambda^{''}(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))d\Theta - \int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))$$
$$\times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right]d\Theta$$
$$- \frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta\right)dv, \quad (23)$$

where

$$\tilde{R}_{n-3}(\Theta,v) = \begin{cases} 2\frac{(\Theta-v)^{n-3}}{(n-3)(\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1)} + \frac{\Theta-\hat{\mathbb{b}}_1}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}((\Theta-v)^{n-4}), & \hat{\mathbb{b}}_1 \le v \le \Theta; \\ 2\frac{(\Theta-v)^{n-3}}{(n-3)(\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1)} + \frac{\Theta-\hat{\mathbb{b}}_2}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}((\Theta-v)^{n-4}), & \Theta < v \le \hat{\mathbb{b}}_2, \end{cases} \quad (24)$$

and

$$R_{n-3}(\Theta,v) = \begin{cases} -\frac{(\Theta-v)^{n-3}}{(n-3)(\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1)} + \frac{\Theta-\hat{\mathbb{b}}_1}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}((\Theta-v)^{n-4}), & \hat{\mathbb{b}}_1 \le v \le \Theta; \\ -\frac{(\Theta-v)^{n-3}}{(n-3)(\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1)} + \frac{\Theta-\hat{\mathbb{b}}_2}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}((\Theta-v)^{n-4}), & \Theta < v \le \hat{\mathbb{b}}_2, \end{cases} \quad (25)$$

$$\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}] = \sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon^2 - (\sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon)^2 - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon^2 + (\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon)^2, \quad (26)$$

and

$$\mathbb{M}(G_k(\cdot,\Theta)) = \sum_{\epsilon=1}^{\omega} q_\epsilon G_k(y_\epsilon,\Theta) - G_k\left(\sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon, \Theta\right) - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon G_k(x_\varepsilon,\Theta)$$
$$+ G_k\left(\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon, \Theta\right), \quad (27)$$

$G_k(\cdot,\Theta)$ *(*$k = 1, \ldots 4,$*) are provided in (11)-(14), respectively.*

Proof. (i) Let $k=1,4$, then applying (18) to the identities (7) and (10) and using linearity of $\mathbb{M}(\Lambda(\cdot))$, we get

$$\begin{aligned}\mathbb{M}(\Lambda(\cdot)) &= \frac{1}{2}\left[\sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon^2 - \left(\sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon\right)^2 - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon^2 + \left(\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\right)^2\right] \\ &\quad \times(2\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)) + \int_{\mathfrak{t}} \mathbb{M}(G_k(.,\Theta))\Lambda^{(3)}(\Theta)d\Theta. \qquad (28)\end{aligned}$$

Differentiating (15) with respect to 'Θ', we have

$$\begin{aligned}\Lambda^{(3)}(\Theta) &= 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right) \\ &\quad + \sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left(\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right) \\ &\quad + \frac{1}{(n-4)!}\int_{\mathfrak{t}} \tilde{R}_{n-3}(\Theta,v)\Lambda^{(n)}(v)dv. \qquad (29)\end{aligned}$$

Putting (29) in (28), we get

$$\begin{aligned}\mathbb{M}(\Lambda(\cdot)) &= \frac{1}{2}\left[\sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon^2 - (\sum_{\epsilon=1}^{\omega} q_\epsilon y_\epsilon)^2 - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon^2 + (\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon)^2\right] \\ &\quad \times (2\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)) + 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot,\Theta))d\Theta \\ &\quad + \int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot,\Theta)) \\ &\quad \times\left[\sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left(\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\right]d\Theta \\ &\quad + \frac{1}{(n-4)!}\int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot,\Theta))\left(\int_{\mathfrak{t}} \tilde{R}_{n-3}(\Theta,v)\Lambda^{(n)}(v)dv\right)d\Theta.\end{aligned}$$

Using Fubini's Theorem in the final term yields (20), for $k=1,4$.
(ii) Applying (15) on $\Lambda^{(3)}$, n be replaced with $n-3$ and the indices can be rearranged to get

$$\begin{aligned}\Lambda^{(3)}(\Theta) &= 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right) + \sum_{\mathfrak{c}=4}^{n-1} \frac{\mathfrak{c}}{(\mathfrak{c}-2)(\mathfrak{c}-4)!} \\ &\quad \times\left(\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right) \\ &\quad + \frac{1}{(n-4)!}\int_{\mathfrak{t}} R_{n-3}(\Theta,v)\Lambda^{(n)}(v)dv. \qquad (30)\end{aligned}$$

Applying (30) in (28) and Fubini's Theorem, we reached at (21) for $k=1,4$.
For $k=2,3$, the similar steps are taken to obtain (iii) and (iv). □

The subsequent result provides an extended version of Bullen-type inequalities containing new Green functions, for the convex functions of higher order.

Theorem 2.2. *Considering all the suppositions of Theorem 2.1 for the n-convex function Λ.*
If

$$\int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta,v)d\Theta \geq 0,\ \ v\in\mathfrak{t}, \tag{31}$$

then for $k=1,4$,

$$\begin{aligned}\mathbb{M}\left(\Lambda(\cdot)\right) \geq{}& \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}](2\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)) \\ &+2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta)) \\ &\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\end{aligned} \tag{32}$$

and for $k=2,3$,

$$\begin{aligned}\mathbb{M}\left(\Lambda(\cdot)\right) \leq{}& \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}](2\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)) \\ &-2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta)) \\ &\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta.\end{aligned} \tag{33}$$

and if

$$\int_{\mathfrak{t}} \mathbb{M}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta \geq 0,\ \ v\in\mathfrak{t}, \tag{34}$$

then for $k=1,4$,

$$\begin{aligned}\mathbb{M}\left(\Lambda(\cdot)\right) \geq{}& \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}](2\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)) \\ &+2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta)) \\ &\times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\end{aligned} \tag{35}$$

and for $k=2,3$,

$$\begin{aligned}\mathbb{M}\left(\Lambda(\cdot)\right) \leq{}& \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}](2\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)) \\ &-2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\mathbb{M}(G_k(\cdot,\Theta)) \\ &\times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta.\end{aligned} \tag{36}$$

Proof. Since Λ is n-convex($n\geq 3$), consequently n-th order derivative of Λ exists, preserving generality. This implies that

$$\Lambda^{(n)}(\Theta)\geq 0,\ \ \forall\ \Theta\in\mathfrak{t},$$

then by using (31) in (20) and (22), to obtain (32) and (33) respectively, also using (34) in (21) and (23), we get (35) and (36), respectively. □

Remark 2.1. (*i*) Inequalities in (32), (33), (35) and (36) are conversed, if the inequalities (31) and (34) are reversed.
(*ii*) If Λ is n-concave then (32), (33), (35) and (35) are conversed.

Remark 2.2. $\mathbb{M}(\cdot)$ is reduced in $\mathbb{D}(\cdot)$,if $\omega = \hat{\varrho}$, $q_\epsilon = p_\varepsilon$ and weights are positive. Then (20), (21), (22), (23), (31), (32), (33), (34), (35) and (36) become

$$\begin{aligned}\mathbb{D}\left(\Lambda(\cdot)\right) &= 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\\ &+\frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta,v)d\Theta\right)dv, \qquad (37)\end{aligned}$$

$$\begin{aligned}\mathbb{D}\left(\Lambda(\cdot)\right) &= 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\\ &+\frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta\right)dv, \qquad (38)\end{aligned}$$

$$\begin{aligned}\mathbb{D}\left(\Lambda(\cdot)\right) &= -2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\\ &-\frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta,v)d\Theta\right)dv, \qquad (39)\end{aligned}$$

$$\begin{aligned}\mathbb{D}\left(\Lambda(\cdot)\right) &= -2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\\ &-\frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta\right)dv, \qquad (40)\end{aligned}$$

$$\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta,v)d\Theta\geq 0,\ \ v\in\mathfrak{t}, \qquad (41)$$

$$\mathbb{D}\left(\Lambda(\cdot)\right)\geq 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))$$

$$\times \sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{f^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta, \tag{42}$$

$$\mathbb{D}\left(\Lambda(\cdot)\right) \leq -2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_1)-\Lambda^{''}(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))$$
$$\times \sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta, \tag{43}$$

$$\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta \geq 0,\ \ v\in\mathfrak{t}, \tag{44}$$

$$\mathbb{D}\left(\Lambda(\cdot)\right) \geq 2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_2)-\Lambda^{''}(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))$$
$$\times \sum_{\mathfrak{c}=4}^{n-1} \frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta, \tag{45}$$

$$\mathbb{D}\left(\Lambda(\cdot)\right) \leq -2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_2)-\Lambda^{''}(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\mathbb{D}(G_k(\cdot,\Theta))$$
$$\times \sum_{\mathfrak{c}=4}^{n-1} \frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta, \tag{46}$$

where

$$\mathbb{D}(G_k(\cdot,\Theta)) = \sum_{\varepsilon=1}^{\hat{\varrho}} q_\varepsilon G_k(y_\varepsilon,\Theta) - G_k\left(\sum_{\varepsilon=1}^{\hat{\varrho}} q_\varepsilon y_\varepsilon,\Theta\right) - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon G_k(x_\varepsilon,\Theta)$$
$$+G_k\left(\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon,\Theta\right), \tag{47}$$

Theorem 2.3. *Assume $\mathfrak{G}$ holds. Then, the following statements are true for the functional $\mathbb{D}(.)$ given in (5).*

(i) If n is odd then for $k=1,4$, the inequalities (42) and (45) hold and for $k=2,3$, (43) and (46) are true.

(ii) For $k=1,4$, let the inequalities (42) and (45), be satisfied and

$$(\Lambda^{''}(\hat{\mathbb{b}}_1)-\Lambda^{''}(\hat{\mathbb{b}}_2))$$
$$+\sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right] \geq 0\ \forall\Theta\in\mathbb{I}_1, \tag{48}$$

then

$$\mathbb{D}(\Lambda(.)) \geq 0. \tag{49}$$

- *Or for* $k = 2, 3$, (43) *and* (46) *be satisfied and*

$$\sum_{\mathfrak{c}=4}^{n-1} \frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right] \geq 0.$$

then

$$\mathbb{D}(\Lambda(.)) \leq 0. \tag{50}$$

Proof. Since the the weights are positive and Green functions $G_k(\cdot, \Theta)(k = 1, \ldots, 4)$ are 3-convex. So, apply Theorem **B**(ii), to obtain $\mathbb{D}(G_k(\cdot, \Theta)) \geq 0,(k = 1 \ldots, 4)$.

(i) For $n = 5, 7, \ldots$, $\tilde{R}_n(\Theta, v)$ and $R_n(\Theta, v)$ are non-negative, thus (41) and (44) hold. As function Λ is n-convex, therefore applying Theorem 2.2, to obtain (42) and (45), for $k = 1, 4$, and (43) and (46), for $k = 2, 3$,.

(ii) Using (48) in (42) and (45), and (50) in (43) and (46), we get (49) and (50), respectively.

□

Theorem 2.4. *Assume* $\mathfrak{G}$ *holds and* $x_\varepsilon, y_\varepsilon \in \mathfrak{t}$ *be such that* $x_\varepsilon + y_\varepsilon = 2\check{c}$, *for* $\varepsilon = 1, \ldots, \hat{\varrho}$, $x_\varepsilon + x_{\hat{\varrho}-\varepsilon+1} \leq 2\check{c}$ *and* $\frac{p_\varepsilon x_\varepsilon + p_{\hat{\varrho}-\varepsilon+1} x_{\hat{\varrho}-\varepsilon+1}}{p_\varepsilon + p_{\hat{\varrho}-\varepsilon+1}} \leq \check{c}$. *Then, we have the following:*

(i) *For* $k = 1, 4$, *the inequalities* (42) *and* (45) *are valid and for* $k = 2, 3$, (43) *and* (46) *are true, provided* n *is odd.*

(ii) *For* $k = 1, 4$, *let the inequalities* (42) *and* (45) *be satisfied and the inequality* (48) *hold, or for* $k = 2, 3$, (43) *and* (46) *be satisfied and the inequality* (50) *is true then* (49) *and* (50) *holds, respectively.*

Proof. Proof is same as of Theorem 2.3. □

Under the condition defined by (6), generalized form of Bullen-type inequality (for positive weights) is presented as follows:

Corollary 2.5. *Assume* $\mathfrak{G}$ *holds and* x_ε, y_ε *satisfy* (6) *and* $\max\{x_1 \ldots x_{\hat{\varrho}}\} \leq \min\{y_1 \ldots y_{\hat{\varrho}}\}$ *then (20)*, (21), (22) *and (23) hold.*

Proof. By applying the conditions stated in the statement and using the same method as in proof of Theorem 2.1, to get (20), (21), (22) and (23). □

Theorem 2.6. *Assume* $\mathfrak{G}$ *holds. Let* $\max\{x_1, \ldots, x_{\hat{\varrho}}\} \leq \min\{y_1, \ldots, y_{\hat{\varrho}}\}$ *and*

$$\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \left(x_\varepsilon - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\right)^2 = \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \left(y_\varepsilon - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon y_\varepsilon\right)^2. \tag{51}$$

Then,

(i) *For* $k = 1, 4$, *the inequalities* (42) *and* (45) *hold and for* $k = 2, 3$, (43) *and* (46) *are true, provided* n *is odd.*

(ii) *For* $k = 1, 4$, *if the inequalities* (42), (45) *and* (48) *are valid, or for* $k = 2, 3$, *if* (43), (46) *and* (50) *are satisfied then* (49) *and* (50), *holds, repectively.*

Proof. Applying Theorem **D**, and follow the same steps such as proof of Theorem 2.3, we get the desired results. □

2.2. Levinson-type inequalities for n-convex ($n \geq 3$) functions. In this section, results associated with Levinson-type inequality employing new Green functions $G_k(k = 1, \dots, 4)$ are generalized via Montgomery identity. For this, first we define:
$\mathcal{I}$**:** Suppose $\Lambda : \mathbb{L}_2 = [0, 2\beta] \to \mathbb{R}$ be a function. Choose $x_1, \dots, x_{\hat{\varrho}} \in (0, \beta)$, $(p_1, \dots, p_{\hat{\varrho}}) \in \mathbb{R}^{\hat{\varrho}}$ and $(q_1, \dots, q_\omega) \in \mathbb{R}^\omega$ such that $\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon = 1$ and $\sum_{\epsilon=1}^{\omega} q_\epsilon = 1$. Let x_ε, $\sum_{\epsilon=1}^{\omega} q_\epsilon(2\beta - x_\epsilon)$ and $\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \in \mathbb{L}_2$. Then

$$\breve{\mathfrak{U}}(\Lambda(\cdot)) = \sum_{\epsilon=1}^{\omega} q_\epsilon \Lambda(2\beta - x_\epsilon) - \Lambda\bigg(\sum_{\epsilon=1}^{\omega} q_\epsilon(2\beta - x_\epsilon)\bigg) - \sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon \Lambda(x_\varepsilon) + \Lambda\bigg(\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon\bigg). \tag{52}$$

For the next results, we construct the following identities:

Theorem 2.7. *Assume $\mathcal{I}$ and $\mathfrak{G}$. Then for $0 \leq \hat{\mathbb{b}}_1 < \hat{\mathbb{b}}_2 \leq 2\beta$ and $k = 1, 4$ we have Then, for $k = 1, 4$, we have*

$$(i) \quad \breve{\mathfrak{U}}(\Lambda(\cdot)) = \frac{1}{2}\mathbb{P}[\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, 2\beta - \boldsymbol{x}](2\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)) + 2\bigg(\frac{\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\bigg) \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) d\Theta + \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) \times \sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!} \bigg[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\bigg] d\Theta + \frac{1}{(n-4)!} \int_{\mathfrak{t}} \Lambda^{(n)}(v) \bigg(\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) \tilde{R}_{n-3}(\Theta, v) d\Theta\bigg) dv, \tag{53}$$

and

$$(ii) \quad \breve{\mathfrak{U}}(\Lambda(\cdot)) = \frac{1}{2}\mathbb{P}[\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, 2\beta - \boldsymbol{x}](2\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)) + 2\bigg(\frac{\Lambda''(\hat{\mathbb{b}}_2) - \Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\bigg) \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) d\Theta + \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) \times \sum_{\mathfrak{c}=4}^{n-1} \frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!} \bigg[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\bigg] d\Theta + \frac{1}{(n-4)!} \int_{\mathfrak{t}} \Lambda^{(n)}(v) \bigg(\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) R_{n-3}(\Theta, v) d\Theta\bigg) dv, \tag{54}$$

For $k = 2, 3$,

$$(iii) \quad \breve{\mathfrak{U}}(\Lambda(\cdot)) = \frac{1}{2}\mathbb{P}[\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, 2\beta - \boldsymbol{x}](2\Lambda''(\hat{\mathbb{b}}_2) - \Lambda''(\hat{\mathbb{b}}_1)) - 2\bigg(\frac{\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\bigg) \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) d\Theta - \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) \times \sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!} \bigg[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\bigg] d\Theta - \frac{1}{(n-4)!} \int_{\mathfrak{t}} \Lambda^{(n)}(v) \bigg(\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot, \Theta)) \tilde{R}_{n-3}(\Theta, v) d\Theta\bigg) dv, \tag{55}$$

and

$$(iv) \qquad \breve{\mathfrak{U}}(\Lambda(\cdot)) = \frac{1}{2}\mathbb{P}[\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, 2\beta - \boldsymbol{x}](2\Lambda''(\hat{\mathbb{b}}_2) - \Lambda''(\hat{\mathbb{b}}_1))$$
$$-2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2) - \Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))d\Theta - \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))$$
$$\times \sum_{\mathfrak{c}=4}^{n-1} \frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right]d\Theta$$
$$-\frac{1}{(n-4)!}\int_{\mathfrak{t}} \Lambda^{(n)}(v)\left(\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))R_{n-3}(\Theta, v)d\Theta\right)dv, \tag{56}$$

where $\breve{\mathfrak{U}}(\Lambda(\cdot))$ *is defined in (52) and*

$$\breve{\mathfrak{U}}(G_k(\cdot,\Theta)) = \sum_{\epsilon=1}^{\omega} q_\epsilon G_k(2\beta - x_\epsilon, \Theta) - G_k\left(\sum_{\epsilon=1}^{\omega} q_\epsilon(2\beta - x_\epsilon, \Theta)\right)$$
$$-\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon G_k(x_\varepsilon, \Theta) + G_k\left(\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon, \Theta\right). \tag{57}$$

Proof. To obtain the desired results, substitute $\mathfrak{t}$, $\mathbb{M}(\cdot)$ and y_ϵ with $\mathbb{I}_2$, $\breve{\mathfrak{U}}(\cdot)$ and $(2\beta - x_\epsilon)$ in Theorem 2.1, respectively. □

The result for n-convex functions is as follows:

Theorem 2.8. *Assume that* Λ *is* n*-convex function and satisfying all the requirements of Theorem 2.1.*
If

$$\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta, v)d\Theta \geq 0, \;\; \Theta \in \mathfrak{t}, \tag{58}$$

then for $k = 1, 4$,

$$\breve{\mathfrak{U}}(\Lambda(\cdot)) \geq \frac{1}{2}\mathbb{P}[\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, 2\beta - \boldsymbol{x}](2\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2))$$
$$+ 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))d\Theta + \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))$$
$$\times \sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right]d\Theta \tag{59}$$

and for $k = 2, 3$,

$$\breve{\mathfrak{U}}(\Lambda(\cdot)) \leq \frac{1}{2}\mathbb{P}[\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{x}, 2\beta - \boldsymbol{x}](2\Lambda''(\hat{\mathbb{b}}_2) - \Lambda''(\hat{\mathbb{b}}_1))$$
$$- 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1) - \Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))d\Theta - \int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))$$
$$\times \sum_{\mathfrak{c}=3}^{n-1} \frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta - \hat{\mathbb{b}}_1)^{\mathfrak{c}-2} - \Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta - \hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2 - \hat{\mathbb{b}}_1}\right]d\Theta. \tag{60}$$

and if

$$\int_{\mathfrak{t}} \breve{\mathfrak{U}}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta \geq 0,\ \ \Theta \in \mathfrak{t}, \tag{61}$$

then for $k = 1, 4$,

$$\begin{aligned}\breve{\mathfrak{U}}\left(\Lambda(\cdot)\right) \geq \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},2\beta-\boldsymbol{x}](2\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)) \\ +2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathfrak{U}}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\breve{\mathfrak{U}}(G_k(\cdot,\Theta)) \\ \times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\end{aligned} \tag{62}$$

and for $k = 2, 3$,

$$\begin{aligned}\breve{\mathfrak{U}}\left(\Lambda(\cdot)\right) \leq \frac{1}{2}\mathbb{P}[\boldsymbol{p},\boldsymbol{q},\boldsymbol{x},2\beta-\boldsymbol{x}](2\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)) \\ -2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathfrak{U}}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\breve{\mathfrak{U}}(G_k(\cdot,\Theta)) \\ \times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta.\end{aligned} \tag{63}$$

Proof. Proof is identical to that of Theorem 2.2. □

Remark 2.3. $\breve{\mathfrak{U}}(\cdot)$ is reduced in $\breve{\mathbb{U}}(\cdot)$, if $\omega = \hat{\varrho}$, $p_{\hat{\varrho}} = q_\omega$ and weights are positive. Then (53), (54), (55), (56), (58), (59), (60), (61), (62) and (63) become

$$\begin{aligned}\breve{\mathbb{U}}\left(\Lambda(\cdot)\right) &= 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta)) \\ &\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta \\ &+\frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta,v)d\Theta\right)dv,\end{aligned} \tag{64}$$

$$\begin{aligned}\breve{\mathbb{U}}\left(\Lambda(\cdot)\right) = 2\left(\frac{\Lambda''(\hat{\mathbb{b}}_2)-\Lambda''(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta)) \\ \times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta \\ +\frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta\right)dv,\end{aligned} \tag{65}$$

$$\begin{aligned}\breve{\mathbb{U}}\left(\Lambda(\cdot)\right) &= -2\left(\frac{\Lambda''(\hat{\mathbb{b}}_1)-\Lambda''(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta)) \\ &\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\end{aligned}$$

$$-\frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta,v)d\Theta\right)dv, \tag{66}$$

$$\begin{aligned}\breve{\mathbb{U}}\left(\Lambda(\cdot)\right) &= -2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_2)-\Lambda^{''}(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta\\ &-\frac{1}{(n-4)!}\int_{\mathfrak{t}}\Lambda^{(n)}(v)\left(\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta\right)dv,\end{aligned} \tag{67}$$

$$\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))\tilde{R}_{n-3}(\Theta,v)d\Theta\geq 0,\ \ \Theta\in\mathfrak{t}, \tag{68}$$

$$\begin{aligned}\breve{\mathbb{U}}\left(\Lambda(\cdot)\right) &\geq 2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_1)-\Lambda^{''}(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta,\end{aligned} \tag{69}$$

$$\begin{aligned}\breve{\mathbb{U}}\left(\Lambda(\cdot)\right) &\leq -2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_1)-\Lambda^{''}(\hat{\mathbb{b}}_2)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=3}^{n-1}\frac{\mathfrak{c}(\mathfrak{c}-1)}{(\mathfrak{c}-1)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta,\end{aligned} \tag{70}$$

$$\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))R_{n-3}(\Theta,v)d\Theta\geq 0,\ \ \Theta\in\mathfrak{t}, \tag{71}$$

$$\begin{aligned}\breve{\mathbb{U}}\left(\Lambda(\cdot)\right) &\geq 2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_2)-\Lambda^{''}(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))d\Theta+\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta,\end{aligned} \tag{72}$$

$$\begin{aligned}\breve{\mathbb{U}}\left(\Lambda(\cdot)\right) &\leq -2\left(\frac{\Lambda^{''}(\hat{\mathbb{b}}_2)-\Lambda^{''}(\hat{\mathbb{b}}_1)}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right)\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))d\Theta-\int_{\mathfrak{t}}\breve{\mathbb{U}}(G_k(\cdot,\Theta))\\ &\times\sum_{\mathfrak{c}=4}^{n-1}\frac{1}{(\mathfrak{c}-2)(\mathfrak{c}-4)!}\left[\frac{\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_1)(\Theta-\hat{\mathbb{b}}_1)^{\mathfrak{c}-2}-\Lambda^{(\mathfrak{c})}(\hat{\mathbb{b}}_2)(\Theta-\hat{\mathbb{b}}_2)^{\mathfrak{c}-2}}{\hat{\mathbb{b}}_2-\hat{\mathbb{b}}_1}\right]d\Theta,\end{aligned} \tag{73}$$

where

$$\breve{\mathbb{U}}(G_k(\cdot,\Theta)) = \sum_{\varepsilon=1}^{\hat{\varrho}}q_\varepsilon G_k(2\beta-x_\varepsilon,\Theta)-G_k\left(\sum_{\varepsilon=1}^{\hat{\varrho}}q_\varepsilon(2\beta-x_\varepsilon),\Theta\right)$$

$$-\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon G_k(x_\varepsilon, \Theta) + G_k\left(\sum_{\varepsilon=1}^{\hat{\varrho}} p_\varepsilon x_\varepsilon, \Theta\right), \tag{74}$$

Theorem 2.9. *Assume $\mathfrak{G}$ holds. Then, we have the subsequent:*

(*i*) *If n is odd then for $k = 1, 4$, the inequalities (69) and (72) hold and for $k = 2, 3$, (70) and (73) are true.*

(*ii*) *For $k = 1, 4$, let the inequalities (69) and (72) be satisfied and (48) is true,then*

$$\breve{\mathbb{U}}(\Lambda(.)) \geq 0. \tag{75}$$

or for $k = 2, 3$, (70) and (73) be satisfied and (50) is valid then

$$\breve{\mathbb{U}}(\Lambda(.)) \leq 0. \tag{76}$$

Proof. Proof is same as Theorem 2.3. □

Conclusion

The Montgomery identity, for the family of n-convex ($n \geq 3$) functions is used to generalize Bullen-type and Levinson-type inequalities (with real weights) by applying four 3-convex Green functions. Additionally, various inequalities and bounds are obtained for positive real weights by utilizing Montgomery identity. Future research may explore comparable results by using different entropies and divergences, such as Ciszár divergence, Shannon entropy, Bhattacharyya coefficient, Kullback-Leibler divergence and Triangular discrimination.

Authors contribution

AR initiated the work and made calculations. KAK supervised and validated the draft. JP dealt with the formal analysis and investigation. GP included the applications to information theory. All the authors read and approved the final manuscript.

References

[1] J. Pečarić, F. Proschan, Y.L. Tong, *Convex functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.

[2] N. Levinson, Generalization of an inequality of Kay Fan, *Journal of Mathematical Analysis and Applications* **6** (1969), 133-134.

[3] T. Popoviciu, Sur une inegalite de N. Levinson, *Mathematica (Cluj)* **6** (1969), 301-306.

[4] A. Aljinovič, J. Pečarić, A. Vukelič, On some Ostrowski type inequalities via Montgomery identity and Taylor's formula II, *Tamkang J. Math.* **36** (2005), no. 4, 279-301.

[5] D.S. Mitrinovič, J. Pečarić, A.M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[6] P.S. Bullen, An inequality of N. Levinson, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* (1973), 109-112.

[7] J. Pečarić, On an inequality on N. Levinson, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* (1980), 71-74.

[8] A.M. Mercer, 94.33 Short proofs of Jensen's and Levinson's Inequalities, *The Mathematical Gazette* **94** (2010), no. 531, 492-495.

[9] N. Mehmood, R.P. Agarwal, S.I. Butt, J. Pečarić, New generalizations of Popoviciu type inequalities via new Green's functions and Montgomery identity, *Journal of Inequalities and Applications* **2017** (2017), 1–21.

[10] M. Adeel, K.A. Khan, Đ. Pečarić, J. Pečarić, Levinson-type inequalities via new Green functions and Montgomery identity, *Open Mathematics* **18** (2020), no. 1, 632-652.

[11] A. Rasheed, K.A. Khan, J. Pečarić, Đ. Pečarić, Generalization of the Levinson inequality via new Green functions with applications to information theory, *Journal of Inequalities and Applications* **2023** (2023), no. 1, 124.

[12] A. Rasheed, K.A. Khan, J. Pečarić, Đ. Pečarić,. Generalizations of Levinson-type inequalities via new Green functions and Hermite interpolating polynomial, *Journal of inequalities and applications* **2024** (2024), no. 1, 70.

[13] M. Adeel, K.A. Khan, Đ. Pečarić, J. Pečarić, Generalization of the Levinson inequality with applications to information theory, *Journal of Inequalities and Applications* **2019**(2019), no. 1, 230.

[14] M. Adeel, K.A. Khan, Đ. Pečarić, J. Pečarić, Levinson type inequalities for higher order convex functions via Abel-Gontscharoff interpolation, *Advances in Difference equations* **2019** (2019), no. 1, 430.

[15] M. Adeel, K.A. Khan, Đ. Pečarić, J. Pečarić, Estimation of f-divergence and Shannon entropy by Levinson type inequalities via new green's functions and Lidstone polynomial, *Advances in Difference equations* **2020** (2020), no. 1, 27.

[16] M. Adeel, K.A. Khan, Đ. Pečarić, J. Pečarić, Estimation of f-divergence and Shannon entropy by using Levinson type Inequalities for higher order convex functions via Hermite interpolating polynomial, *Journal of Inequalities and Applications* **2020** (2020), no. 1, 137.

[17] M. Adeel, K.A. Khan, Đ. Pečarić, J. Pečarić, Estimation of f-divergence and Shannon entropy by Bullen type inequalities via Fink's identity, *Filomat* **36** (2022), no. 2, 527-538.

[18] M. Adeel, K.A. Khan, Đ. Pečarić, J. Pečarić, Entropy results for Levinson-type inequalities via Green functions and Hermite interpolating polynomial, *Aequationes mathematicae* **96** (2022), no. 1, 1-16.

[19] M. Adeel, K.A. Khan, Đ. Pečarić, J. Pečarić, Estimation of f-divergence and shannon entropy by levinson type inequalities via lidstone interpolating polynomial, *Transactions of A. Razmadze Mathematical Institute* **175** (2021), no. 1, 1-11.

[20] A. Rasheed, K.A. Khan, J. Pečarić, Đ. Pečarić, Estimations of Levinson-type inequalities using novel 3-convex Green functions with Taylor's formula,*Journal of Applied Analysis* **31** (2024), no. 2.

[21] M. Bilal, K.A. Khan, A. Nosheen, J. Pečarić, Generalizations of Shannon type inequalities via diamond integrals on time scales, *Journal of Inequalities and Applications* **2023** (2023), no. 1, 24.

[22] M. Bilal, K.A. Khan, A. Nosheen, J. Pečarić, Some inequalities related to Csiszár divergence via diamond integral on time scales, *Journal of Inequalities and Applications* **2023** (2023), no. 1, 55.

(Awais Rasheed) Department of Mathematics, Govt. Graduate College Sillanwali Sargodha 40100 Pakistan
E-mail address: awais10mathematicians@gmail.com

(Khuram Ali Khan) Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan
E-mail address: khuramsms@gmail.com

(Josip Pečarić) Croatian Academy of Science, Zagreb, Croatia
E-mail address: jopecaric@gmail.com

(Đilda Pečarić) Department of Communication, Media and Journalism University North, Trg Dr. Žarka Dolinara 1, Koprivnica, Croatia
E-mail address: gildapeca@gmail.com