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Classes of residuated lattices

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Abstract. The commutative residuated lattices were first introduced by M. Ward and R.P. Dilworth as generalization of ideal lattices of rings. Non-commutative residuated lattices, called sometimes pseudo-residuated lattices, biresiduated lattices or generalized residuated lattices are algebraic counterpart of substructural logics, that is, logics which lack some of the three structural rules, namely contraction, weakening and exchange. Complete studies on residuated lattices were developed by H. Ono, T. Kowalski, P. Jipsen and C. Tsinakis. The aim of this paper is to study some special classes of residuated lattices, such as local, perfect and Archimedean residuated lattices. As an important result of the paper we prove that, generally, the Archimedean residuated lattices are not commutative. Additionally, we study some properties of the lattice of filters of a residuated lattice.

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1. Introduction

The concept of a commutative residuated lattice was firstly introduced by M. Ward and R.P. Dilworth ([24]) as generalization of ideal lattices of rings. The properties of a commutative residuated lattice were presented in [21] and the lattice of filters of a commutative residuated lattice was investigated in [22]. A. Di Nola, G. Georgescu and A. Iorgulescu introduced in [6] and [7] the *pseudo BL-algebras* as non-commutative extension of Hájek's BL-algebras.
Pseudo BL-algebra is an algebra $\mathbf{L} = (L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ of the type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:
(L_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(L_2) $(L, \odot, 1)$ is a non-commutative monoid;
(L_3) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in L$ (pseudo-residuation) ;
(L_4) $(x \to y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ (pseudo divisibility) ;
(L_5) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ (pseudo prelinearity).
If the algebra $\mathbf{L}$ satisfies (L_1), (L_2), (L_3) and (L_5), then $\mathbf{L}$ is called *pseudo MTL-algebra* (also called *weak pseudo BL-algebra*). These structures were studied by P. Flondor, G. Georgescu and A. Iorgulescu in [12] and [17].
If the algebra $\mathbf{L}$ satisfies (L_1), (L_2), (L_3) and (L_4), then $\mathbf{L}$ is called *Rℓ-monoid* and it was investigated by A. Dvurečenskij and J. Rachnek in [10] and [11].
An algebra $\mathbf{L} = (L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ satisfying (L_1), (L_3) and (L_3) is a non-commutative residuated lattices, called sometimes *pseudo-residuated lattice*, *biresiduated lattice* or *generalized residuated lattice*. Studies on these structures were deeply

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developed in [1], [18] and [19]. In this paper we will investigate some classes of residuated lattices, such as local, perfect and Archimedean residuated lattices. Additionally, we study some properties of the lattice of filters of a residuated lattice.

2. Residuated lattices and their basic properties

Definition 2.1. ([1]) *A residuated lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of the type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions:
(L_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(L_2) $(L, \odot, 1)$ is a non-commutative monoid;
(L_3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in L$.
L is called *commutative* if the operation $\odot$ is commutative.

In the sequel we will agree that the operations $\wedge, \vee, \odot$ have higher priority than the operations $\rightarrow, \rightsquigarrow$.

Example 2.2. Let's take $L = \{0, a, b, c, 1\}$ where $0 < a < b, c < 1$ and b, c are incomparable. Consider the operations $\odot, \rightarrow, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	a	b	a	b
c	0	0	0	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

.

Then $\mathbf{L} = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a residuated lattice.

Example 2.3. Let's consider a *pseudo-MV algebra* $\mathbf{M} = (M, \oplus, -, \sim, 0, 1)$ with the additional operation $x \odot y = (y^- \oplus x^-)^\sim$.
The order on M is defined by $x \leq y$ iff $x^- \oplus y = 1$ (iff $y \oplus x^\sim = 1$).
Defining $x \wedge y = x \odot (x^- \oplus y)$ and $x \vee y = x \oplus x^\sim \odot y$, according to [14], (Proposition 1.13), $(M, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.
Applying [14] (Proposition 1.7), $(M, \odot, 1)$ is a non-commutative monoid.
If we define $x \rightarrow y = y \oplus x^\sim$ and $x \rightsquigarrow y = x^- \oplus y$, then according to [14] (Proposition 1.12) we have $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$.
Thus, $\mathbf{M} = (M, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded residuated lattice.

Remark 2.4. (1) If additionally for any $x, y \in L$ the structure $\mathbf{L}$ satisfies the axioms:
(L_4) $(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ (pseudo divisibility) ;
(L_5) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ (pseudo prelinearity)
then $\mathbf{L}$ is a *pseudo-BL algebra.*
(2) If $\mathbf{L}$ satisfies the conditions $(L_1), (L_2), (L_3)$ and (L_5), then it is a *weak pseudo-BL algebra* (or *pseudo-MTL algebra*).
(3) If $\mathbf{L}$ satisfies the conditions $(L_1), (L_2), (L_3)$ and (L_4), then it is a *bounded Rℓ-monoid* or *divisible residuated lattice* ([11]).

One can easy prove that the structure $\mathbf{L} = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ from Example 2.2 is neither pseudo BL-algebra, nor Rℓ-monoid, because

$$(b \rightarrow a) \odot b \neq b \odot (b \rightsquigarrow a),$$

so (L_4) doesn't hold.

Remark 2.5. (1) A residuated lattice is *commutative* iff $x \to y = x \rightsquigarrow y$ for any x, y ;
(2) Let $(L, \wedge, \vee, \odot, 0, 1)$ be a structure satisfying conditions (L_1) and (L_2). If exists a pair $(\to, \rightsquigarrow)$ of operations such that $(L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice, then the pair is unique ;
(3) If a residuated lattice is a chain, then it is a weak pseudo-BL algebra.

In a residuated lattice $\mathbf{L} = (L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ we define for all $x \in L$:

$$x^{-} = x \to 0 \text{ and } \mathrm{x}^{\sim} = \mathrm{x} \rightsquigarrow 0.$$

In the sequel we will refer to the residuated lattice $\mathbf{L}$ by its universe L.
The following proposition provides some rules of calculus in a residuated lattice (see [1], [3], [6], [17]), [18], [21]).

Proposition 2.6. *In any residuated lattice the following rules of calculus hold:*
(c_1) $x \to (y \to z) = (x \odot y) \to z$;
(c_2) $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$;
(c_3) $x \leq y$ *iff* $x \to y = 1$ *iff* $x \rightsquigarrow y = 1$;
(c_4) $x \to x = x \rightsquigarrow x = 1$;
(c_5) $x \to 1 = x \rightsquigarrow 1 = 1$;
(c_6) $0 \to x = 0 \rightsquigarrow x = 1$;
(c_7) $x \odot 0 = 0 \odot x = 0$;
(c_8) $x \odot y \leq x \wedge y$;
(c_9) $(x \to y) \odot x \leq y$ *and* $x \odot (x \rightsquigarrow y) \leq y$;
(c_{10}) $x \leq y \to (x \odot y)$ *and* $x \leq y \rightsquigarrow (y \odot x)$;
(c_{11}) $x \leq y$ *implies* $x \odot z \leq y \odot z$ *and* $z \odot x \leq z \odot y$ *for any* $z \in L$;
(c_{12}) $(x \to y) \odot x \leq x \wedge y$ *and* $x \odot (x \rightsquigarrow y) \leq x \wedge y$;
(c_{13}) $(x \to y) \odot x \leq x \leq y \to (x \odot y)$ *and* $(x \to y) \odot x \leq y \leq x \to (y \odot x)$;
(c_{14}) $x \odot (x \rightsquigarrow y) \leq y \leq x \rightsquigarrow (x \odot y)$ *and* $x \odot (x \rightsquigarrow y) \leq x \leq y \rightsquigarrow (y \odot x)$;
(c_{15}) *If* $x \leq y$ *then* $z \to x \leq z \to y$ *and* $z \rightsquigarrow x \leq z \rightsquigarrow y$;
(c_{16}) *If* $x \leq y$ *then* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;
(c_{17}) $1 \to x = x$ *and* $1 \rightsquigarrow x = x$;
(c_{18}) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$;
(c_{19}) $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;
(c_{20}) $x \to y \leq (z \to x) \rightsquigarrow (z \to y)$;
(c_{21}) $x \rightsquigarrow y \leq (z \rightsquigarrow x) \to (z \rightsquigarrow y)$;
(c_{22}) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;
(c_{23}) $x \to (x \rightsquigarrow y) = x \rightsquigarrow (x \to y)$;
(c_{24}) $x \to y = x \to (x \wedge y)$;
(c_{25}) $x \rightsquigarrow y = x \rightsquigarrow (x \wedge y)$;
(c_{26}) $y \leq x \to y$ *and* $y \leq x \rightsquigarrow y$;
(c_{27}) *If* $x \leq y$ *then* $x \leq z \to y$ *and* $x \leq z \rightsquigarrow y$;
(c_{28}) $z \odot (x \wedge y) \leq (z \odot x) \wedge (z \odot y)$;
(c_{29}) $(x \wedge y) \odot z \leq (x \odot z) \wedge (y \odot z)$;
(c_{30}) $x \to y \leq (x \odot z) \to (y \odot z)$;
(c_{31}) $x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y)$;
(c_{32}) $(x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leq x \rightsquigarrow z$;
(c_{33}) $x \odot (y \to z) \leq y \to (x \odot z)$;
(c_{34}) $(y \rightsquigarrow z) \odot x \leq y \rightsquigarrow (z \odot x)$;
(c_{35}) $(y \to z) \odot (x \to y) \leq x \to z$;
(c_{36}) $(x_1 \rightsquigarrow x_2) \odot (x_2 \rightsquigarrow x_3) \odot ... \odot (x_{n-1} \rightsquigarrow x_n) \leq x_1 \rightsquigarrow x_n$;

(c_{37}) $(x_{n-1} \to x_n) \odot (x_{n-2} \to x_{n-1}) \odot ... \odot (x_2 \to x_3) \odot (x_1 \to x_2) \le x_1 \to x_n$;
(c_{38}) $(x \to y) \odot (x' \to y') \le (x \vee x') \to (y \vee y')$;
(c_{39}) $(x \rightsquigarrow y) \odot (x' \rightsquigarrow y') \le (x \vee x') \rightsquigarrow (y \vee y')$;
(c_{40}) $(x \to y) \odot (x' \to y') \le (x \wedge x') \to (y \wedge y')$;
(c_{41}) $(x \rightsquigarrow y) \odot (x' \rightsquigarrow y') \le (x \wedge x') \rightsquigarrow (y \wedge y')$;
(c_{42}) $1^- = 1^\sim = 0$ *and* $0^- = 0^\sim = 1$;
(c_{43}) $x^- \odot x = 0$ *and* $x \odot x^\sim = 0$;
(c_{44}) $x \le y^-$ *iff* $x \odot y = 0$ *and* $x \le y^\sim$ *iff* $y \odot x = 0$;
(c_{45}) $x \le x^{-\sim}$ *and* $x \le x^{\sim -}$;
(c_{46}) $x \to y^- = (x \odot y)^-$ *and* $x \rightsquigarrow y^\sim = (y \odot x)^\sim$;
(c_{47}) $x \le y^-$ *iff* $y \le x^\sim$;
(c_{48}) *If* $x \le y$, *then* $y^- \le x^-$ *and* $y^\sim \le x^\sim$;
(c_{49}) $x \le x^\sim \to y$ *and* $x \le x^- \rightsquigarrow y$;
(c_{50}) $x \to y \le y^- \rightsquigarrow x^-$ *and* $x \rightsquigarrow y \le y^\sim \to x^\sim$;
(c_{51}) $x \to y^\sim = y \rightsquigarrow x^-$ *and* $x \rightsquigarrow y^- = y \to x^\sim$;
(c_{52}) $x^{\sim -\sim} = x^\sim$ *and* $x^{-\sim -} = x^-$;
(c_{53}) $x \to x^\sim = x \rightsquigarrow x^-$;
(c_{54}) $x \odot (\vee_{i\in I} y_i) = \vee_{i\in I}(x \odot y_i)$;
(c_{55}) $(\vee_{i\in I} y_i) \odot x = \vee_{i\in I}(y_i \odot x)$;
(c_{56}) $(\vee_{i\in I} x_i) \rightsquigarrow y = \wedge_{i\in I}(x_i \rightsquigarrow y)$;
(c_{57}) $(\vee_{i\in I} x_i) \to y = \wedge_{i\in I}(x_i \to y)$;
(c_{58}) $y \rightsquigarrow (\wedge_{i\in I} x_i) = \wedge_{i\in I}(y \rightsquigarrow x_i)$;
(c_{59}) $y \to (\wedge_{i\in I} x_i) = \wedge_{i\in I}(y \to x_i)$;
(c_{60}) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$;
(c_{61}) $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$;
(c_{62}) $(x \vee y) \to (x \wedge y) = (x \to y) \wedge (y \to x)$;
(c_{63}) $(x \vee y) \rightsquigarrow (x \wedge y) = (x \rightsquigarrow y) \wedge (y \rightsquigarrow x)$;
(c_{64}) $(x \vee y)^- = x^- \wedge y^-$ *and* $(x \vee y)^\sim = x^\sim \wedge y^\sim$;
(c_{65}) $(x \wedge y)^- \ge x^- \vee y^-$ *and* $(x \wedge y)^\sim \ge x^\sim \vee y^\sim$;
(c_{66}) $(x \vee y)^{-\sim} \ge x^{-\sim} \vee y^{-\sim}$ *and* $(x \vee y)^{\sim -} \ge x^{\sim -} \vee y^{\sim -}$;
(c_{67}) $y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim} = x \to y^{-\sim}$;
(c_{68}) $y^\sim \to x^\sim = x^{\sim -} \rightsquigarrow y^{\sim -} = x \rightsquigarrow y^{\sim -}$;
(c_{69}) *If* $x \vee y = 1$, *then for each* $n \in N$, $n \ge 1$, $x^n \vee y^n = 1$.

Corollary 2.7. *For any* $g, h, k \in L$ *we have*
(c_{70}) $g \vee (h \odot k) \ge (g \vee h) \odot (g \vee k)$.

Proof.

$$\begin{aligned}(g \vee h) \odot (g \vee k) &= ((g \vee h) \odot g) \vee ((g \vee h) \odot k) = \\ &= ((g \odot g) \vee (h \odot g)) \vee ((g \odot k) \vee (h \odot k)) \le g \vee g \vee (g \vee (h \odot k)) = \\ &= g \vee (h \odot k).\end{aligned}$$

□

Corollary 2.8. (c_{71}) $g \vee (h_1 \odot h_2 \odot ...h_n) \ge (g \vee h_1) \odot (g \vee h_2) \odot ... \odot (g \vee h_n)$.

Corollary 2.9. (c_{72}) $g \vee h^n \ge (g \vee h)^n$.

Proposition 2.10. *In any residuated lattice the following properties hold:*
(c_{73}) $x \vee y \le [(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x]$;
(c_{74}) $x \vee y \le [(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x]$;

(c_{75}) $x \vee y \leq [(x \to y) \rightsquigarrow y] \wedge [(y \to x) \rightsquigarrow x]$;
(c_{76}) $x \vee y \leq [(x \rightsquigarrow y) \to y] \wedge [(y \rightsquigarrow x) \to x]$.

Proof. (c_{73}) : From $(x \to y) \odot x \leq y$ we have $x \leq (x \to y) \rightsquigarrow y$.
Taking in consideration that $y \leq (x \to y) \rightsquigarrow y$, we get

$$x \vee y \leq (x \to y) \rightsquigarrow y.$$

Similarly, $x \vee y \leq (y \rightsquigarrow x) \to x$.
Thus, $x \vee y \leq [(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x]$.
(c_{74}),(c_{75}),(c_{76}) : Similarly as (c_{73}). □

Corollary 2.11. *If $x \vee y = 1$, then $x \to y = x \rightsquigarrow y = y$.*

Corollary 2.12. *If $x \vee y = 1$ for any $x, y \in L$, then L is commutative.*

The next result extends to residuated lattices a property presented in [9] for Rℓ-monoids and it can provide other examples of residuated lattices.

Proposition 2.13. *Let $L = (L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ be a residuated latice. Then, the algebra $L'_a = ([a, 1], \odot_a^1, \wedge, \vee, \to, \rightsquigarrow, a, 1)$ is a residuated lattice, where*
$x \odot_a^1 y = (x \odot y) \vee a$.

Proof. We will check conditions $(L_1) - (L_3)$ from the definition of a residuated lattice:
(L_1) Obviously, $([a, 1], \wedge, \vee, a, 1)$ is a bounded lattice with the smallest element a and the greatest element 1;
(L_2) Since $x \odot_a^1 1 = (x \odot 1) \vee a = x \vee a = x$ and $1 \odot_a^1 x = (1 \odot x) \vee a = x \vee a = x$, it follows that 1 is the unity element in $([a, 1], \odot_a^1, 1)$.
For any $x, y, z \in [a, 1]$ we have:

$$\begin{aligned} (x \odot_a^1 y) \odot_a^1 z &= (((x \odot y) \vee a) \odot z) \vee a \\ &= (x \odot y \odot z) \vee (a \odot z)) \vee a = \\ &= (x \odot y \odot z) \vee ((a \odot z) \vee a) = \\ &= (x \odot y \odot z) \vee a. \end{aligned}$$

$$\begin{aligned} x \odot_a^1 (y \odot_a^1 z) &= (x \odot (y \odot z) \vee a) \vee a \\ &= ((x \odot y \odot z) \vee (x \odot a)) \vee a = \\ &= (x \odot y \odot z) \vee ((x \odot a) \vee a) \\ &= (x \odot y \odot z) \vee a. \end{aligned}$$

Thus, $([a, 1], \odot_a^1, 1)$ is a monoid.
(L_3) $x \odot_a^1 y \leq z \Rightarrow (x \odot y) \vee a \leq z \Rightarrow x \odot y \leq z \Rightarrow x \leq y \to z$ and $y \leq x \rightsquigarrow z$.
Conversely, $x \leq y \to z \Rightarrow x \odot y \leq z$ and considering that $a \leq z$ we get
$(x \odot y) \vee a \leq z \vee a = z \Rightarrow x \odot_a^1 y \leq z$.
Similarly, from $y \leq x \rightsquigarrow z$ we get $x \odot_a^1 y \leq z$.
We conclude that L_a^1 is a residuated lattice.
Trivially, $L = L_0^1$ and $\{1\} = L_1^1$. □

Definition 2.14. A residuated lattice L is called *good* if $x^{-\sim} = x^{\sim -}$ for any $x \in L$.

Remark 2.15. In [13] it was formulated as open problem to find a pseudo-BL algebra which is not good. In the case of a residuated lattice we have a such example.
Indeed, the residuated lattice in Example 2.2 is not good :

$$a^{-\sim} = c^{\sim} = b, \text{ but } \mathrm{a}^{\sim -} = \mathrm{b}^{-} = \mathrm{c}.$$

Proposition 2.16. *In any good residuated lattice the following properties hold:*
(c_{77}) $(x^{\sim} \odot y^{\sim})^{-} = (x^{-} \odot y^{-})^{\sim}$;
(c_{78}) $x^{-\sim} \odot y^{-\sim} \leq (x \odot y)^{-\sim}$.

Proof. (c_{77}) : Applying (c_{46}),(c_{50}),(c_{51}) we have:

$$\begin{aligned}(x^{\sim} \odot y^{\sim})^{-} &= x^{\sim} \to y^{\sim-} = x^{\sim} \to y^{-\sim} = y^{-\sim-} \rightsquigarrow x^{\sim-} \\ &= y^{-} \rightsquigarrow x^{\sim-} = y^{-} \rightsquigarrow x^{-\sim} = (x^{-} \odot y^{-})^{\sim}.\end{aligned}$$

(In the last equality we also applied (c_{46})).
(c_{78}) : Because the residuated lattice is good and by (c_9), we have:

$$\begin{aligned}(x \odot y)^{-\sim} &= (x \odot y)^{\sim-} \geq (x \odot y)^{\sim-} \wedge x^{\sim-} \geq x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \odot y)^{\sim-}) \\ &= x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \odot y)^{-\sim}) = x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \to y^{-}))^{\sim}.\end{aligned}$$

But, applying (c_{46}) and (c_2) we have:

$$\begin{aligned}&x^{\sim-} \rightsquigarrow (x \to y^{-})^{\sim} = \\ &x^{\sim-} \rightsquigarrow ((x \to y^{-}) \rightsquigarrow 0) = (x^{\sim-} \to y^{-}) \odot x^{\sim-} \rightsquigarrow 0 \\ &= ((x^{\sim-} \to y^{-}) \odot x^{\sim-})^{\sim} \geq (x^{\sim-} \wedge y^{-})^{\sim} \geq x^{\sim-\sim} \vee y^{-\sim} = x^{\sim} \vee y^{-\sim}.\end{aligned}$$

(By (c_{46}) we have $(x^{\sim-} \to y^{-}) \odot x^{\sim-} \leq (x^{\sim-} \wedge y^{-})$, so
$((x^{\sim-} \to y^{-}) \odot x^{\sim-})^{\sim} \geq (x^{\sim-} \wedge y^{-})^{\sim})$.
It follows that

$$\begin{aligned}(x \odot y)^{-\sim} &\geq x^{\sim-} \odot (x^{\sim} \vee y^{-\sim}) = (x^{\sim-} \odot x^{\sim}) \vee (x^{\sim-} \odot y^{-\sim}) \\ &= 0 \vee (x^{\sim-} \odot y^{-\sim}) = x^{\sim-} \odot y^{\sim-} = x^{-\sim} \odot y^{-\sim}.\end{aligned}$$

(we applied (c_{54}), (c_{55}) and (c_{43})). □

Proposition 2.17. *Let L be a good residuated lattice. We define a binary operation $\oplus$ on L by $x \oplus y := (y^{\sim} \odot x^{\sim})^{-}, x, y \in L$. Then for all $x, y, z \in L$ we have:*
(c_{79}) $x \oplus y := (y^{-} \odot x^{-})^{\sim}$;
(c_{80}) $\oplus$ *is associative;*
(c_{81}) $x, y \leq x \oplus y$;
(c_{82}) $x \oplus 0 = x^{-\sim} = 0 \oplus x$;
(c_{83}) $x \oplus 1 = 1 = 1 \oplus x$;
(c_{84}) $x \oplus y = x^{-} \rightsquigarrow y^{-\sim} = y^{\sim} \to x^{-\sim}$.

Proof. (c_{79}) : It follows from (c_{77}).
(c_{80}) : We have :

$$\begin{aligned}(x \oplus y) \oplus z &= (y^{\sim} \odot x^{\sim})^{-} \odot z = (z^{\sim} \odot (y^{\sim} \odot x^{\sim})^{-\sim})^{-} \\ &= z^{\sim} \to (y^{\sim} \odot x^{\sim})^{-\sim-} = z^{\sim} \to (y^{\sim} \odot x^{\sim})^{-} = z^{\sim} \to (y^{\sim} \to x^{\sim-})\end{aligned}$$

(we applied (c_{46}) and (c_{52});

$$\begin{aligned}&x \oplus (y \oplus z) = x \oplus (z^{\sim} \odot y^{\sim})^{-} = ((z^{\sim} \odot y^{\sim})^{-\sim} \odot x^{\sim})^{-} \\ &= (z^{\sim} \odot y^{\sim})^{-\sim} \to x^{\sim-} = (z^{\sim} \odot y^{\sim})^{-\sim} \to x^{-\sim} \\ &= (z^{\sim} \odot y^{\sim}) \to x^{-\sim} = z^{\sim} \to (y^{\sim} \to x^{-\sim}) = z^{\sim} \to (y^{\sim} \to x^{\sim-}) = (x \oplus y) \oplus z.\end{aligned}$$

(we applied (c_{50}) and (c_1));
(c_{81}) : By (c_{46}) and (c_{49}) we have:

$$x \oplus y = (y^{-} \odot x^{-})^{\sim} = x^{-} \rightsquigarrow y^{-\sim} \geq x$$

$$x \oplus y = (y^{\sim} \odot x^{\sim})^{-} = y^{\sim} \to x^{\sim-} \geq y;$$

(c_{82}) : Applying $(rl - c_{43})$ and $(rl - c_{17})$ we get:
$x \oplus 0 = (0^{\sim} \odot x^{\sim})^{-} = 0^{\sim} \to x^{-\sim} = 1 \to x^{-\sim} = x^{\sim-} = x^{-\sim}$;

$0 \oplus x = (x^- \odot 0^-)^\sim = x^- \rightsquigarrow 0^{-\sim} = x^- \rightsquigarrow 0 = x^{-\sim}$.
$(c_{83}) : x \oplus 1 = (1^- \odot x^-)^\sim = x^- \rightsquigarrow 1^{-\sim} = x^- \rightsquigarrow 1 = 1$ ((c_{46}) and (c_5));
$1 \oplus x = (x^- \odot 1^-)^\sim = 1^- \rightsquigarrow x^{-\sim} = 0 \rightsquigarrow x^{-\sim} = 1$ ((c_{46}) and (c_6)).
$(c_{84}) : x \oplus y = (y^\sim \odot x^\sim)^- = y^\sim \to x^{\sim -} = y^\sim \to x^{-\sim}$.
$x \oplus y = (y^- \odot x^-)^\sim = x^- \rightsquigarrow y^{-\sim}$.
It follows that $x \oplus y = x^- \rightsquigarrow y^{-\sim} = y^\sim \to x^{-\sim}$. □

Other properties of the operation " $\oplus$" can be established similarly as in the above proposition.

3. Lattice of filters of a residuated lattice

In this section we extend to a residuated lattice the results investigated in [6], [7], [3] and [15] for the case of lattice of filters of a pseudo BL-algebra.

Definition 3.1. Let L be a residuated lattice. A nonempty set F of L is called *filter* of L if the following conditions hold:
(F_1) If $x, y \in F$, then $x \odot y \in F$;
(F_2) If $x \in F$, $y \in L$, $x \leq y$ then $y \in F$.

Proposition 3.2. *([13]) If F is a filter of L then:*
(F_3) *$1 \in F$;*
(F_4) *If $x \in F$, $y \in L$ then $y \to x \in F$, $y \rightsquigarrow x \in F$;*
(F_5) *If $x, y \in F$, then $x \wedge y \in F$.*

Proposition 3.3. *([13]) For a subset F of L the following are equivalent:*
(i) *F is a filter;*
(ii) *$1 \in F$ and if $x, x \to y \in F$, then $y \in F$;*
(iii) *$1 \in F$ and if $x, x \rightsquigarrow y \in F$, then $y \in F$.*

A filter F of L is *proper* if $F \neq L$. Clearly, F is a proper filter iff $0 \notin F$.

Proposition 3.4. *If L is a residuated lattice, then the sets*

$$L_0^- = \{x \in L \mid^- = 0\} \,\text{and}\, L_0^\sim = \{x \in L \mid x^\sim = 0\}$$

are proper filters of L.

Proof. If $x, y \in L_0^-$, then $(x \odot y)^- = x \to y^- = x \to 0 = x^- = 0$, so $x \odot y \in L_0^-$.
If $x \in L_0^-, y \in A$ such that $x \leq y$, then $y^- \leq x^- = 0$, so $y^- = 0$, that is $y \in L_0^-$.
Because $0 \notin L_0^-$, we conclude that L_0^- is a proper filter of L.
Similarly for the case of $L_0^\sim$. □

We will denote by $\mathcal{F}_s(L)$ the set of filter of L.

Definition 3.5. For every subset $X \subseteq L$, the smallest filter of L containing X (i.e. the intersection of all filters $F \in \mathcal{F}_s(L)$ such that $X \subseteq F$) is called the filter *generated by* X and will be denoted by $[X)$.

For any $n \in \mathbb{N}$, $x \in L$ we put $x^0 = 1$ and $x^{n+1} = x^n \odot x = x \odot x^n$. The *order* of $x \in L$, denoted $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$.
If there is no such n, then $ord(x) = \infty$.

Definition 3.6. A residuated lattice L is *locally finite* if for any $x \in L$, $x \neq 1$ implies $ord(x) < \infty$.

Example 3.7. Consider the residuated lattice L from Example 2.2.
Since $ord(b) = \infty$ and $ord(c) = \infty$, it follows that L is not locally finite.

Example 3.8. Let's consider $L = \{0, a, b, c, 1\}$ where $0 < a < b < c < 1$ and the operations $\odot, \rightarrow, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	a	a	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

.

Then $\mathbf{L} = (L, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a residuated lattice and we have :

$$ord(0) = 1, \ \ ord(a) = 2, \ \ ord(b) = 2, \ \ ord(c) = 3.$$

Thus, L is a locally finite residuated lattice.

Remark 3.9. (1) In [20] it is proved that every locally finite pseudo MV-algebra is commutative ;
(2) In [15] it is proved that every locally finite pseudo BL-algebra is a MV-algebra, so it is commutative ;
(3) By the above example we have proved that there exist non-commutative locally finite residuated lattices.

Lemma 3.1. *([15]) Let L be a residuated lattice and $x, y \in L$. Then :*
(1) $[x)$ is proper iff $ord(x) = \infty$;
(2) if $x \leq y$ and $ord(y) < \infty$, then $ord(x) < \infty$;
(3) if $x \leq y$ and $ord(x) = \infty$, then $ord(y) = \infty$.

Proposition 3.10. *([6]) If $X \subseteq L$, then*

$$[X) = \{y \in L \mid y \geq x_1 \odot x_2 \odot \cdots \odot x_n \, for\, some\, n \geq 1\, and\, x_1, x_2, \ldots, x_n \in X\}.$$

Proposition 3.11. *([6]) If $X \subseteq L$, then*

$$\begin{aligned}[X) &= \{y \in L \mid x_1 \rightarrow (x_2 \rightarrow (\ldots (x_n \rightarrow y)) \ldots) \\ &= 1 \,\text{for some}\ n \geq 1 \,\text{and}\, x_1, x_2, \ldots, x_n \in X\} = \\ &= \{y \in L \mid x_1 \rightsquigarrow (x_2 \rightsquigarrow (\ldots (x_n \rightsquigarrow y)) \ldots) \\ &= 1 \,\text{for some}\ n \geq 1 \,\text{and}\, x_1, x_2, \ldots, x_n \in X\}.\end{aligned}$$

Remark 3.12. The following results proved in ([6]) for a pseudo BL-algebra are also valid in the case of a residuated lattice :
(1) If X is a filter of L, then $[X) = X$;
(2) If $X = \{x\}$ we write $[x)$ instead of $[\{x\})$ and $[x) = \{y \in X \mid y \geq x^n \,\text{for some}\, n \geq 1\}$. $[x)$ is called *principal* filter.
(3) If F is a filter of L and $x \in L$, then
$F(x) = [F \cup \{x\}) = \{y \in L \mid y \geq (f_1 \odot x^{n_1}) \odot (f_2 \odot x^{n_2}) \odot \cdots \odot (f_m \odot x^{n_m})$ for some $m \geq 1, \ n_1, n_2, ..., n_m \geq 0, \ f_1, f_2, ..., f_m \in F\}$.

Proposition 3.13. *([3], [6]) In any residuated lattice the following hold:*
(1) If F is a filter of L and $x \in L \setminus F$, then $F(x) = F \vee [x)$;
(2) $[x)$ is a proper filter iff $ord(x) = \infty$;
(3) $[x \vee y) = [x) \cap [y)$;

(4) *If* $x \leq y$, *then* $[y) \subseteq [x)$;
(5) $[x) \vee [y) = [x \vee y) = [x \odot y)$;
(6) $[x \odot y) = [y \odot x)$;
(7) $[x \to y) \vee [x) = [x \rightsquigarrow y) \vee [x)$.

Proposition 3.14. *(*[3]*) If* F_1, F_2 *are nonempty sets of* L *such that* $1 \in F_1 \cap F_2$ *then* $[F_1 \cup F_2) = \{x \in L \mid x \geq (f_1 \odot f_1') \odot (f_2 \odot f_2') \odot \cdots \odot (f_n \odot f_n') \,\text{for some}\, n \geq 1, f_1, f_2, \ldots, f_n \in F_1,\ f_1', f_2', \ldots, f_n' \in F_2\}$

Corollary 3.15. *(*[3]*) If* F_1, F_2 *are filters of* L*, then* $[F_1 \cup F_2) = \{x \in Lx \geq (f_1 \odot f_1') \odot (f_2 \odot f_2') \odot \cdots \odot (f_n \odot f_n') \,\text{for some}\, n \geq 1, f_1, f_2, \ldots, f_n \in F_1,\ f_1', f_2', \ldots, f_n' \in F_2\}$

If $(F_i)_{i \in I}$ is a family of all filters of L, we define $\wedge_{i \in I} F_i = \cap_{i \in I} F_i$ and $\vee_{i \in I} F_i = [\cup_{i \in I} F_i)$

Definition 3.16. ([3], [16]) A complete lattice $(L, \wedge, \vee)$ is called *Browerian* if it satisfies the identity $a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$; the element a of L is called *compact* if $a \leq \vee X$ for some $X \subseteq L$ implies $a \leq \vee X_1$ for some finite subset $X_1 \subseteq X$. A complete lattice is called *algebraic* if every element is the join of compact elements of L.

Theorem 3.17. *(*[3], [16]*)* $(\mathcal{F}_s(L), \wedge, \vee)$ *is a complete Browerian algebraic lattice, the compact elements being exactly the principal filter of* L.

Definition 3.18. ([7]) A filter H of L is called *normal* if for any $x, y \in L$,
(N) $x \to y \in H$ iff x $\rightsquigarrow$ y $\in$ H.
We denote by $\mathcal{F}_n(L)$ the set of all normal filters of L.

Example 3.19. Consider $L = \{0, a, b, c, 1\}$ where $0 < a < b < c < 1$ and the operations $\odot, \to, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	a	a	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $\mathbf{L} = (L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice and $H = \{b, c, 1\}$ is a normal filter of L.

Remark 3.20. ([7]) Let H be a normal filter of L. Then:
(1) $x^- \in H$ iff $x^\sim \in H$;
(2) $x \in H$ implies $(x^-)^- \in H$ and $(x^\sim)^\sim \in H$.

Remark 3.21. In case of a pseudo BL-algebra A, it is proved in [13] that a filter H is normal if and only if $x \odot H = H \odot x$ for any $x \in A$. This equality doesn't hold in the case of residuated latices as we can see in the next example.

Example 3.22. Consider $L = \{0, a, b, c, 1\}$ where $0 < a < b < c < 1$ and the operations $\odot, \to, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	a	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

.

Then $\mathbf{L} = (L, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a residuated lattice and $H = \{a, b, c, 1\}$ is a normal filter of L.
One can see that $c \odot H = \{a, b, c\}$ and $H \odot c = \{a, c\}$, so $c \odot H \neq H \odot c$.

Lemma 3.2. *Let H be a normal filter of L. Then:*
(1) *For any $x \in L$ and $h \in H$ there is $h' \in H$ such that $x \odot h \geq h' \odot x$;*
(2) *For any $x \in L$ and $h \in H$ there is $h' \in H$ such that $h \odot x \geq x \odot h'$.*

Proof. (1) Let $y = x \odot h$. Then $x \odot h = y = x \wedge y \geq (x \to y) \odot x$.
But $h \leq x \rightsquigarrow x \odot h = x \rightsquigarrow y$. Since $h \in H$, it follows that $x \rightsquigarrow y \in H$.
Because H is a normal filter we have $h' = x \to y \in H$. Thus $x \odot h \geq h' \odot x$.
(2) Let $y = h \odot x$. Then $h \odot x = y = x \wedge y \geq x \odot (x \rightsquigarrow y)$. But $h \leq x \to h \odot x = x \to y$.
Since $h \in H$, it follows that $x \to y \in H$.
Because H is a normal filter we have $h' = x \rightsquigarrow y \in H$.
Thus $h \odot x \geq x \odot h'$. □

Proposition 3.23. *Let H be a normal filter of L and $x \in L$. Then*

$$\begin{aligned}[H \cup \{x\}) &= \{y \in L \mid y \geq h \odot x^n \,\text{for some}\, n \in \mathbb{N},\, h \in H\} \\ &= \{y \in L \mid y \geq x^n \odot h \,\text{for some}\, n \in \mathbb{N},\, h \in H\} \\ &= \{y \in L \mid x^n \to y \in H \,\text{for some}\, n \geq 1\} \\ &= \{y \in L \mid x^n \rightsquigarrow y \in H \,\text{for some}\, n \geq 1\}.\end{aligned}$$

Proof. Let $y \in [H \cup \{x\})$. Then $y \geq (h_1 \odot x^{n_1}) \odot (h_2 \odot x^{n_2}) \odot \cdots \odot (h_m \odot x^{n_m})$ for some $m \geq 1$, $n_1, n_2, \ldots, n_m \geq 0$, $h_1, h_2, \ldots, h_m \in F$.
If $m = 1$ then $y \geq h_1 \odot x^{n_1}$ and we take $h = h_1$ and $n = n_1$.
If $m = 2$ then $y \geq (h_1 \odot x^{n_1}) \odot (h_2 \odot x^{n_2}) = h_1 \odot (x^{n_1} \odot h_2) \odot x^{n_2}$.
According to the above lemma, there is $h_2' \in H$ such that $x^{n_1} \odot h_2 \geq h_2' \odot x^{n_1}$.
Hence $y \geq h_1 \odot (h_2' \odot x^{n_1}) \odot x^{n_2} = (h_1 \odot h_2') \odot x^{n_1+n_2}$ and we take $h = h_1 \odot h_2'$ and $n = n_1 + n_2$.
By induction we get $y \geq h \odot x^n$ for some $n \in \mathbb{N}$, $h \in H$.
Similarly $y \geq x^n \odot h$ for some $n \in \mathbb{N}$, $h \in H$.
Thus $[H \cup \{x\}) = \{y \in L \mid y \geq h \odot x^n \,\text{for some}\, n \in \mathbb{N}, h \in H\} = \{y \in L \mid y \geq x^n \odot h \,\text{for some}\, n \in \mathbb{N}, h \in H\}$. If $y \in [H \cup \{x\})$, then $h \odot x^n \leq y$ for some $n \geq 1$, $h \in H$. Thus $h \leq x^n \to y$, hence $x^n \to y \in H$.
Conversely, assume that $h = x^n \to y \in H$ for some $n \geq 1$.
We also have $(h \odot x^n) \to y = h \to (x^n \to y) = h \to h = 1$, hence $h \odot x^n \leq y$.
Therefore, $y \in [H \cup \{x\})$ and we conclude that

$$[H \cup \{x\}) = \{y \in L \mid x^n \to y \in H \,\text{for some}\, n \geq 1\}.$$

Similarly $[H \cup \{x\}) = \{y \in L \mid x^n \rightsquigarrow y \in H \,\text{for some}\, n \geq 1\}$. □

Proposition 3.24. *If $F_1, F_2 \in \mathcal{F}_n(L)$ then*

$$[F_1 \cup F_2) = \{x \in L \mid x \geq u \odot v \,\text{for some}\, u \in F_1, v \in F_2\}.$$

Proof. We have $[F_1 \cup F_2) = \{x \in L \mid x \geq (f_1 \odot f_1') \odot (f_2 \odot f_2') \odot \cdots \odot (f_n \odot f_n')$ for some $n \geq 1, f_1, f_2, \ldots, f_n \in F_1, f_1', f_2', \ldots, f_n' \in F_2\}$.
Put $f = (f_1 \odot f_1') \odot (f_2 \odot f_2') \odot \cdots \odot (f_n \odot f_n') = f_1 \odot (f_1' \odot f_2) \odot \cdots \odot (f_{n-1}' \odot f_n) \odot f_n'$.
There is $f_2'' \in F_2$ such that $f_1' \odot f_2 \geq f_2 \odot f_2''$. Hence

$$f \geq f_1 \odot f_2 \odot (f_2'' \odot f_3) \odot \cdots \odot (f_n \odot f_n').$$

Similarly, there is $f_3'' \in F_2$ such that $f_2'' \odot f_3 \geq f_3 \odot f_3''$ so

$$f \geq f_1 \odot f_2 \odot f_3 \odot (f_3'' \odot f_4) \odot \cdots \odot (f_n \odot f_n').$$

Finally $f \geq f_1 \odot f_2 \odot f_3 \odot \cdots \odot f_n \odot f_n''$ with $f_1, f_2, \ldots, f_n \in D_1$, $f_n'' \in F_2$. Taking $u = f_1 \odot f_2 \odot f_3 \odot \cdots \odot f_n$, $v = f_n''$ we get $x \geq f \geq u \odot v$ with $u \in F_1$, $v \in F_2$. □

Proposition 3.25. *If $F_1, F_2 \in \mathcal{F}_n(L)$ then:*
(1) *$F_1 \wedge F_2 \in \mathcal{F}_n(L)$;*
(2) *$F_1 \vee F_2 \in \mathcal{F}_n(L)$.*

Proof. (1) We have $F_1 \wedge F_2 = F_1 \cap F_2$. Consider $x, y \in L$ such that $x \to y \in F_1 \cap F_2$, that is, $x \to y \in F_1$ and $x \to y \in F_2$. It follows that $x \rightsquigarrow y \in F_1$ and $x \rightsquigarrow y \in F_2$, hence $x \rightsquigarrow y \in F_1 \cap F_2$.
Similarly, $x \rightsquigarrow y \in F_1 \cap F_2$ implies $x \to y \in F_1 \cap F_2 = F_1 \wedge F_2$.
(2) Let $x, y \in L$ such that $x \to y \in F_1 \vee F_2$.
There are $u \in F_1, v \in F_2$ such that $u \odot v \leq x \to y$.
Hence $(u \odot v) \odot x \leq y$ so $u \odot (v \odot x) \leq y$. Since there is $v' \in F_2$ such that $v \odot x \geq x \odot v'$, we get $y \geq (u \odot x) \odot v'$.
Similarly, there is $u' \in F_1$ such that $u \odot x \geq x \odot u'$, so $y \geq x \odot (u' \odot v')$. We get $u' \odot v' \leq x \rightsquigarrow y$, hence $x \rightsquigarrow y \in F_1 \vee F_2$.
Similarly $x \rightsquigarrow y \in F_1 \vee F_2$ implies $x \to y \in F_1 \vee F_2$. □

Proposition 3.26. *If $(F_i)_{i \in I}$ is a family of normal filter of L then:*
(1) *$\bigwedge_{i \in I} F_i \in \mathcal{F}_n(L)$;*
(2) *$\bigvee_{i \in I} F_i \in \mathcal{F}_n(L)$;*

Proof. Similarly as above. □

As a consequence of the above result we get:

Theorem 3.27. *([3]) $\mathcal{F}_n(L)$ is a complete sublattice of $(\mathcal{F}_s(L), \subseteq)$.*

With any normal filter H of L we associate a binary relation $\equiv_H$ on L by defining $x \equiv_H y$ iff $x \to y, y \to x \in H$ iff $x \rightsquigarrow y, y \rightsquigarrow x \in H$.

Proposition 3.28. *([13]) For a given normal filter H of L the relation $\equiv_H$ is a congruence on L.*

For any $x \in L$, let x/L be the equivalence class $x/_{\equiv_H}$ and $L/H = \{x/H | x \in L\}$. L/H becomes a residuated lattice with the natural operations induced from those of L. If $x, y \in L$, then $x/H \leq y/H$ *iff* $x \to y \in H$ *iff* $x \rightsquigarrow y \in H$.

Definition 3.29. A proper filter of L is called *maximal* or *ultrafilter* if it is not strictly contained in any other proper filter of L.
Denote $Max(L) = \{F \mid F \text{ is maximal filter of } L\}$ and
$Max_n(L) = \{F \mid F \text{ is maximal and normal filter of } L\}$.

Proposition 3.30. *Any proper filter of a residuated lattice L can be extended to a maximal filter of L.*

Proof. It is an immediate consequence of Zorn's lemma. □

Theorem 3.31. *If F is a proper filter of L then the following are equivalent :*
(1) F is a maximal filter ;
(2) For any $x \notin F$ there is $f \in F$, $n, m \in \mathbb{N}$, $n, m \geq 1$ such that $(f \odot x^n)^m = 0$.

Proof. $(a) \Rightarrow (b)$: Since F is a maximal filter of L and $x \notin F$, then $[F \cup \{x\}) = L$, so $0 \in [F \cup \{x\})$. By Remark 3.12 it follows tha there exist $m \geq 1,\ n_1, n_2, ..., n_m \geq 0,\ f_1, f_2, ..., f_m \in F\}$ such that

$$(f_1 \odot x^{n_1}) \odot (f_2 \odot x^{n_2}) \odot \cdots \odot (f_m \odot x^{n_m}) \leq 0, \text{ so}$$
$$(f_1 \odot x^{n_1}) \odot (f_2 \odot x^{n_2}) \odot \cdots \odot (f_m \odot x^{n_m}) = 0.$$

Taking $n = max\{n_1,\ n_2,\ \cdots, n_m\}$ and $f = f_1 \odot f_2 \odot\ \cdots\ \odot f_m \in F$ we get

$$(f \odot x^n)^m \leq (f_1 \odot x^{n_1}) \odot (f_2 \odot x^{n_2}) \odot \cdots \odot (f_m \odot x^{n_m}) = 0.$$

It follows that $(f \odot x^n)^m = 0$.
$(b) \Rightarrow (a)$: Assume that there is a proper filter E of L such that $F \subset E$. Then there exists $x \in E$ such that $x \notin F$. By hypothesis, there exist $f \in F$, $n, m \in \mathbb{N}$ such that $(f \odot x^n)^m = 0$. Since $x, f \in E$, it follows that $0 \in E$, hence $E = L$ which is a contradiction.
Thus, F is a maximal filter of L. □

Theorem 3.32. *If H is a proper normal filter of L then the following are equivalent:*
(1) $H \in Max(L)$;
(2) For any $x \in L$, $x \notin H$ iff $(x^n)^-$ for some $n \in \mathbb{N}$;
(3) For any $x \in L$, $x \notin H$ iff $(x^n)^\sim$ for some $n \in \mathbb{N}$.

Proof. $(1) \Rightarrow (2)$: If $x \notin H$ then $[H \cup \{x\}) = L$, hence $0 \in [H \cup \{x\})$.
Then $h \leq x^n \to 0 = (x^n)^-$, i.e. $(x^n)^- \in H$.
Conversely, if $(x^n)^- \in H$, then $x^n \to 0 \in H$, so $0 \in [H \cup\ \{x\}$.
This means that $[H \cup \{x\}) = L$, i.e. H is not a proper filter, which is a contradiction.
Hence, $x \notin H$.
$(2) \Rightarrow (1)$: Assume there is a proper filter F such that $H \subseteq F$ and $H \neq F$.
Then there is $x \in F$ such that $x \notin H$. Hence, $(x^n)^- \in H \subseteq F$ for some $n \in N$.
It follows that $x^n \in F$ and $x^n \odot (x^n)^- = 0 \in F$, wich is a contradiction with the fact that F is a proper filter. Thus, $H \in Max(L)$.
$(1) \Leftrightarrow (3)$: Similarly as $(1) \Leftrightarrow (2)$. □

Proposition 3.33. *If H is a proper normal filter of L then the following are equivalent:*
(1) $H \in Max_n(L)$;
(2) L/H is locally finite.

Proof. It follows from the fact that the condition (2) is equivalent with :
for any $x \in L, x/H \neq 1/H$ iff $(x^n)^-/H = 1/H$ for some $n \in N$ iff $(x/H)^n = 0/H$ for some $n \in N$. □

Proposition 3.34. *If $F = L \setminus \{0\}$ is a maximal filter of the residuated lattice L, then L is good.*

Proof. Obviously $(0^-)^\sim = (0^\sim)^- = 0$. Assume $x > 0$, that is, $x \in F$. If $x^-, x^\sim \in F$ it follows that $x^- \odot x, x \odot x^- \in F$, that is, $0 \in F$, a contradiction.
Thus $x^- = x^\sim = 0$, hence $(x^-)^\sim = (x^\sim)^- = 1$.
Therefore, L is a good residuated lattice. □

Proposition 3.35. *If F is a maximal filter of the residuated lattice L, then:*
(1) *$y \notin F$ and $y \odot x = x$ implies $x = 0$;*
(2) *$y \notin F$ and $x \odot y = x$ implies $x = 0$.*

Proof. (1) Assume $x > 0$ and consider $E = \{z \in L \mid z \odot x = x\}$.
First we prove that E is a proper filter. Obviously, $1 \in E$ and $0 \notin E$.
Consider $y, y \to z \in E$, that is, $y \odot x = x$ and $(y \to z) \odot x = x$.
Since $(y \to z) \odot y \odot x = (y \to z) \odot x$, it follows that

$$x = [(y \to z) \odot y] \odot x \leq (y \wedge z) \odot x \leq (y \odot x) \vee (z \odot x) = x \vee (z \odot x) = z \odot x \leq x$$

Thus $z \odot x = x$, hence $z \in E$. Therefore E is a proper filter. Since $y \in E$ and F is maximal, it follows that $y \in F$, a contradiction. Thus $x = 0$.
(2) Similarly as (1). □

In the next section we will study the local and perfect residuated lattices.

4. Local residuated lattices

In this section we exdend to residuated lattices the results established in [15] for pseudo BL-algebras.

Definition 4.1. A residuated lattice is called *local* if and only if it has a unique maximal filter.

If L is a residuated lattice, we will denote :

$$D(L) = \{x \in L \mid ord(x) = \infty\}$$
$$D(L)^* = \{x \in L \mid ord(x) < \infty\}.$$

Obviously, $D(L) \cap D(L)^* = \oslash$ and $D(L) \cup D(L)^* = L$.
We also can remark that $1 \in D(L)$ and $0 \in D(L)^*$.

Let L be a residuated lattice and F a filter of L. We will use the following notations:

$$F^*_- = \{x \in L \mid x \leq y^- \text{ for some y} \in F\}$$
$$F^*_\sim = \{x \in L \mid x \leq y^\sim \text{ for some y} \in F\}.$$

Remark 4.2. Let L be a residuated lattice. Then :
(1) $F^*_- = \{x \in L \mid x \odot y = 0 \text{ for some } y \in F\}$;
(2) $F^*_\sim = \{x \in L \mid y \odot x = 0 \text{ for some } y \in F\}$;
(3) $F^*_- = \{x \in L \mid x^\sim \in F\}$;
(4) $F^*_\sim = \{x \in L \mid x^- \in F\}$.

Proof. (1) : $x \in F^*_-$ iff $x \leq y^-$ for some $y \in F$ iff $x \leq y \to 0$ for some $y \in F$ iff $x \odot y = 0$ for some $y \in F$;
(2) Similarly as (1) ;
(3) Let's denote $E = \{x \in L \mid x^\sim \in F\}$. We must prove that $F^*_- = E$.
Consider $x \in F^*_-$. It follows that $x \leq y^-$ for some $y \in F$. We have $y \leq y^{-\sim} \leq x^\sim$.
Since F is a filter of L, we get $x^\sim \in F$. Hence, $x \in E$.
Conversely, suppose $x \in E$, that is $x^\sim \in F$. Since $x \leq (x^\sim)^-$, taking $y = x^\sim$, we get $x \leq y^-$ with $y \in F$, so $x \in F^*_-$;
(4) Simlilarly as (3). □

Proposition 4.3. *Let L be a local residuated lattice. Then :*
(1) *any proper filter of L is included in the unique maximal filter of L ;*
(2) *L_0^- and $L_0^\sim$ are included in the unique maximal filter of L.*

Proof. (1) It follows applying the Proposition 3.30 and taking into consideration that L has a unique maximal filter ;
(2) Apply Proposition 3.4 and (1). □

Proposition 4.4. *Let L be a residuated lattice. Then the following are equivalent:*
(i) *$D(L)$ is a filter of L ;*
(ii) *$D(L)$ is a proper filter of L ;*
(iii) *L is local ;*
(iv) *$D(L)$ is the unique maximal filter of L ;*
(v) *for all $x, y \in L$, $ord(x \odot y) < \infty$ implies $ord(x) < \infty$ or $ord(y) < \infty$.*

Proof. $(i) \Rightarrow (ii)$ Since $ord(0) = 1$, we have $0 \notin D(L)$, so $D(L)$ is a proper filter of L.
$(ii) \Rightarrow (i)$ Obviously.
$(i) \Rightarrow (v)$ Consider $x, y \in L$ such that $ord(x \odot y) < \infty$, so $x \odot y \notin D(L)$. Since $D(L)$ is a filter of L, it follows that $x \notin D(L)$ or $y \notin D(L)$. Hence, $ord(x) < \infty$ or $ord(y) < \infty$.
$(v) \Rightarrow (i)$ Because $1 \in D(L)$ it follows that $D(L)$ is nonempty. Consider $x, y \in D(L)$, that is, $ord(x) = \infty$ and $ord(y) = \infty$. By (v) we get $ord(x \odot y) = \infty$, so $x \odot y \in D(L)$. Consider $x \in D(L)$ and $y \in L$ such that $x \leq y$. It follows that $x^n > 0$ for all $\in \mathbb{N}$. Since $x^n \leq y^n$ we get $y^n > 0$ for all $\in \mathbb{N}$, so $ord(y) = \infty$, that is $y \in D(L)$.
Thus, $D(L)$ is a filter of L.
$(iv) \Rightarrow (iii)$ It follows by the definition of a local residuated lattice.
$(iii) \Rightarrow (iv)$ If M is the unique maximal filter of L, then by Lemma 3.1 and Proposition 4.3 we have $x \in M$ iff $[x) \subseteq M$ iff $[x)$ is proper iff $ord(x) = \infty$ iff $x \in D(L)$. Hence, $M = D(L)$.
$(iv) \Rightarrow (i)$ Obviously.
$(i) \Rightarrow (iv)$ Since $0 \notin D(L)$, it follows that $D(L)$ is a proper filter of L. Let F be a proper filter of L. Consider $x \in F$. Since $[x) \subseteq F$, we have that $[x)$ is proper filter of L, so by Lemma 3.1 it follows that $ord(x) = \infty$. Hence, $x \in D(L)$, so $F \subseteq D(L)$. Thus, $D(L)$ is the unique maximal filter of L. □

Example 4.5. Consider the residuated lattice from Example 3.19. One can check that $D(L) = \{b, c, 1\}$ is a filter of L, so L is a local residuated lattice.

Example 4.6. Consider the residuated lattice from Example 3.22. One can check that $D(L) = \{a, b, c, 1\}$ is a filter of L, so L is a local residuated lattice.

Example 4.7. In the case of the residuated lattice in Example 2.2 we have $D(L) = \{b, c, 1\}$. Since $b \odot c = a \notin D(L)$, it follows that $D(L)$ is not a filter of L, so L is not a local residuated lattice.

Corollary 4.8. *If L is a local residuated lattice, then :*
(1) *for any $x \in L$, $ord(x) < \infty$ or $[ord(x^-) < \infty$ and $ord(x^\sim) < \infty]$;*
(2) *$D(L)^*_- \subseteq D(L)^*$ and $D(L)^*_\sim \subseteq D(L)^*$;*
(3) *$D(L) \cap D(L)^*_- = D(L) \cap D(L)^*_\sim = \oslash$.*

Proof. (1) Let $x \in L$. Since $x \odot x^- = x^\sim \odot x = 0$, it follows that $ord(x \odot x^-) = ord(x^\sim \odot x) = ord(0) = 1 < \infty$. By Proposition 4.4 (iv) we get (1).
(2) Consider $x \in D(L)^*_-$. It means that there is $y \in D(L)$ such that $x \leq y^-$. Since $ord(y) = \infty$, by (1) we get that $ord(y^-) < \infty$. Hence, $ord(x) < \infty$, so $x \in D(L)^*$. Thus, $D(L)^*_- \subseteq D(L)^*$. Similarly, $D(L)^*_\sim \subseteq D(L)^*$.
(3) It follows by (2) taking into consideration that $D(L) \cap D(L)^* = \oslash$. □

Proposition 4.9. *Any residuated lattice chain is a local residuated lattice.*

Proof. Assume that L is a residuated lattice chain and consider $x, y \in L$ such that $ord(x \odot y) < \infty$. Since L is a chain, we have $x \leq y$ or $y \leq x$. Let's suppose that $x \leq y$. It follows that $x \odot x \leq x \odot y$, so $ord(x \odot x) \leq \infty$. Hence, $ord(x) < \infty$. Similarly, from $y \leq x$ we get $ord(y) < \infty$. Thus, according to Proposition 4.4, L is a local residuated lattice. □

Definition 4.10. A residuated lattice L is called *perfect* if it satisfies the following conditions :
(1) L is a local good residuated lattice ;
(2) for any $x \in L$, $ord(x) < \infty$ iff $[ord(x^-) = \infty \text{ and } \mathrm{ord}(x^\sim) = \infty]$.

Proposition 4.11. *Let L be a local good residuated lattice. Then the following are equivalent :*
(i) L is perfect ;
(ii) $D(L)^*_- = D(L)^*$ *and* $D(L)^*_\sim = D(L)^*$.

Proof. $(i) \Rightarrow (ii)$: Since L is a local residuated lattice, applying Corollary 4.8 (2) we get $D(L)^*_- \subseteq D(L)^*$ and $D(L)^*_\sim \subseteq D(L)^*$.
Conversely, consider $x \in D(L)^*$, that is $ord(x) < \infty$. By the definition of a perfect residuated lattice we get $ord(x^-) = \infty$ and $ord(x^\sim) = \infty$, that is $x^-, x^\sim \in D(L)$.
Applying the properties $x \leq x^{\sim -}$ and $x \leq x^{-\sim}$ we get $x \in D(L)^*_-$ and $x \in D(L)^*_\sim$.
It follows that $D(L)^* \subseteq D(L)^*_-$ and respectively $D(L)^* \subseteq D(L)^*_\sim$.
Thus, $D(L)^*_- = D(L)^*$ and $D(L)^*_\sim = D(L)^*$.
$(ii) \Rightarrow (i)$: Consider $x \in L$ such that $ord(x) < \infty$, that is $x \in D(L)^*$.
Since $D(L)^*_- = D(L)^*$, there exists $y \in D(L)$ such that $x \leq y^-$, so $y^{-\sim} \leq x^\sim$.
By $y \leq y^{-\sim}$ and $ord(y) = \infty$, we get $ord(y^{-\sim}) = \infty$.
¿From $y^{-\sim} \leq x^\sim$ we get $ord(x^\sim) = \infty$.
Since $D(L)^*_\sim = D(L)^*$, there exists $y \in D(L)$ such that $x \leq y^\sim$, so $y^{\sim -} \leq x^-$.
By $y \leq y^{\sim -}$ and $ord(y) = \infty$, we get $ord(y^{\sim -}) = \infty$.
¿From $y^{\sim -} \leq x^-$ we get $ord(x^-) = \infty$.
Conversely, consider $x \in L$ such that $(ord(x^-) = \infty \text{ and } \mathrm{ord}(x^\sim) = \infty)$.
Since L is local, by Corollary 4.8 (1) it follows that $ord(x) < \infty$.
Thus, L is a perfect residuated lattice. □

Corollary 4.12. *If L is a perfect residuated lattice, then*

$$D(L)^* = \{x^- \mid x \in D(L)\} = \{x^\sim \mid x \in D(L)\}.$$

Example 4.13. Let's consider the residuated lattice L in Example 2.2. Since L is not a good residuated lattices, it is not a perfect residuated lattice.

Example 4.14. Consider the local residuated lattice in Example 3.22.
One can easy check that L is a good residuated. Moreover, $D(L) = \{a, b, c, 1\}$ and $D(L)^* = \{0\}$.
Since $ord(0^-) = ord(0^\sim) = \infty$, it follows that L is a perfect residuated lattice.

Definition 4.15. Let L be a residuated lattice. The intersection of all maximal filters of L is called the *radical* of L and it is denoted by $Rad(L)$.
The intersection of all maximal normal filters of L is called the *normal radical* of L and it is denoted by $Rad_n(L)$.

Proposition 4.16. *If L is a perfect residuated lattice, then* $Rad(L) = D(L)$.

Proof. By Proposition 4.4 it follows that $D(L)$ is the unique maximal filter of L, so $Rad(L) = D(L)$. □

Example 4.17. Let's consider the perfect residuated lattice L in Example 3.22. One can easily check that $Rad(L) = Rad_n(L) = D(L) = \{a, b, c, 1\}$.

Remark 4.18. If L is a local residuated lattice and $x \in Rad(L)^*$, $y \in L$ such that $y \le x$, then $y \in Rad(L)^*$.

Theorem 4.19. *If A is a perfect residuated lattice then $Rad(L)$ is a normal filter of L.*

Proof. We have to prove that $x \to y \in Rad(L)$ iff $x \rightsquigarrow y \in Rad(L)$ for all $x, y \in L$.
Consider $x, y \in L$ such that $x \to y \in Rad(L)$ and suppose $x \rightsquigarrow y \notin Rad(L)$.
¿From $y \le y^{-\sim}$ we get $x \to y \le x \to y^{-\sim}$ (by (c_{45}) and (c_{15})). Since $Rad(L)$ is a filter of L, it follows that $x \to y^{-\sim} \in Rad(L)$, that is, $(x \odot y^{\sim})^{-} \in Rad(L)$ (by (c_{46})). Hence, $x \odot y^{\sim} \in Rad(L)^*$.
On the other hand, from $x \rightsquigarrow y \notin Rad(L)$, it follows that $x \rightsquigarrow y \in Rad(L)^*$.
Since $x \le x^{-\sim}$, by (c_{16}) we get $x^{-\sim} \rightsquigarrow y \le x \rightsquigarrow y$, so $x^{-\sim} \rightsquigarrow y \in Rad(L)^*$ (by Remark 3.20). By (c_{49}) we have $x^{\sim} \le x^{\sim -} \rightsquigarrow y$, so $x^{\sim} \in Rad(L)^*$, that is, $x \in Rad(L)$. But $y \le x \rightsquigarrow y$, so $y \in Rad(L)^*$, that is, $y^{\sim} \in Rad(L)$.
Since $Rad(L)$ is a filter of L and $x, y^{\sim} \in Rad(L)$, we get $x \odot y^{\sim} \in Rad(L)$ which is a contradiction. Thus, $x \to y \in Rad(L)$ implies $x \rightsquigarrow y \in Rad(L)$.
Similarly, $x \rightsquigarrow y \in Rad(L)$ implies $x \to y \in Rad(L)$ and we conclude that $Rad(L)$ is a normal filter of L. □

Remark 4.20. If the residuated lattice L is not perfect, then the above result is not allways valid. Indeed, let's consider the residuated lattice L in Example 2.2.
Since L is not good, it is not a perfect residuated lattice. Moreover, $H = \{c, 1\}$ is the unique maximal filter of L, so $Rad(L) = H$. Since $b \to 0 = c \in H$ and $b \rightsquigarrow 0 = 0 \notin H$, it follows that H is not a normal filter of L.

5. Archimedean residuated lattices

In this section we will introduce the class of Archimedean residuated lattices in the same way it was introduced in [3] in the case of pseudo BL-algebras.

Proposition 5.1. *In any residuated lattice the following are equivalent:*
(i) $x^n \ge x^{-} \vee x^{\sim}$ for any $n \in \mathbb{N}$ implies $x = 1$;
(ii) $x^n \ge y^{-} \vee y^{\sim}$ for any $n \in \mathbb{N}$ implies $x \vee y = 1$.

Proof. $(i) \Rightarrow (ii)$ Take $x, y \in L$ such that $x^n \ge y^{-} \vee y^{\sim}$ for any $n \in \mathbb{N}$.
By the properties of residuated lattices and by hypothesis we have

$$(x \vee y)^{-} = x^{-} \vee y^{-} \le y^{-} \le y^{-} \vee y^{\sim} \le x^n \le (x \vee y)^n \text{ and}$$

$$(x \vee y)^{\sim} = x^{\sim} \vee y^{\sim} \le y^{\sim} \le y^{-} \vee y^{\sim} \le x^n \le (x \vee y)^n,$$

hence $(x \vee y)^n \ge (x \vee y)^{-} \vee (x \vee y)^{\sim}$ for any $n \in \mathbb{N}$. Thus, by hypothesis we get $x \vee y = 1$.
$(ii) \Rightarrow (i)$ Consider $x \in L$ such that $x^n \ge x^{-} \vee x^{\sim}$ for any $n \in \mathbb{N}$.
Taking $y = x$ in (ii) we get $x \vee x = 1$, hence $x = 1$. □

Definition 5.2. A residuated lattice is called *Archimedean* if one of the equivalent conditions from the above proposition is satisfied.

Proposition 5.3. *If in an Archimedean residuated lattice L $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$, then $x \to y = x \rightsquigarrow y = y$.*

Proof. Similarly as in the above proposition, if $x, y \in L$ we have

$$(x \vee y) \leq [(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x]$$
$$(x \vee y) \leq [(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x].$$

Since $x \vee y = 1$, it follows that:

$$[(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x] = 1,$$
$$[(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x] = 1,$$

hence $(x \to y) \rightsquigarrow y = 1$ and $(x \rightsquigarrow y) \to y = 1$.
From $(x \to y) \rightsquigarrow y = 1$ we have $x \to y \leq y$ and taking in consideration that $y \leq x \to y$, we obtain $x \to y = y$.
Similarly, $x \rightsquigarrow y = y$. □

Example 5.4. Consider the residuated lattice L from Example 3.19.
Since $c^n = c \geq c^- \vee c^\sim = a$ for all $n \in \mathbb{N}$, it follows that L ist not an Archimedean residuated lattice.

Example 5.5. Consider the residuated lattice L from Example 2.2. We have :

$$0^n = 0 \ngeq 0^- \vee 0^\sim = 1 \vee 1 = 1,\ n \geq 1$$
$$a^n = 0 \ngeq a^- \vee a^\sim = c \vee b = 1,\ n \geq 2$$
$$b^n = b \ngeq b^- \vee b^\sim = c \vee 0 = c,\ n \geq 1$$
$$c^n = c \ngeq c^- \vee c^\sim = 0 \vee b = b,\ n \geq 1$$
$$1^n = 1 \geq 1^- \vee 1^\sim = 0 \vee 0 = 0,\ n \geq 1.$$

Obviously : $a \ngeq a^- \vee a^\sim = c \vee b = 1$.
We conclude that, $x^n \geq x^- \vee x^\sim$ for all $n \in \mathbb{N}$, $n \geq 1$, implies $x = 1$. Hence, L is an Archimedean residuated lattice.

Let's consider the commutative case of a residuated lattice.

Proposition 5.6. *In any commutative residuated lattice the following properties are equivalent:*
(a) $x^n \geq \overline{x}$ for any $n \in N$ implies $x = 1$;
(b) $x^n \geq \overline{y}$ for any $n \in N$ implies $x \vee y = 1$;
(c) $x^n \geq \overline{y}$ for any $n \in N$ implies $x \to y = y$ and $y \to x = x$.

Proof. $(a) \Rightarrow (b)$: Let $x, y \in L$ such that $x^n \geq \overline{y}$ for any $n \in N$.
By (c_{64}) and by the hypothesis we have:

$$(x \vee y)^- = \overline{x} \wedge \overline{y} \leq \overline{y} \leq x^n \leq (x \vee y)^n,$$

hence $(x \vee y)^n \geq (x \vee y)^-$ for any $n \in N$. Thus, by the hypothesis we get $x \vee y = 1$.
$(b) \Rightarrow (a)$: Consider $x \in L$ such that $x^n \geq \overline{y}$ for any $n \in N$.
By (b), taking $y = x$ we get $x \vee x = 1$, hence $x = 1$.
$(a) \Rightarrow (c)$. Let $x, y \in L$ such that $x^n \geq \overline{y}$ for any $n \in N$.
Similarly with $(a) \Rightarrow (b)$, if $x, y \in L$ we have
$(x \vee y)^n \geq (x \vee y)^-$ for any $n \in N$,
hence, by the hypothesis, we get $x \vee y = 1$.
By (c_{73}) we have $x \vee y \leq [(x \to y) \to y] \wedge [(y \to x) \to x]$.
Since $x \vee y = 1$, it follows that $[(x \to y) \to y] \wedge [(y \to x) \to x] = 1$,
hence $(x \to y) \to y = 1$ and $(y \to x) \to x = 1$.

¿From $(x \to y) \to y = 1$, we have $x \to y \leq y$ and considering that $y \leq x \to y$ we obtain $x \to y = y$. Similarly, $y \to x = x$.
$(c) \Rightarrow (a)$: Consider $x \in L$ such that $x^n \geq \overline{x}$, for any $n \in N$.
By the hypothesis we obtain $x \to x = x$, hence $x = 1$. □

Definition 5.7. A commutative residuated lattice is called *Archimedean* if one of the equivalent conditions from the above proposition is satisfied.

We will give bellow one example of commutative not Archimedean residuated lattices.

Example 5.8. Consider the residuated lattice $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ defined on the unit interval $L = [0, 1]$ with the operations (see [23]):

$$x \odot y = \begin{cases} 0, \text{ if } x + y \leq \frac{1}{2} \\ x \wedge y, \text{ otherwise} \end{cases}$$

$$x \to y = \begin{cases} 1, \text{ if } x \leq y \\ max\{\frac{1}{2} - x, y\}, \text{ otherwise} \end{cases}$$

Since $(\frac{1}{3})^n = \frac{1}{3} > \frac{1}{6} = (\frac{1}{3})^-$ for al $n \in \mathbb{N}$ and $\frac{1}{3} \neq 1$, it follows that L is not Archimedean.

Remark 5.9. When we study an Archimedean noncommutative ordered structure it is a crucial problem to prove the existence of objects in this class. As answer of this question we have :
(1) Every Archimedean ℓ-group is commutative ([2]) ;
(2) Every Archimedean pseudo MV-algebra is commutative, i.e. an MV-algebra ([8]);
(3) Generally, an Archimedean pseudo MTL-algebra is not commutative ([5]);
(4) By Example 5.5 we proved that, generally, an Archimedean residuated lattice is not commutative.

Remark 5.10. The existence of finite non-commutative residuated structures is also a very important problem.
As we know, every finite peudo MV-algebra is an MV-algebra ([8]).
In this paper we proved that there exist finite non-commutative residuated lattices and finite pseudo MTL-algebras.

Open problem 5.11. Find an example of Archimedean pseudo BL-algebra.

Open problem 5.12. Find an example of Archimedean non-commutative Rℓ-monoid.

Open problem 5.13. Find an example of finite pseudo BL-algebra.

Open problem 5.14. Find an example of finite non-commutative Rℓ-monoid.

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