Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 33, 2006, Pages 208–215
ISSN: 1223-6934

On $\widetilde{g}$-Semi-Homeomorphism in Topological Spaces

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Abstract. In this paper, we first introduce a new class of closed maps called $\widetilde{g}s$-closed maps also introduce a new class of homeomorphisms called $\widetilde{g}s^*$-homeomorphisms and prove that the set of all $\widetilde{g}s^*$-homeomorphisms forms a group under the operation composition of maps.

2000 Mathematics Subject Classification. Primary 54A05, Secondary 54C08.
Key words and phrases. g-closed set, $\widetilde{g}s$-closed set, $\widetilde{g}s$-open set, $\widetilde{g}s$-continuous function, $\widetilde{g}s$-irresolute map.

1. Introduction

The notion homeomorphism plays a very important role in topology. By definition, a homeomorphism between two topological spaces X and Y is a bijective map $f : X \to Y$ when both f and f^{-1} are continuous. Malghan [3] introduced the concept of generalized closed maps in topological spaces. In this paper, we first introduce a new class of closed maps called $\widetilde{g}s$-closed maps in topological space and then we introduce and study $\widetilde{g}s^*$-homeomorphisms and prove that the set of all $\widetilde{g}s^*$-homeomorphisms forms a group under the operation composition of functions.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively.

We recall the following definitions and some results, which are used in the sequel

Definition 2.1. *A subset A of a space (X, τ) is called:*
(i) *semi-open* [2] *if* $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$.
(ii) *semi-closed* [2] *if* $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq A$.

The semi-closure [1] of a subset A of X, denoted by $s\,\mathrm{Cl}_X(A)$ briefly $s\,\mathrm{Cl}(A)$, is defined to be the intersection of all semi-closed sets containing A.

Definition 2.2. *A subset A of a space (X, τ) is called:*
(i) *a $\widehat{g}$-closed set* [7] *if* $\mathrm{Cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is semi-open in* (X, τ). *The complement of a $\widehat{g}$-closed set is called $\widehat{g}$-open.*
(ii) *a $^*g$-closed set* [8] *if* $\mathrm{Cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is $\widehat{g}$-open in* (X, τ). *The complement of a $^*g$-closed set is called $^*g$-open.*
(iii) *a $^\#g$-semi-closed (briefly $^\#gs$-closed) set* [9] *if* $s\,\mathrm{Cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is $^*g$-open in* (X, τ). *The complement of a $^\#gs$-closed set is called $^\#gs$-open.*
(iv) *$\widetilde{g}$-semi-closed set (briefly $\widetilde{g}s$-closed)* [6] *if* $s\,\mathrm{Cl}(A) \subseteq U$ *whenever* $A \subseteq U$ *and* U *is $^\#gs$-open in* (X, τ). *The complement of a $\widetilde{g}s$-closed set is called $\widetilde{g}s$-open.*

Definition 2.3. *A function $f : (X, \tau) \to (Y, \sigma)$ is called:*
(i) *$\widetilde{g}s$-continuous* [5] *if $f^{-1}(V)$ is $\widetilde{g}s$-closed in (X, τ) for every closed set V in (Y, σ).*
(ii) *$\widetilde{g}s$-irresolute* [5] *if $f^{-1}(V)$ is $\widetilde{g}s$-closed in (X, τ) for every $\widetilde{g}s$-closed set V in (Y, σ).*

Proposition 2.1. [5] *If a map $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s$-irresolute, then it is $\widetilde{g}s$-continuous.*

Definition 2.4. *Let (X, τ) be a topological space and $E \subseteq X$. We define the $\widetilde{g}s$-closure* [4] *of E (briefly $\widetilde{g}s$-$\mathrm{Cl}(E)$) to be the intersection of all $\widetilde{g}s$-closed sets containing E. In symbols, $\widetilde{g}s$-$\mathrm{Cl}(E) = \bigcap \{A : E \subseteq A \text{ and } A \in \widetilde{G}SC(X, \tau)\}$.*

Proposition 2.2. [4] *Let (X, τ) be a topological space and $E \subseteq X$. The following properties are hold:*
(i) *$\widetilde{g}s$-$\mathrm{Cl}(E)$ is the smallest $\widetilde{g}s$-closed set containing E and*
(ii) *E is $\widetilde{g}s$-closed if and only if $\widetilde{g}s$-$\mathrm{Cl}(E) = E$.*

Proposition 2.3. [4] *For any two subsets A and B of (X, τ),*
(i) *If $A \subseteq B$, then $\widetilde{g}s$-$\mathrm{Cl}(A) \subseteq \widetilde{g}s$-$\mathrm{Cl}(B)$,*
(ii) *$\widetilde{g}s$-$\mathrm{Cl}(A \cap B) \subseteq \widetilde{g}s$-$\mathrm{Cl}(A) \cap \widetilde{g}s$-$\mathrm{Cl}(B)$.*

Theorem 2.1. [6] *Suppose that $B \subseteq A \subseteq X$, B is a $\widetilde{g}s$-closed set relative to A and that A is open and $\widetilde{g}s$-closed in (X, τ). Then B is $\widetilde{g}s$-closed in (X, τ).*

Corollary 2.1. [6] *If A is a $\widetilde{g}s$-closed set and F is a closed set, then $A \cap F$ is a $\widetilde{g}s$-closed set.*

Theorem 2.2. [6] *A set A is $\widetilde{g}s$-open in (X, τ) if and only if $F \subseteq s\,\mathrm{Int}(A)$ whenever F is $^{\#}gs$-closed in (X, τ) and $F \subseteq A$.*

Definition 2.5. [4] *Let (X, τ) be a topological space and $E \subseteq X$. We define the $\widetilde{g}s$-interior of E (briefly $\widetilde{g}s$-$\mathrm{Int}(E)$) to be the union of all $\widetilde{g}s$-open sets contained in E.*

Lemma 2.1. [4] *For any $E \subseteq X$, $\mathrm{Int}(E) \subseteq \widetilde{g}s$-$\mathrm{Int}(E) \subseteq E$.*

Proof. Since every open set is $\widetilde{g}s$-open, the proof follows immediately. □

3. $\widetilde{g}s$-Closed Maps

In this section, we introduce the notions of $\widetilde{g}s$-closed maps, $\widetilde{g}s$-open maps, $\widetilde{g}s^*$-closed maps, $\widetilde{g}s^*$-open maps in topological spaces and obtain certain characterizations of these maps.

Definition 3.1. *The map $f : (X, \tau) \to (Y, \sigma)$ is called $\widetilde{g}s$-closed if the image of every closed set in (X, τ) is $\widetilde{g}s$-closed in (Y, σ).*

Example 3.1. *Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, X\}$ and $\sigma = \{\varnothing, \{b\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is a $\widetilde{g}s$-closed map.*

Proposition 3.1. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s$-closed if and only if $\widetilde{g}s$-$\mathrm{Cl}(f(A)) \subseteq f(\mathrm{Cl}(A))$ for every subset A of (X, τ).*

Proof. Suppose that f is $\widetilde{g}s$-closed and $A \subseteq X$. Then $f(\mathrm{Cl}(A))$ is $\widetilde{g}s$-closed in (Y, σ). We have $f(A) \subseteq f(\mathrm{Cl}(A))$ and by Propositions 2.2 and 2.3, $\widetilde{g}s$-$\mathrm{Cl}(f(A)) \subseteq \widetilde{g}s$-$\mathrm{Cl}(f(\mathrm{Cl}(A))) = f(\mathrm{Cl}(A))$.

Conversely, let A be any closed set in (X, τ). Then $A = \mathrm{Cl}(A)$ and so $f(A) = f(\mathrm{Cl}(A)) \supseteq \widetilde{g}s\text{-}\mathrm{Cl}(f(A))$, by hypothesis. We have $f(A) \subseteq \widetilde{g}s\text{-}\mathrm{Cl}(f(A))$ by Proposition 2.2. Therefore, $f(A) = \widetilde{g}s\text{-}\mathrm{Cl}(f(A))$. i.e., $f(A)$ is $\widetilde{g}s$-closed by Proposition 2.2 and hence f is $\widetilde{g}s$-closed. □

Theorem 3.1. *A map $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s$-closed if and only if for each subset S of (Y, σ) and for each open set U containing $f^{-1}(S)$ there exists a $\widetilde{g}s$-open set V of (Y, σ) such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose that f is a $\widetilde{g}s$-closed map. Let $S \subseteq Y$ and U be an open subset of (X, τ) such that $f^{-1}(S) \subseteq U$. Then $V = (f(U^c))^c$ is a $\widetilde{g}s$-open set containing S such that $f^{-1}(V) \subseteq U$.

For the converse, let S be a closed set of (X, τ). Then $f^{-1}((f(S))^c) \subseteq S^c$ and S^c is open. By assumption, there exists a $\widetilde{g}s$-open set V of (Y, σ) such that $(f(S))^c \subseteq V$ and $f^{-1}(V) \subseteq S^c$ and so $S \subseteq (f^{-1}(V))^c$. Hence $V^c \subseteq f(S) \subseteq f((f^{-1}(V))^c) \subseteq V^c$ which implies $f(S) = V^c$. Since V^c is $\widetilde{g}s$-closed, $f(S)$ is $\widetilde{g}s$-closed and therefore f is $\widetilde{g}s$-closed. □

The following example shows that the composition of two $\widetilde{g}s$-closed maps need not be $\widetilde{g}s$-closed.

Example 3.2. *Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b, c\}, X\}$, $\sigma = \{\varnothing, \{a, c\}, Y\}$ and $\eta = \{\varnothing, \{b\}, \{a, c\}, Z\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = f(b) = b$ and $f(c) = a$ and a map $g : (Y, \sigma) \to (Z, \eta)$ by $g(a) = c$, $g(b) = b$ and $g(c) = a$. Then both f and g are $\widetilde{g}s$-closed maps but their composition $g \circ f$: $(X, \tau) \to (Z, \eta)$ is not a $\widetilde{g}s$-closed map, since for the closed set $\{b, c\}$ in (X, τ), $(g \circ f)(\{b, c\}) = \{a, b\}$, which is not a $\widetilde{g}s$-closed set in (Z, η).*

Proposition 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a closed map and $g : (Y, \sigma) \to (Z, \eta)$ be a $\widetilde{g}s$-closed map, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is $\widetilde{g}s$-closed.*

Proof. Obvious. □

Remark 3.1. *If $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s$-closed and $g : (Y, \sigma) \to (Z, \eta)$ is closed, then their composition need not be a $\widetilde{g}s$-closed map as seen from the following example.*

Example 3.3. *Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b, c\}, X\}$, $\sigma = \{\varnothing, \{a, c\}, Y\}$ and $\eta = \{\varnothing, \{b\}, \{a, c\}, Z\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = f(b) = b$ and $f(c) = a$ and $g : (Y, \sigma) \to (Z, \eta)$ be the identity map. Then f is a $\widetilde{g}s$-closed map and g is a closed map. But their composition $g \circ f : (X, \tau) \to (Z, \sigma)$ is not a $\widetilde{g}s$-closed map, since for the closed set $\{b, c\}$ in (X, τ), $(g \circ f)(\{b, c\}) = \{a, b\}$, which is not $\widetilde{g}s$-closed in (Z, σ).*

Theorem 3.2. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two mappings such that their composition $g \circ f : (X, \tau) \to (Z, \eta)$ be a $\widetilde{g}s$-closed mapping. Then the following statements are true.*

(i) *If f is continuous and surjective, then g is $\widetilde{g}s$-closed.*
(ii) *If g is $\widetilde{g}s$-irresolute and injective, then f is $\widetilde{g}s$-closed.*

Proof. (i). Let A be a closed set of (Y, σ). Since f is continuous, $f^{-1}(A)$ is closed in (X, τ) and since $g \circ f$ is $\widetilde{g}s$-closed, $(g \circ f)(f^{-1}(A))$ is $\widetilde{g}s$-closed in (Z, σ). i.e., $g(A)$ is $\widetilde{g}s$-closed in (Z, σ), since f is surjective. Therefore, g is a $\widetilde{g}s$-closed map.

(ii). Let B be a closed set of (X, τ). Since $g \circ f$ is $\widetilde{g}s$-closed, $(g \circ f)(B)$ is $\widetilde{g}s$-closed in (Z, σ). Since g is $\widetilde{g}s$-irresolute, $g^{-1}((g \circ f)(B))$ is $\widetilde{g}s$-closed in (Y, σ). i.e., $f(B)$ is $\widetilde{g}s$-closed in (Y, σ), since g is injective. Thus, f is a $\widetilde{g}s$-closed map. □

As for the restriction f_A of a map $f : (X, \tau) \to (Y, \sigma)$ to a subset A of (X, τ), we have the following:

Theorem 3.3. *Let (X, τ) and (Y, σ) be any topological spaces. Then*

(i) *If $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s$-closed and A is a closed subset of (X, τ), then $f_A : (A, \tau_A) \to (Y, \sigma)$ is $\widetilde{g}s$-closed.*

(ii) *If $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s$-closed (resp. closed) and $A = f^{-1}(B)$ for some closed (resp. $\widetilde{g}s$-closed) set B of (Y, σ), then f_A: $(A, \tau_A) \to (Y, \sigma)$ is $\widetilde{g}s$-closed.*

Proof. (i). Let B be a closed set of A. Then $B = A \cap F$ for some closed set F of (X, τ) and so B is closed in (X, τ). By hypothesis, $f(B)$ is $\widetilde{g}s$-closed in (Y, σ). But $f(B) = f_A(B)$ and therefore f_A is a $\widetilde{g}s$-closed map.

(ii). Let D be a closed set of A. Then $D = A \cap H$ for some closed set H in (X, τ). Now $f_A(D) = f(D) = f(A \cap H) = f(f^{-1}(B) \cap H) = B \cap f(H)$. Since f is $\widetilde{g}s$-closed, $f(H)$ is $\widetilde{g}s$-closed and so $B \cap f(H)$ is $\widetilde{g}s$-closed in (Y, σ) by Corollary 2.1. Therefore, f_A is a $\widetilde{g}s$-closed map. □

Analogous to a $\widetilde{g}s$-closed map, we define a $\widetilde{g}s$-open map as follows:

Definition 3.2. *A map $f : (X, \tau) \to (Y, \sigma)$ is said to a $\widetilde{g}s$-open map if the image $f(A)$ is $\widetilde{g}s$-open in (Y, σ) for each open set A in (X, τ).*

Proposition 3.3. *For any bijective $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *$f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\widetilde{g}s$-continuous,*

(ii) *f is a $\widetilde{g}s$-open map and*

(iii) *f is a $\widetilde{g}s$-closed map.*

Proof. (i) ⇒ (ii): Let U be an open set of (X, τ). By assumption $(f^{-1})^{-1}(U) = f(U)$ is $\widetilde{g}s$-open in (Y, σ) and so f is $\widetilde{g}s$-open.

(ii) ⇒ (iii): Let F be a closed set of (X, τ). Then F^c is open in (X, τ). By assumption, $f(F^c)$ is $\widetilde{g}s$-open in (Y, σ). i.e., $f(F^c) = (f(F))^c$ is $\widetilde{g}s$-open in (Y, σ) and therefore $f(F)$ is $\widetilde{g}s$-closed in (Y, σ). Hence f is $\widetilde{g}s$-closed.

(iii) ⇒ (i): Let F be a closed set in (X, τ). By assumption $f(F)$ is $\widetilde{g}s$-closed in (Y, σ). But $f(F) = (f^{-1})^{-1}(F)$ and therefore f^{-1} is $\widetilde{g}s$-continuous on Y. □

Definition 3.3. *Let x be a point of (X, τ) and V be a subset of X. Then V is called a $\widetilde{g}s$-neighbourhood of x in (X, τ) if there exists a $\widetilde{g}s$-open set U of (X, τ) such that $x \in U \subseteq V$.*

Theorem 3.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping. Then the following statements are equivalent:*

(i) *f is a $\widetilde{g}s$-open mapping.*

(ii) *For a subset A of (X, τ), $f(int(A) \subseteq \widetilde{g}s\text{-Int}(f(A)))$.*

(iii) *For each $x \in X$ and for each neighborhood U of x in (X, τ), there exists a $\widetilde{g}s$-neighbourhood W of $f(x)$ in (Y, σ) such that $W \subseteq f(U)$.*

Proof. (i) ⇒ (ii): Suppose f is $\widetilde{g}s$-open. Let $A \subseteq X$. Since $\text{Int}(A)$ is open in (X, τ), $f(\text{Int}(A))$ is $\widetilde{g}s$-open in (Y, σ). Hence $f(\text{Int}(A)) \subseteq f(A)$ and we have, $f(\text{Int}(A)) \subseteq \widetilde{g}s$-$\text{Int}(f(A))$.

(ii) ⇒ (iii): Suppose (ii) holds. Let $x \in X$ and U be an arbitrary neighborhood of x in (X, τ). Then there exists an open set G such that $x \in G \subseteq U$. By assumption, $f(G) = f(\text{Int}(G)) \subseteq \widetilde{g}s\text{-Int}(f(G))$. This implies $f(G) = \widetilde{g}s\text{-Int}(f(G))$. Therefore, $f(G)$ is $\widetilde{g}s$-open in (Y, σ). Further, $f(x) \in f(G) \subseteq f(U)$ and so (iii) holds, by taking $W = f(G)$.

(iii) $\Rightarrow$ (i): Suppose (iii) holds. Let U be any open set in (X, τ), $x \in U$ and $f(x) = y$. Then for each $x \in U$, $y \in f(U)$, by assumption there exists a $\widetilde{g}s$-neighbourhood W_y of y in (Y, σ) such that $W_y \subseteq f(U)$. Since W_y is a $\widetilde{g}s$-neighbourhood of y, there exists a $\widetilde{g}s$-open set V_y in (Y, σ) such that $y \in V_y \subseteq W_y$. Therefore, $f(U) = \bigcup\{V_y : y \in f(U)\}$. Since any union of $\widetilde{g}s$-open sets is $\widetilde{g}s$-open set, $f(U)$ is a $\widetilde{g}s$-open set of (Y, σ). Thus, f is a $\widetilde{g}s$-open mapping. □

Theorem 3.5. *A function $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s$-open if and only if for any subset B of (Y, σ) and for any closed set S containing $f^{-1}(B)$, there exists a $\widetilde{g}s$-closed set A of (Y, σ) containing B such that $f^{-1}(A) \subseteq S$.*

Proof. Similar to Theorem 3.1. □

Corollary 3.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s$-open if and only if $f^{-1}(\widetilde{g}s\text{-Cl}(B)) \subseteq \text{Cl}(f^{-1}(B))$ for every subset B of (Y, σ).*

Proof. Suppose that f is $\widetilde{g}s$-open. Then for any $B \subseteq Y$, $f^{-1}(B) \subseteq \text{Cl}(f^{-1}(B))$. By Theorem 3.5, there exists a $\widetilde{g}s$-closed set A of (Y, σ) such that $B \subseteq A$ and $f^{-1}(A) \subseteq \text{Cl}(f^{-1}(B))$. Therefore, $f^{-1}(\widetilde{g}s\text{-Cl}(B)) \subseteq f^{-1}(A) \subseteq \text{Cl}(f^{-1}(B))$, since A is a $\widetilde{g}s$-closed set in (Y, σ).

Conversely, let S be any subset of (Y, σ) and F be any closed set containing $f^{-1}(S)$. Put $A = \widetilde{g}s\text{-Cl}(S)$. Then A is a $\widetilde{g}s$-closed set and $S \subseteq A$. By assumption, $f^{-1}(A) = f^{-1}(\widetilde{g}s\text{-Cl}(S)) \subseteq \text{Cl}(f^{-1}(S)) \subseteq F$ and therefore by Theorem 3.5, f is $\widetilde{g}s$-open. □

Finally in this section, we define another new class of maps called $\widetilde{g}s^*$-closed maps which are stronger than $\widetilde{g}s$-closed maps.

Definition 3.4. *A map $f : (X, \tau) \to (Y, \sigma)$ is said to be a $\widetilde{g}s^*$-closed map if the image $f(A)$ is $\widetilde{g}s$-closed in (Y, σ) for every $\widetilde{g}s$-closed set A in (X, τ).*

For example, the map f in Example 3.1 is a $\widetilde{g}s^*$-closed map.

Remark 3.2. *Since every closed set is a $\widetilde{g}s$-closed set, we have every $\widetilde{g}s^*$-closed map is a $\widetilde{g}s$-closed map. The converse is not true in general as seen from the following example.*

Example 3.4. *Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, \{a, b\}, X\}$, $\sigma = \{\varnothing, \{a\}, \{a, b\}, Y\}$ and $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is a $\widetilde{g}s$-closed map but not a $\widetilde{g}s^*$-closed map, since $\{a, c\}$ is a $\widetilde{g}s$-closed set in (X, τ), but its image under f is $\{a, c\}$, which is not $\widetilde{g}s$-closed in (Y, σ).*

Proposition 3.4. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is $\widetilde{g}s^*$-closed if and only if $\widetilde{g}s\text{-Cl}(f(A)) \subseteq f(\widetilde{g}s\text{-Cl}(A))$ for every subset A of (X, τ).*

Proof. Similar to Proposition 3.1 □

Analogous to $\widetilde{g}s^*$-closed map we can also define $\widetilde{g}s^*$-open map.

Proposition 3.5. *For any bijection $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:*
(i) *$f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\widetilde{g}s$-irresolute,*
(ii) *f is a $\widetilde{g}s^*$-open and*
(iii) *f is a $\widetilde{g}s^*$-closed map.*

Proof. Similar to Proposition 3.3. □

Lemma 3.1. *Let A be a subset of X. Then $p \in \widetilde{g}s\text{-Cl}(A)$ if and only if for any $\widetilde{g}s$-neighborhood N of p in X, $A \cap N \neq \varnothing$.*

Definition 3.5. *Let A be a subset of X. A maplication $r : X \to A$ is called a $\widetilde{g}s$-continuous retraction if r is $\widetilde{g}s$-continuous and the restriction r_A is the identity mapping on A.*

Definition 3.6. *A topological space (X, τ) is called a $\widetilde{g}s$-Hausdorff if for each pair x, y of distinct points of X, there exists $\widetilde{g}s$-neighborhoods U_1 and U_2 of x and y, respectively, that are disjoint.*

Theorem 3.6. *Let A be a subset of X and $r : X \to A$ be a $\widetilde{g}s$-continuous retraction. If X is $\widetilde{g}s$-Hausdorff, then A is a $\widetilde{g}s$-closed set of X.*

Proof. Suppose that A is not $\widetilde{g}s$-closed. Then there exists a point x in X such that $x \in \widetilde{g}s\text{-Cl}(A)$ but $x \notin A$. It follows that $r(x) \neq x$ because r is $\widetilde{g}s$-continuous retraction. Since X is $\widetilde{g}s$-Hausdorff, there exists disjoint $\widetilde{g}s$-open sets U and V in X such that $x \in U$ and $r(x) \in V$. Now let W be an arbitrary $\widetilde{g}s$-neighborhood of x. Then $W \cap U$ of x. Since $x \in \widetilde{g}s\text{-Cl}(A)$, by Lemma 3.1, we have $(W \cap U) \cap A \neq \varnothing$. Therefore there exists a point y in $W \cap U \cap A$. Since $y \in A$, we have $r(y) = y \in U$ and hence $r(y) \notin V$. This implies that $r(W) \nsubseteq V$ because $y \in W$. This is contrary to the $\widetilde{g}s$-continuity of r. Consequently, A is a $\widetilde{g}s$-closed set of X. □

Theorem 3.7. *Let $\{X_i | i \in I\}$ be any family of topological spaces. If $f : X \to \Pi X_i$ is a $\widetilde{g}s$-continuous mapping, then $P_{r_i} \circ f : X \to X_i$ is $\widetilde{g}s$-continuous for each $i \in I$, where P_{r_i} is the projection of ΠX_j on X_i.*

Proof. We shall consider a fixed $i \in I$. Suppose U_i is an arbitrary open set in X_i. Then $P_{r_i}^{-1}(U_i)$ is open in ΠX_i. Since f is $\widetilde{g}s$-continuous, we have $f^{-1}(P_{r_i}^{-1}(U_i)) = (P_{r_i} \circ f)^{-1}(U_i)$ $\widetilde{g}s$-open in X. Therefore, $P_{r_i} \circ f$ is $\widetilde{g}s$-continuous. □

4. $\widetilde{g}s^*$-Homeomorphisms

In this section we introduce the following definition.

Definition 4.1. *A bijection $f : (X, \tau) \to (Y, \sigma)$ is said to be $\widetilde{g}s^*$-homeomorphisms if both f and f^{-1} are $\widetilde{g}s$-irresolute.*

We denote the family of all $\widetilde{g}s^*$-homeomorphism of a topological space (X, τ) onto itself by $\widetilde{g}s^*\text{-}h(X, \tau)$.

Proposition 4.1. *If $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ are $\widetilde{g}s^*$-homeomorphisms, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is also $\widetilde{g}s^*$-homeomorphism.*

Proof. Let U be $\widetilde{g}s$-open set in (Z, η). Now, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$, where $V = g^{-1}(U)$. By hypothesis, V is $\widetilde{g}s$-open in (Y, σ) and so again by hypothesis, $f^{-1}(V)$ is $\widetilde{g}s$-open in (X, τ). Therefore, $g \circ f$ is $\widetilde{g}s$-irresolute.

Also for a $\widetilde{g}s$-open set G in (X, τ), we have $(g \circ f)(G) = g(f(G)) = g(W)$, where $W = f(G)$. By hypothesis $f(G)$ is $\widetilde{g}s$-open in (Y, σ) and so again by hypothesis, $g(f(G))$ is $\widetilde{g}s$-open in (Z, η). i.e., $(g \circ f)(G)$ is $\widetilde{g}s$-open in (Z, η) and therefore $(g \circ f)^{-1}$ is $\widetilde{g}s$-irresolute. Hence $g \circ f$ is a $\widetilde{g}s^*$-homeomorphism. □

Theorem 4.1. *The set $\widetilde{g}s^*\text{-}h(X, \tau)$ is a group under the composition of maps.*

Proof. Define a binary operation $* : \widetilde{g}s^*\text{-}h(X, \tau) \times \widetilde{g}s^*\text{-}h(X, \tau) \to \widetilde{g}s^*\text{-}h(X, \tau)$ by $f * g = g \circ f$ for all f, $g \in \widetilde{g}s^*\text{-}h(X, \tau)$ and $\circ$ is the usual operation of composition of maps. Then by Proposition 4.1, $g \circ f \in \widetilde{g}s^*\text{-}h(X, \tau)$. We know that the composition of maps is associative and the identity map I: $(X, \tau) \to (X, \tau)$ belonging to $\widetilde{g}s^*\text{-}h(X, \tau)$

servers as the identity element. If $f \in \widetilde{g}s^*\text{-}h(X,\tau)$, then $f^{-1} \in \widetilde{g}s^*\text{-}h(X,\tau)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of $\widetilde{g}s^*\text{-}h(X,\tau)$. Therefore, $(\widetilde{g}s^*\text{-}h(X,\tau), \circ)$ is a group under the operation of composition of maps. □

Theorem 4.2. *Let $f : (X,\tau) \to (Y,\sigma)$ be a $\widetilde{g}s^*$-homeomorphism. Then f induces an isomorphism from the group $\widetilde{g}s^*\text{-}h(X,\tau)$ onto the group $\widetilde{g}s^*\text{-}h(Y,\sigma)$.*

Proof. Using the map f, we define a map θ_f: $\widetilde{g}s^*\text{-}h(X,\tau) \to \widetilde{g}s^*\text{-}(Y,\sigma)$ by $\theta_f(h) = f \circ h \circ f^{-1}$ for every $h \in \widetilde{g}s^*\text{-}h(X,\tau)$. Then θ_f is a bijection. Further, for all h_1, $h_2 \in \widetilde{g}s^*\text{-}h(X,\tau)$, $\theta_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \theta_f(h_1) \circ \theta_f(h_2)$. Therefore, θ_f is a homeomorphism and so it is an isomorphism induced by f. □

Theorem 4.3. *$\widetilde{g}s^*$-homeomorphism is an equivalence relation in the collection of all topological spaces.*

Proof. Reflexivity and symmetry are immediate and transitivity follows from Proposition 4.1. □

Theorem 4.4. *If $f : (X,\tau) \to (Y,\sigma)$ is a $\widetilde{g}s^*$-homeomorphism, then $\widetilde{g}s\text{-Cl}(f^{-1}(B)) = f^{-1}(\widetilde{g}s\text{-Cl}(B))$ for all $B \subseteq Y$.*

Proof. Since f is a $\widetilde{g}s^*$-homeomorphism, f is $\widetilde{g}s$-irresolute. Since $\widetilde{g}s\text{-Cl}(f(B))$ is a $\widetilde{g}s$-closed set in (Y,σ), $f^{-1}(\widetilde{g}s\text{-Cl}(f(B)))$ is $\widetilde{g}s$-closed in (X,τ). Now, $f^{-1}(B) \subseteq f^{-1}(\widetilde{g}s\text{-Cl}(B))$ and so by Proposition 2.3, $\widetilde{g}s\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\widetilde{g}s\text{-Cl}(B))$.

Again since f is a $\widetilde{g}s^*$-homeomorphism, f^{-1} is $\widetilde{g}s$-irresolute. Since $\widetilde{g}s\text{-Cl}(f^{-1}(B))$ is $\widetilde{g}s$-closed in (X,τ), $(f^{-1})^{-1}(\widetilde{g}s\text{-Cl}(f^{-1}(B))) = f(\widetilde{g}s\text{-Cl}(f^{-1}(B)))$ is $\widetilde{g}s$-closed in (Y,σ). Now, $B \subseteq (f^{-1})^{-1}(f^{-1}(B)) \subseteq (f^{-1})^{-1}(\widetilde{g}s\text{-Cl}(f^{-1}(B))) = f(\widetilde{g}s\text{-Cl}(f^{-1}(B)))$ and so $\widetilde{g}s\text{-Cl}(B) \subseteq f(\widetilde{g}s\text{-Cl}(f^{-1}(B)))$. Therefore, $f^{-1}(\widetilde{g}s\text{-Cl}(B)) \subseteq f^{-1}(f(\widetilde{g}s\text{-Cl}(f^{-1}(B))))$ $\subseteq \widetilde{g}s\text{-Cl}(f^{-1}(B))$ and hence the equality holds. □

Corollary 4.1. *If $f : (X,\tau) \to (Y,\sigma)$ is a $\widetilde{g}s^*$-homeomorphism, then $\widetilde{g}s\text{-Cl}(f(B)) = f(\widetilde{g}s\text{-Cl}(B))$ for all $B \subseteq X$.*

Proof. Since $f : (X,\tau) \to (Y,\sigma)$ is a $\widetilde{g}s^*$-homeomorphism, $f^{-1} : (Y,\sigma) \to (X,\tau)$ is also a $\widetilde{g}s^*$-homeomorphism. Therefore, by Theorem 4.4, $\widetilde{g}s\text{-Cl}((f^{-1})^{-1}(B)) = (f^{-1})^{-1}(\widetilde{g}s\text{-Cl}(B))$ for all $B \subseteq X$. i.e., $\widetilde{g}s\text{-Cl}(f(B)) = f(\widetilde{g}s\text{-Cl}(B))$. □

Corollary 4.2. *If $f : (X,\tau) \to (Y,\sigma)$ is a $\widetilde{g}s^*$-homeomorphism, then $f(\widetilde{g}s\text{-Int}(B)) = \widetilde{g}s\text{-Int}(f(B))$ for all $B \subseteq X$.*

Proof. For any set $B \subseteq X$, $\widetilde{g}s\text{-Int}(B) = (\widetilde{g}s\text{-Cl}(B^c))^c$.

$$\begin{aligned} Thus, f(\widetilde{g}s - \text{Int}(B)) &= f((\widetilde{g}s - \text{Cl}(B^c))^c) \\ &= (f(\widetilde{g}s - \text{Cl}(B^c)))^c \\ &= (\widetilde{g}s - \text{Cl}(f(B^c)))^c, \text{ by Corollary 4.1} \\ &= (\widetilde{g}s - \text{Cl}((f(B))^c))^c = \widetilde{g}s - \text{Int}(f(B)). \end{aligned}$$

□

Corollary 4.3. *If $f : (X,\tau) \to (Y,\sigma)$ is a $\widetilde{g}s^*$-homeomorphism, then $f^{-1}(\widetilde{g}s\text{-Int}(B)) = \widetilde{g}s\text{-Int}(f^{-1}(B))$ for all $B \subseteq Y$.*

Proof. Since $f^{-1} : (Y,\sigma) \to (X,\tau)$ is a $\widetilde{g}s^*$-homeomorphism, the proof follows from Corollary 4.2. □

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