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A Class of Compact Complex Manifolds Without Complex Submanifolds

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Abstract. This paper provides a comprehensive survey of the study of complex submanifolds in compact non-Kähler complex manifolds, focusing on the existence and non-existence of such submanifolds, particularly curves and surfaces. It explores classical constructions, including Inoue surfaces, as well as their higher-dimensional generalizations by Oeljeklaus and Toma (OT-manifolds), and more recent families introduced by Endo and Pajitnov (EP-manifolds). These manifolds exhibit a variety of geometric structures and present distinct behaviors concerning the presence of complex submanifolds. The paper revisits key constructions, examining their algebraic, and topological properties, and provides insights into how these properties influence the existence of complex subvarieties. In particular, this study highlights the interplay between number-theoretic data and geometric properties, such as the lack of complex curves in certain OT-manifolds and the nuanced behavior of Endo–Pajitnov manifolds, where the presence of complex submanifolds is sensitive to algebraic parameters. The aim is to offer a unified perspective on the rigidity phenomena characterizing these manifolds, with an emphasis on the interplay between algebraic structures and complex geometry, while also suggesting avenues for future research in this fascinating area of non-Kähler geometry.

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1. Introduction

The study of compact complex manifolds and the existence of their complex submanifolds has been a central topic in complex geometry, touching deep aspects of both the topology and differential geometry of complex spaces. The existence of complex subvarieties often reflects profound properties of the underlying manifold, including its cohomological structure, potential theory, and metric geometry.

This paper serves as a survey of several classes of compact complex manifolds, focusing on the existence or non-existence of complex submanifolds, particularly curves and surfaces. We revisit the classical constructions introduced by Inoue [8], as well as their higher-dimensional generalizations proposed by Oeljeklaus and Toma [11], and more recent families constructed by Endo and Pajitnov [15]. The aim is to provide a unified perspective on how algebraic, arithmetic, and topological structures impact the presence of complex submanifolds in these examples.

Inoue surfaces, introduced in [8], were among the first examples of compact complex surfaces that are non-Kähler and, significantly, contain no complex curves. These

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surfaces are constructed as quotients of $\mathbb{C} \times \mathbb{H}$ by discrete groups of affine transformations, and they possess a solvmanifold structure. Their lack of complex curves was established by a careful analysis of holomorphic sections of line bundles over these surfaces, showing that certain cohomology groups vanish. More details about metric properties and cohomology of these surfaces can be found in [16], [14], [1].

Building on these ideas, Oeljeklaus and Toma introduced in [11] a family of higher-dimensional compact complex manifolds, now known as OT-manifolds, associated with number fields possessing prescribed real and complex embeddings. These manifolds generalize the construction of Inoue surfaces to arbitrary dimensions and provide examples of compact, non-Kähler manifolds whose geometric structure is deeply intertwined with arithmetic properties of the underlying number field. The de Rham and twisted cohomology of these manifolds was computed in [9]. A remarkable feature of certain OT-manifolds is their complete lack of compact complex submanifolds, a phenomenon that first systematically investigated. With respect to the study of complex submanifolds, it has been established that these manifolds do not admit any compact complex curves ([18]). Furthermore, in the event that compact complex surfaces exist, they must be Inoue surfaces S^M ([17]). Moreover, Oeljeklaus–Toma manifolds admitting a locally conformally Kähler metric do not contain any nontrivial complex subvarieties ([12]).

Further developments in this direction were provided by Endo and Pajitnov ([15]), who introduced new families of compact complex manifolds. These manifolds can be viewed as generalizations of Inoue surfaces, whose construction relies heavily on linear algebra techniques and it is based on choosing a matrix with special properties of its spectrum. The authors proved that these new manifolds are non-Kähler and, if the matrix M is diagonalizable, then some of these manifolds are biholomorphic to OT manifolds. Further topological and metric properties of the Endo–Pajitnov manifolds were discussed in [4]. These manifolds, depending on the algebraic properties of associated matrices, can either admit or exclude complex submanifolds. This observation introduces a new layer of richness to the classification problem: while Inoue surfaces and certain OT-manifolds systematically lack complex subvarieties, the Endo–Pajitnov manifolds present a more nuanced behavior, where the existence of complex submanifolds can depend sensitively on number-theoretic data.

In this survey, we systematically present these constructions, summarize the known results regarding the existence or absence of complex subvarieties, and highlight the intricate relations between the algebraic data defining the manifolds (such as eigenvalues and eigenvectors of associated matrices) and the complex geometric properties they exhibit. Special attention is paid to necessary conditions for the absence of complex curves in Endo–Pajitnov manifolds, as well as to the role played by the geometry of their LCK structures.

The structure of the paper is as follows. In Section 2, we recall fundamental notions on complex manifolds. In Section 3 , we review the construction of Inoue surfaces, providing explicit details on the group actions and the methods used to establish the absence of complex curves. Section 4 is devoted to the study of OT-manifolds, emphasizing the role of number-theoretic data in their geometry and the conditions under which they admit LCK metrics or lack complex subvarieties. In Section 5, we turn to Endo–Pajitnov manifolds, where we present the construction, analyze their metric properties, and discuss results concerning the (non-)existence of complex submanifolds.

Through this overview, we aim to underline both the unity and the diversity present in the field of non-Kähler complex geometry, particularly in relation to the problem of complex subvarieties, and to stimulate further investigations into the rigidity phenomena that characterize these fascinating manifolds.

2. Preliminaries

This section presents several definitions that will prove useful in the sections that follow.

Let M be a differentiable manifold of dimension $2n$ with a structure of a complex manifold. A structure of complex manifold is equivalent to an integrable almost complex structure J on M, i.e. an endomorphism of the tangent bundle TM such that $J^2 = -Id_{TM}$ and the Nijenhuis tensor N_J identically vanishes, that is

$$N_J(X,Y) = [X,Y] - J[JX,Y] - J[X,JY] - [JX,JY] = 0$$

for every $X, Y \in TM$. We will denote the extension of J to the complexification of tangent bundle $TM_{\mathbb{C}}$ by J as well and the eigenvalues of this operator are $\pm i$. Hence, we will obtain a direct sum decomposition of $TM_{\mathbb{C}}$, $TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$ in terms of the $\pm i$-eigenspaces of J.

As a first step, we will rigorously establish the necessary conventions. Throughout the paper we shall use the conventions from [2, (2.1)] for the complex structure J acting on complex forms on a complex manifold (M, J). Namely:

- $J\alpha = i^{q-p}\alpha$, for any $\alpha \in A^{p,q}_{\mathbb{C}}M$, or equivalently

$$J\eta(X_1, \ldots, X_p) = (-1)^p \eta(JX_1, \ldots, JX_p);$$

- the fundamental form of a Hermitian metric is given by $\omega(X,Y) := g(JX,Y)$;
- the operator d^c is defined as $d^c := -J^{-1}dJ$, where $J^{-1} = (-1)^{\deg \alpha} J$.

Let $z_1, \ldots, z_n$ be local complex coordinates in an open neighborhood of the point $y \in M$.

Definition 2.1. A $(1,1)$-*form* on M is a 2-form ω, such that $\omega(JX,Y) = -\omega(X,JY) = i\omega(X,Y)$ for each $X, Y \in T_yM$.

Definition 2.2. A $(1,1)$-form ω on M is *semipositive* if $\omega(X,JX) \geqslant 0$ for each tangent vector $X \in T_yM$.

3. Inoue surfaces

In 1974, M. Inoue ([8]) introduced three types of complex compact surfaces, S^M (or S^0), S^+, S^-. These surfaces are cocompact quotients of $\mathbb{C} \times \mathbb{H}$ (where $\mathbb{H}$ stands for the Poincaré half-plane) by a discrete, cocompact group of complex affine transforms. Moreover, they are diffeomorphic to solvmanifolds.

3.1. Inoue surfaces of class S^M. We recall the construction of S^M, following the linear algebraic approach from Inoue's original paper.

Let $M = (m_{ij})$ be a matrix in $SL_3(\mathbb{Z})$ with one real eigenvalue $\alpha > 1$ and two complex eigenvalues β and $\bar{\beta}$. We denote by $(a_1, a_2, a_3)^t$ a real eigenvector of α and by $(b_1, b_2, b_3)^t$ a complex eigenvector of β. Let G_M be the group of affine transformations of $\mathbb{C} \times \mathbb{H}$ generated by the transformations g_0, g_1, g_2, g_3, where:

$$\begin{aligned} g_0(z, w) &= (\beta z, \alpha w), \\ g_i(z, w) &= (z + b_i, w + a_i), \end{aligned}$$

for all $i = 1, 2, 3$. We also denote by H_M the subgroup of $\mathrm{Aut}(\mathbb{C} \times \mathbb{H})$ generated only by translations, $H_M =< g_1, g_2, g_3 >$.

Definition 3.1. The surface $S^M := (\mathbb{C} \times \mathbb{H})/G_A$ is called an Inoue surface of class S^M. It is a compact complex surface, where the complex structure, which we shall denote by J, is the the one inherited from $\mathbb{C} \times \mathbb{H}$.

Remark 3.1. It is easy to see that S^M is a mapping torus of a $3-$dimensional torus $\mathbb{T}^3$ and we have the following relations between the generators of G_M:

$$\begin{aligned} g_i g_j = g_j g_i, \quad &\text{for} \quad i, j = \overline{1, 3}, \\ g_0 g_i g_0^{-1} = g_1^{m_{i1}} g_2^{m_{i2}} g_3^{m_{i3}}, \quad &\text{for} \quad i = \overline{1, 3}. \end{aligned}$$

Remark 3.2. From the metric viewpoint, Inoue surfaces are non-Kähler (they have the first Betti number equals one); however, in [16], Triceri provides an explicit construction of a locally conformally Kähler (LCK) metric on S^M. The LCK structure is defined on $\mathbb{C} \times \mathbb{H}$ as a globally conformally Kähler structure that is invariant under the action of the group G_M. The expressions for the metric and the Lee form, respectively, are:

$$\begin{aligned} g &= -\mathrm{i} \left(\frac{dw \otimes d\overline{w}}{\mathrm{Im}(w)^2} + \mathrm{Im}(w) dz \otimes d\bar{z} \right), \\ \theta &= \frac{d\,\mathrm{Im}(w)}{\mathrm{Im}(w)}. \end{aligned}$$

We now present the following result concerning the existence of one-dimensional complex submanifolds.

Theorem 3.1. ([8, Proposition 2]) S^M *contains no complex curves.*

Recall from [7, Chapter I.1] that for any hypersurface in a complex manifold, there is an associated holomorphic line bundle which admits a global holomorphic section such that the divisor associated to this section is precisely the divisor of the starting hypersurface ([7], ch.1.1, p 136).

Inoue proved that $H^0(S^M, \mathcal{O}(F)) = 0$, for any nontrivial complex line bundle F on S^M, and as a direct consequence S^M contains no curves. To establish this result, he used the fact that any holomorphic section $\psi \in H^0(S^M, \mathcal{O}(F))$ can be seen in terms of holomorphic function f on $\mathbb{C} \times \mathbb{H}$, which is G_M-invariant. However, such a function must be constant.

3.2. Inoue surfaces of class S^+. Let $N = (n_{ij}) \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix with real eigenvalues $\alpha > 1$ and $1/\alpha$ and $(a_1, a_2)^t, (b_1, b_2)^t$ real eigenvectors corresponding to α and $1/\alpha$. Let us fix some integers p, q, r with $r \neq 0$ and a complex number t. Let e_1, e_2 be defined as

$$e_i = \tfrac{1}{2} n_{i1}(n_{i1} - 1)a_1 b_1 + \tfrac{1}{2} n_{i2}(n_{i2} - 1)a_2 b_2 + n_{i1} n_{i2} b_1 a_2$$

and c_1, c_2 defined by

$$(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q).$$

We denote by $G^+_{N,p,q,r,t}$ the group of affine transformations of $\mathbb{H} \times \mathbb{C}$ generated by the following:

$$\begin{aligned} g_0(w, z) &= (\alpha w, z + t), \\ g_i(w, z) &= (w + a_i, z + b_i w + c_i), \qquad i = 1, 2, \\ g_3(w, z) &= (w, z + \frac{b_1 a_2 - b_2 a_1}{r}). \end{aligned}$$

The action of $G^+_{N,p,q,r,t}$ on $\mathbb{H} \times \mathbb{C}$ is properly discontinuous and without fixed points (see [8]).

Definition 3.2. The compact complex surface $S^+_{N,p,q,r,t} := (\mathbb{H} \times \mathbb{C})/G^+_{N,p,q,r,t}$ is called an Inoue surface of class S^+.

Remark 3.3. The following relations hold:

$$\begin{aligned} g_3 g_i &= g_i g_3, \qquad \text{for} \qquad i = 0, 1, 2, \\ g_1^{-1} g_2^{-1} g_1 g_2 &= g_3^r, \\ g_0 g_1 g_0^{-1} &= g_1^{n_{11}} g_2^{n_{12}} g_3^p, \\ g_0 g_2 g_0^{-1} &= g_1^{n_{21}} g_2^{n_{22}} g_3^q. \end{aligned}$$

Remark 3.4. As in the case of S^M, $b_1(S^+_{N,p,q,r,t}) = 1$, and hence can not be Kähler. Depending on the choice of t, two distinct cases arise:

(1) If $t \in \mathbb{R}$, Tricerri ([16]) found a LCK $G^+_{N,p,q,r,t}-$invariant metric on$S^+_{N,p,q,r,t}$. If we consider $(w, z) = ((w_1, w_2), (z_1, z_2))$ the coordinates on $\mathbb{H} \times \mathbb{C}$ this metric is given by:

$$\begin{aligned} g &= \frac{1}{w_2^2}((w_2 dz) \otimes (w_2 d\bar{z} - z_2 d\bar{w}) + dw \otimes d\bar{w}), \\ \theta &= \frac{dw_2}{w_2}. \end{aligned}$$

(2) if $t \in \mathbb{C} \setminus \mathbb{R}$, $S^+_{N,p,q,r,t}$ does not carry an LCK metric ([1]).

By a similar argument to that used in the case of S^M surfaces, Inoue obtained the following result:

Theorem 3.2. [8, Proposition 3] *The surface $\mathcal{S}^+_{N,p,q,r,t}$ contains no complex curves.*

3.3. Inoue surfaces of class S^-. Let $N \in \mathrm{GL}_2(\mathbb{Z})$ with $\det N = -1$ and eigenvalues $\alpha > 1$ and $-\frac{1}{\alpha}$. We consider $(a_1, a_2)^t$ and $(b_1, b_2)^t$ real eigenvectors corresponding to α and $-\frac{1}{\alpha}$ and let c_1, c_2 be defined by

$$-(c_1, c_2) = (c_1, c_2) \cdot N^t + (e_1, e_2) + \frac{b_1 a_2 - b_2 a_1}{r}(p, q),$$

where e_1 and e_2 are defined as in the case of S^+ and p, q, r $(r \neq 0)$ are integers. We denote by $G^-_{N,p,q,r}$ the group of analytic automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by:

$$\begin{aligned} g_0(w, z) &= (\alpha w, -z), \\ g_i(w, z) &= (w + a_i, z + b_i w + c_i), \qquad i = 1, 2, \\ g_3(w, z) &= (w, z + \frac{b_1 a_2 - b_2 a_1}{r}). \end{aligned}$$

Definition 3.3. The compact complex surface $S^-_{N,p,q,r} := (\mathbb{H} \times \mathbb{C})/G^-_{N,p,q,r}$ is called an Inoue surface of class S^-.

To prove that $\mathcal{S}^-_{N,p,q,r}$ contains no complex curves, we use the elementary fact that

$$\langle g_0^2, g_1, g_2, g_3 \rangle = G^+_{N^2,p_1,p_2,r,0},$$

for some integers p_1, p_2. This implies that $S^+_{N^2,p_1,p_2,r,0}$ is an unramified double covering of $S^-_{N,p,q,r}$. Since the covering space contains no complex curves, the same holds for $S^-_{N,p,q,r}$.

4. Oeljeklaus–Toma manifolds

Oeljeklaus–Toma manifolds, introduced in 2005 by Karl Oeljeklaus and Matei Toma [11], form a remarkable class of compact complex manifolds that generalize the construction of Inoue surfaces S^M to higher dimensions. They arise as quotients of $\tilde{X} := \mathbb{H}^s \times \mathbb{C}^t$ by discrete, properly discontinuous actions of groups built from the ring of integers and unit group of a number field.

4.1. Construction of Oeljeklaus–Toma manifolds. We will recall the construction of Oeljeklaus–Toma manifolds, following [11].

Let K be an algebraic number field and let $n := [K : \mathbb{Q}]$ be its degree. The field K admits n distinct embeddings into $\mathbb{C}$: $\sigma_1, \ldots, \sigma_n$, where $\sigma_1, \ldots, \sigma_s$ are the real embeddings and $\sigma_{s+1}, \ldots, \sigma_n$ are the complex embeddings. Because the complex embeddings of K into $\mathbb{C}$ always occur in conjugate pairs, the number $n-s$ is necessarily even; we denote it by $2t$ (we will use the convention that $\sigma_{s+t+i} = \overline{\sigma_{s+i}}$, for any $1 \leq i \leq t$).

Definition 4.1. The ring of algebraic integers O_K is a subring of K that consists of all roots of polynomials with integer coefficients which lie in K.

The unit group O_K^* is the multiplicative subgroup of invertible elements of O_K.

We denote by $O_K^{*,+}$ the group of units which are positive in all the real embeddings of K.

The ring of integers O_K acts on $\mathbb{H}^s \times \mathbb{C}^t$, and the action is given by translations via the first $s+t$ embeddings:

$$T_a(w,z) = (w_1 + \sigma_1(a), \dots, w_s + \sigma_s(a), z_1 + \sigma_{s+1}(a), \dots, z_t + \sigma_{s+t}(a)), \quad a \in O_K.$$

It is a free and proper action, and as a smooth manifold, the quotient is given by:

$$\hat{X} := \mathbb{H}^s \times \mathbb{C}^t / O_K \simeq (\mathbb{R}^*_+)^s \times \mathbb{T}^n.$$

Moreover, the group $O_K^{*,+}$ acts on $\mathbb{H}^s \times \mathbb{C}^t$ by dilatations:

$$R_u(w,z) = (\sigma_1(u)w_1, \dots, \sigma_s(u)w_s, \sigma_{s+1}(u)z_1, \dots, \sigma_{s+t}(u)z_t), \quad u \in O_K^{*,+}.$$

This action is free, but not properly discontinuous. However, one can choose a rank s subgroup $U \subset O_K^{*,+}$, such that U acts properly discontinuously on $\mathbb{H}^s \times \mathbb{C}^t$. Also, U acts on O_K so that one gets a free, properly discontinuous action of the semi-direct product $U \rtimes O_K$ on $\mathbb{H}^s \times \mathbb{C}^t$.

Definition 4.2. The manifold $X(K,U) := (\mathbb{H}^s \times \mathbb{C}^t)/(U \rtimes O_K)$ is an Oeljeklaus–Toma manifold of type (s,t). It is a compact complex manifold of dimension $s+t$.

Remark 4.1. For $s = t = 1$ and $U = O_K^{*,+}$, $X(K,U)$ is an Inoue surface of class S^0.

Remark 4.2. Oeljeklaus and Toma proved that the first Betti number $b_1 X(K,U) = s$ and if U is of simple type $b_2(X,U) = \binom{s}{2}$ ([11, Proposition 2.3]).

The de Rham cohomology and twisted cohomology of OT-manifolds was computed in terms of numerical invariants coming from number field K (for details, see [9, Theorem 3.1, Theorem 5.1]).

From the LCK geometry viewpoint, we have that if $t = 1$, all OT-manifolds $X(K,U)$ has an LCK metric ([11]). The existence of LCK metric for OT-manifolds which are not of type (s,1) was intensively studied and solved by combining the result from [11], [6] and [5]. Moreover, Kasuya proved that OT-manifolds cannot carry any Vaisman metric ([10]).

4.2. Submanifolds of Oeljelkaus-Toma manifolds. Since Oeljeklaus–Toma manifolds generalize Inoue surfaces of class S^0, and given that Inoue surfaces are known to admit no complex curves, it is natural to investigate whether OT-manifolds admit nontrivial complex submanifolds.

In the study of one-dimensional submanifolds, the following theorem holds:

Theorem 4.1. [18, Theorem 2.9] *Let X be an Oeljeklaus–Toma manifold. There are no compact complex curves on X.*

The idea of the proof is to consider, on the universal cover $\tilde{X} := \mathbb{H}^s \times \mathbb{C}^t$ a semipositive $(1,1)-$ form $\tilde{\omega}$, defined by $\tilde{\omega} = i\sum_{i=1}^s \frac{dz_i \wedge d\bar{z}_i}{4(\operatorname{im} z_i)^2}$. Since $\tilde{\omega}$ is invariant under the action of $U \rtimes O_K$ it descends to an exact $(1,1)-$form ω on X. The leaves of the zero foliations of this $\tilde{\omega}$ are biholomorphic with $\mathbb{C}^t$. Moreover, since $\tilde{\omega}$ is invariant, each leaf of the zero foliation of ω on X is isomorphic to a component of the leaf of the zero foliation of $\tilde{\omega}$ on $\tilde{X}$. Using semipositivity, any compact complex curve must be contained entirely within a leaf of the zero foliation of ω. However, since $\mathbb{C}^t$ admits no compact complex submanifolds, we conclude that X does not contain any compact complex curves.

There exists, by the same author, an extension of this theorem, to

Theorem 4.2. [17, Theorem 3.5] *Let X be an Oeljeklaus–Toma manifold. X could not contain any nontrivial compact complex submanifolds of dimension 2, except the Inoue surfaces.*

Using the same $(1,1)-$form as in the previous theorem, the author proved that every surface in an Oeljeklaus-Toma manifold is of Kähler rank 1. Then, by applying Brunella's theorem ([3]), which classifies compact connected complex surfaces of Kähler rank 1, and using the fact that OT-manifolds contain no compact complex curves, the desired conclusion was obtained.

Based on the properties of the number field K, we can determine which Oeljeklaus–Toma manifolds contain an Inoue surface.

Claim 4.1: [17, Claim 3.6] *For each number field K which contains a subfield K_1 with exactly one real embedding and two complex embeddings, there exists an Oeljeklaus-Toma manifold which contains an Inoue surface.*

It is well known that, with suitable choices of the number field K and the admissible subgroup U, the corresponding Oeljeklaus–Toma manifold $X(K,U)$ may contain proper complex submanifolds. For example, if K is a proper extension of another number field L and if $U \subset O_L^{*,+}$, then $X(L,U) \subset X(K,U)$ (see [11, Remark 1.7]).

Also, there exists an example of an OT submanifold embedded in an OT-manifold which is *of simple type*.

Example 4.1. [13, Example 3.1] Take $L = \mathbb{Q}[X]/(X^3-2)$; then L has one real embedding τ_1 and $2t = 2$ complex ones $\tau_2, \tau_3 (= \overline{\tau}_2)$. Note that $U_L = \mathcal{O}_L^{*,+}$ is a free group of rank one, and denote u_1 be a generator for U_L. Then U_L is an admissible group, and let $X(L,U_L)$ is the corresponding OT-manifold (an Inoue surface S^0).

Now take $K = \mathbb{Q}[X]/(X^6-2)$. The field K is an extension of degree 2 of L which has two real embeddings σ_1, σ_2 (which both extend the embedding τ_1 of L) and four complex embeddings: σ_3, σ_4 (which extend τ_2) and $\sigma_5 = \overline{\sigma}_3, \sigma_6 = \overline{\sigma}_4$ (which extend $\tau_3 = \overline{\tau}_2$). Consider the unit $u_2 \in \mathcal{O}_K^{*+}$ such that $\sigma_1(u_2) = (\sqrt[6]{2}-1)^2$. Then $\sigma_2(u_2) = (\sqrt[6]{2}+1)^2$, and hence the subgroup $U_K \subset \mathcal{O}_K^{*+}$ generated by u_1 and u_2 is admissible.

Let $X = X(K,U_K)$ be the corresponding OT-manifold. Define the map $i : X(L,U_L) \longrightarrow X(K,U_K)$ by

$$i([w,z]) = [w,w,z,z],$$

where we denoted by $[x]$ the equivalence classe of x. Clearly, i is well-defined and injective.

We retrict to the case when $t = 1$, where LCK metrics exist. In this setting, the following result holds

Theorem 4.3. [12, Theorem 3.1] *Let $[K : \mathbb{Q}]$ be a number field of degree $n = s + 2$, with s real embeddings and two complex embeddings, and let $X(K,U)$ be the corresponding Oeljeklaus-Toma manifold. Then $X(K,U)$ has no non-trivial complex subvarieties.*

In the proof, the authors analyzed the zero foliation Σ of the same $(1,1)$-form ω as in 4.1, showing that all its leaves are Zariski dense in X. They proved that any complex subvariety $Z \subset X$ must contain, for each point $z \in Z$, the entire leaf Σ_z passing through z. Since all leaves of Σ are Zariski dense, it follows that Z itself is Zariski dense. The density argument is based on the application of the Strong Approximation Theorem.

5. Endo-Pajitnov manifolds

The Endo–Pajitnov manifolds are a higher-dimensional generalization of Inoue surfaces, constructed in the same spirit as Inoue's original construction. They are a quotient of $\mathbb{H} \times \mathbb{C}^n$ by a freely and properly discontinuously acting group of automorphisms, resulting in a compact non-Kähler complex manifold. Depending on the initial choice of the matrix M, on which the construction is based, some EP manifolds are biholomorphic to OT manifolds. What follows is a brief overview of the construction of these manifolds, along with a summary of existing results on their submanifolds.

5.1. Construction of Endo-Pajitnov manifolds. In this section, we recall the construction of the Endo–Pajitnov manifolds, as introduced in [15].

Let $n > 1$ and $M = (m_{ij})_{i,j} \in \mathrm{SL}(2n+1, \mathbb{Z})$ such that the eigenvalues of M are $\alpha, \beta_1, \cdots, \beta_k, \overline{\beta_1}, \cdots, \overline{\beta_k}$ with $\alpha > 0, \alpha \neq 1$ and $\mathrm{Im}(\beta_j) > 0$.

Denote by V the eigenspace corresponding to α and set:

$$W(\beta_j) = \{x \in \mathbb{C}^{2n+1} \mid \exists\, N \in \mathbb{N} \text{ such that } (M - \beta_j I)^N x = 0\},$$

$$W = \bigoplus_{j=1}^{k} W(\beta_j), \qquad \overline{W} = \bigoplus_{j=1}^{k} W(\overline{\beta_j}).$$

We then have $\mathbb{C}^{2n+1} = V \bigoplus W \bigoplus \overline{W}$.

Let $a \in \mathbb{R}^{2n+1}$ be a non-zero eigenvector corresponding to α and fix a basis $\{b_1, \ldots, b_n\}$ in W:

$$a = (a^1, a^2, \ldots, a^{2n+1})^T, \quad b_i = (b_i^1, b_i^2 \ldots, b_i^{2n+1})^T, \quad i = \overline{1, n}.$$

For any $1 \leq i \leq 2n+1$, we let $u_i = (a^i, b_1^i, ..., b_n^i) \in \mathbb{R} \times \mathbb{C}^n \simeq \mathbb{R}^{2n+1}$.

Note that $\{u_1, ..., u_{2n+1}\}$ are linearly independent over $\mathbb{R}$, since $\{a, b_1, \ldots, b_n, \overline{b_1} \ldots, \overline{b_n}\}$ is a basis in $\mathbb{C}^{2n+1}$.

Let now $f_M : W \longrightarrow W$ be the restriction of the multiplication with M on W and $R = (r_{ij})_{i,j}$ the matrix of f_M with respect to the basis $\{b_1, ..., b_n\}$. Let $\mathbb{H}$ be the Poincaré upper half-plane, and consider the automorphisms $g_0, g_1, \ldots, g_{2n+1} : \mathbb{H} \times \mathbb{C}^n \longrightarrow \mathbb{H} \times \mathbb{C}^n$,

$$g_0(w, z) = (\alpha w, R^T z), \quad g_i(w, z) = (w, z) + u_i, \quad w \in \mathbb{H}, z \in \mathbb{C}^n, \quad i = 1, \ldots, 2n+1.$$

These automorphisms are well defined because $\alpha > 0$ and the first component of u_i is $a^i \in \mathbb{R}$.

Let G_M be the subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{C}^n)$ generated by $g_0, g_1, \ldots, g_{2n+1}$.

Theorem 5.1. *([15]) The action of G_M on $\mathbb{H} \times \mathbb{C}^n$ is free and properly discontinous. Hence, the quotient $T_M := G_M \backslash (\mathbb{H} \times \mathbb{C}^n)$ is a compact complex manifold of complex dimension $n+1$, with $\pi_1(T_M) \simeq G_M$.*

Definition 5.1. The above quotient $T_M := G_M \backslash (\mathbb{H} \times \mathbb{C}^n)$ is called an *Endo–Pajitnov manifold.*

Remark 5.1. In the same paper, the authors prove that:

- If M is diagonalizable, then some T_M are biholomorphic to OT manifolds ([15, Proposition 5.3]);
- If M is not diagonalizable, then T_M cannot be biholomorphic to any OT manifold ([15, Proposition 5.6]).

5.2. Submanifolds in Endo-Pajitnov manifolds. In contrast to the situation for Inoue surfaces, where the absence of complex curves is a general property, the existence of complex curves in Endo–Pajitnov manifolds T_M depends crucially on the choice of the matrix M. In particular, for certain matrices M satisfying appropriate conditions, one can explicitly construct examples of Endo–Pajitnov manifolds that admit nontrivial compact complex subvarieties of dimension one.

Example 5.1. [3, Example 3.2]
Let $n = 2$, $k = 1$, and a diagonalizable matrix M

$$M = \begin{pmatrix} 1 & 2 & -1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We now express the matrix M in block form as follows:

$$M = \begin{pmatrix} N & 0 \\ 0 & P \end{pmatrix},$$

where

$$N = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix}, \quad \text{and} \quad P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that $M \in \mathrm{SL}(5, \mathbb{Z})$. The characteristic polynomial of M is $P_M = (X^3 + 3X - 1)(X^2 + 1)$, hence,

$$\mathrm{Spec}(M) = \left\{\alpha, \beta_1, \beta_2, \overline{\beta_1}, \overline{\beta_2} \mid \alpha \in \mathbb{R}, \alpha \in (0,1), \mathrm{Im}(\beta_j) > 0, j = \overline{1,2}\right\}.$$

Therefore, M satisfies the special conditions from construction of Endo–Pajitnov manifold. In this particular case, the matrix R is

$$R = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$$

.

We can explicitly describe the automorphisms $g_0, g_1, g_2, g_3, g_4, g_5 : \mathbb{H} \times \mathbb{C}^2 \longrightarrow \mathbb{H} \times \mathbb{C}^2$,

$$g_0(w, (z_1, z_2)) = (\alpha w, (\beta_1 z_1, \beta_2 z_2)),$$

$$g_i(w, (z_1, z_2)) = (w, (z_1, z_2)) + \begin{cases} (a^i, b_1^i, 0) & \text{for } 1 \le i \le 3 \\ (0, 0, b_2^i) & \text{for } 4 \le i \le 5 \end{cases}, w \in \mathbb{H}, z_1, z_2 \in \mathbb{C}.$$

Consequently, M defines an Endo–Pajitnov manifold T_M of complex dimension 3.
Moreover, $N \in \mathrm{SL}(3, \mathbb{Z})$ defines an Inoue surface of type S^N, which we denote by T_N.

One can define the projection $\pi : T_M \longrightarrow T_N$

$$\pi\left([w, (z_1, z_2)]\right) = [[w, z_1]], \quad w \in \mathbb{H}, z_1, z_2 \in \mathbb{C}.$$

Since π is a holomorphic submersion, it follows that T_M projects over an Inoue surface, with complex curves as fibres.

By extending the previous constructions to higher dimensions, we are able to generalize the results and derive new insights concerning the existence of complex submanifolds within Endo–Pajitnov manifolds.

Theorem 5.2. [3, Theorem 3.1] *Let X be an Endo–Pajitnov manifold associated to a diagonal block matrix, one of the blocks producing a (smaller dimensional) Endo–Pajitnov manifold. Then X contains complex submanifolds.*

The identification of those Endo–Pajitnov manifolds admitting complex submanifolds relies on constructing a holomorphic submersion onto another complex manifold, with fibers that are complex manifolds.

Additionally, a necessary condition for the manifold T_M to not contain complex curves can be identified. This condition is algebraic and is expressed in terms of the components of the eigenvector a associated with the real eigenvalue α of the matrix M.

Theorem 5.3. [3, Theorem 4.1] *Let T_M be an Endo–Pajitnov manifold. If the components of the eigenvector a associated to the real eigenvalue α of the matrix M are linearly independent over $\mathbb{Z}$, then there are no compact complex curves on T_M.*

Remark 5.2. It is clear that 5.1 does not satisfy the condition prescribed in 5.3.

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