Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 33, 2006, Pages 216–226
ISSN: 1223-6934

On Some properties of fractionnal Sobolev Spaces

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Abstract. In this paper we prove some basic properties of fractionnal Sobolev spaces. First, we view $W^{s,p}$ as an intermediate space between L^p and $W^{1,p}$. Then, classical results of $W^{1,p}$ may be extended to $W^{s,p}$. When Ω is an extension domain, we give a complete proof of the existence of a bounded linear and surjective mapping tr : $W^{1,p}(\Omega) \to W^{1-1/p,p}(\Omega)$ which extend the restriction operator defined on the dense subset $\mathcal{D}(\overline{\Omega})$. This was partially done and suggested in [2].

2000 Mathematics Subject Classification. Primary 60J05; Secondary 60J20.
Key words and phrases. fractionnal Sobolev spaces, extension theorem, traces in Sobolev spaces.

1. Fractionnal Sobolev Spaces

Let s be a positive real number such that $s \notin \mathbb{N}$, $1 \leq p < \infty$ and Ω an open set of $\mathbb{R}^N$. When $0 < s < 1$, the fractionnal Sobolev space of order s, p on Ω is defined by

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty\} . \tag{1}$$

Otherwise, denoting by $[s]$ the integer part of s, let

$$W^{s,p}(\Omega) = \{u \in W^{[s],p}(\Omega) : \partial^\alpha u \in W^{s-[s],p}(\Omega), \forall \alpha \in \mathbb{N}^N, |\alpha| = [s]\} .$$

In this paper, we only consider the generic case $0 < s < 1$. Notice that the definition (1) is equivalent to the following

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) : \Lambda_u \in L^p(\Omega \times \Omega)\} ,$$

where

$$\Lambda_u(x, y) = \frac{u(x) - u(y)}{|x-y|^{N/p+s}}$$

is called the $s-$derivative of u. The two next properties are trivial : if u is a constant mapping then the $s-$derivative of u is equal to 0; if u and v are defined on Ω, then a.e. $\Lambda_{uv}(x, y) = u(x)\Lambda_v(x, y) + v(y)\Lambda_u(x, y)$ (product's rule).

An usual norm on $W^{s,p}(\Omega)$ is given by

$$\|u\|_{s,p,\Omega} = \|u\|_{p,\Omega} + \|\Lambda_u\|_{p,\Omega\times\Omega} . \tag{2}$$

The quantity $\|\Lambda_u\|_{p,\Omega\times\Omega}$ is called the semi-norm $W^{s,p}$ of u and is usually denoted by $|||u|||_{s,p,\Omega}$. In order to prove that $W^{s,p}$ is complete and separable, the definition (2) suggest to use the same argument as in $W^{1,p}$ by showing that the isometry

$$\mathcal{A} : W^{s,p}(\Omega) \to L^p(\Omega) \times L^p(\Omega \times \Omega) : u \to (u, \Lambda_u) ,$$

is such that $\mathcal{A}(W^{s,p}(\Omega))$ is closed in $L^p(\Omega) \times L^p(\Omega \times \Omega)$. The answer is given by the following lemma.

Lemma 1.1. *If* $u_n \to u$ *in* $\mathrm{L}^p(\Omega)$ *and* $\Lambda_n \to \Lambda$ *in* $\mathrm{L}^p(\Omega \times \Omega)$*, and if, for all* n*,* $\Lambda_n = \Lambda_{u_n}$*, then* $\Lambda = \Lambda_u$*.*

Proof. Up to a subsequence, one may assume that $u_n \to u$ a.e. and therefore $\Lambda_n \to \Lambda_u$ a.e., hence $\Lambda = \Lambda_u$. □

As consequence, we give the following corollary.

Corollary 1.1. $\mathrm{W}^{s,p}(\Omega)$ *is a separable Banach space. Moreover, if* $1 < p < \infty$*, then* $\mathrm{W}^{s,p}(\Omega)$ *is reflexive.*

The notation of fractionnal Sobolev spaces is partially justified by the following theorem.

Theorem 1.1. *The space* $\mathrm{W}^{1,p}(\mathbb{R}^N)$ *is continuously injected into* $\mathrm{W}^{s,p}(\mathbb{R}^N)$*. Moreover, if* $u \in \mathrm{W}^{1,p}(\mathbb{R}^N)$*, then*

$$|||u|||_{s,p} \leq \Big(\frac{2\mathrm{vol}(\mathrm{S}^{N-1})}{ps(1-s)}\Big)^{1/p} \|\nabla u\|_p^s \; \|u\|_p^{1-s} \ . \tag{3}$$

Proof. Let $u \in \mathcal{D}(\mathbb{R}^N)$. First write

$$\begin{aligned} |||u|||_{s,p}^p &= \int_{\mathbb{R}^N} |\xi|^{-(N+sp)} d\xi \int_{\mathbb{R}^N} |u(x+\xi) - u(x)|^p dx \\ &= \Big[\int_{|\xi|<\lambda} + \int_{|\xi|>\lambda}\Big] |\xi|^{-(N+sp)} d\xi \int_{\mathbb{R}^N} |u(x+\xi)-u(x)|^p dx, \end{aligned}$$

where λ is a positive parameter independant of x and ξ wich will be choosen later. Using polar coordinates,

$$\int_{|\xi|>\lambda} |\xi|^{-(N+sp)} d\xi \int_{\mathbb{R}^N} |u(x+\xi)-u(x)|^p dx \quad \leq \quad 2^p \frac{\mathrm{vol}(\mathrm{S}^{N-1})\lambda^{-sp}}{sp} \|u\|_p^p \ ,$$

Similarly,

$$\begin{aligned} &\int_{|\xi|<\lambda} |\xi|^{-(N+sp)} d\xi \int_{\mathbb{R}^N} |u(x+\xi)-u(x)|^p dx \\ = &\int_{|\xi|<\lambda} |\xi|^{-(N+sp)} d\xi \int_{\mathbb{R}^N} \Big| \int_0^1 \frac{d}{dt} u(x+t\xi) dt \Big|^p dx \\ \leq &\int_{|\xi|<\lambda} |\xi|^{p-(N+sp)} d\xi \int_{\mathbb{R}^N} \Big(\int_0^1 |\nabla u(x+t\xi)| dt \Big)^p dx \\ \leq &\int_{|\xi|<\lambda} |\xi|^{p-(N+sp)} d\xi \int_{\mathbb{R}^N} |\nabla u(y)|^p dy \ , \end{aligned}$$

where the last inequality is obtained by using Holder's inequality. Finally, using again polar coordinates

$$\int_{|\xi|<\lambda} |\xi|^{-(N+sp)} d\xi \int_{\mathbb{R}^N} |u(x+\xi)-u(x)|^p dx \quad \leq \quad \frac{\mathrm{vol}(\mathrm{S}^{N-1})\lambda^{p(1-s)}}{p(1-s)} \|\nabla u\|_p^p \ .$$

Combining these two estimates,

$$|||u|||_{s,p}^p \leq 2^p \ \frac{\mathrm{vol}(\mathrm{S}^{N-1})\lambda^{-sp}}{sp} \|u\|_p^p + \frac{\mathrm{vol}(\mathrm{S}^{N-1})\lambda^{p(1-s)}}{p(1-s)} \|\nabla u\|_p^p \ . \tag{4}$$

Let $u \in \mathrm{W}^{s,p}(\mathbb{R}^N)$ and consider a sequence $(u_n) \subset \mathcal{D}(\mathbb{R}^N)$ such that $u_n \to u$ in $\mathrm{W}^{1,p}$. Since u_n satisfies, for all n, inequality (4), by applying Fatou's lemma, one gets the inequality (4). The proof is concluded by choosing $\lambda = 2\frac{\|u\|_p}{\|\nabla u\|_p}$, which minimize the r.h.s. of (4). □

This theorem remains true where $\mathbb{R}^N$ is replaced by any extension domain Ω for $\mathrm{W}^{1,p}$ (i.e. Ω is such that there exists a bounded linear extension operator to $\mathrm{W}^{1,p}(\mathbb{R}^N)$) .

Corollary 1.2. *Let Ω be an C^1-open subset of $\mathbb{R}^N$ with bounded boundary $\partial\Omega$ or the product of N open intervals. The space $\mathrm{W}^{1,p}(\Omega)$ is continuously injected in $\mathrm{W}^{s,p}(\Omega)$.*

Now, we turn to the problem of density of test functions. Theorem 1.1 partially suggest to transpose the proof of the corresponding result in $\mathrm{W}^{1,p}$.

Theorem 1.2. *$\mathcal{D}(\mathbb{R}^N)$ is dense in $\mathrm{W}^{s,p}(\mathbb{R}^N)$.*

Proof. Let u be a map in $\mathrm{W}^{s,p}(\mathbb{R}^N)$ and $\eta \in \mathrm{C}^\infty(\mathbb{R})$ be such that $\eta = 1$ on $]-\infty, 1]$, $0 \le \eta \le 1$, supp $\eta \subset]-\infty, 2]$. Set $\eta_n(x) = \eta(|x|/n)$. Notice that the sequence $(\eta_n) \subset \mathcal{D}(\mathbb{R}^N)$ is such that for all n, $0 \le \eta_n \le 1$, $\eta_n = 1$ on $B(0,n)$, $\mathrm{supp}(\eta_n) \subset B[0,2n]$ and $|\nabla \eta_n| \le C/n$ where $C > 0$ is independant of n. To prove that $u_n = u\eta_n \in \mathrm{W}^{s,p}$ and $\|u - u_n\|_{s,p} \to 0$ when $n \to \infty$, we use an argument of dominated convergence. Clearly, $\|u - u_n\|_p \to 0$, so that one only has to prove that $\|\Lambda_u - \Lambda_{u_n}\|_p \to 0$. Since $\Lambda_{u_n}(x,y) = \Lambda_{u\eta_n}(x,y) = u(x)\Lambda_{\eta_n}(x,y) + \eta_n(y)\Lambda_u(x,y)$ a.e. ,

$$|\Lambda_{u_n}(x,y)| \le \frac{C|u(x)|}{|x-y|^{N/p+s-1}}\mathbf{1}_{B(x,1)}(y) + \frac{2|u(x)|}{|x-y|^{N/p+s}}\mathbf{1}_{B^c(x,1)}(y) + |\Lambda_u(x,y)| \ , \quad (5)$$

where one has used, in the first term, the mean-value theorem. The use of polar coordinates clearly show that the r.h.s. in (5) is L^p-integrable.
Thus, we may now assume that supp u is a compact of $\mathbb{R}^N$. The end of the proof rely on an argument of regularization. Let (ρ_n) be a regularized sequence. We want to prove that $\rho_n * u \to u$ in $\mathrm{W}^{s,p}$. By analogy with $\mathrm{W}^{1,p}$, we expect that $\Lambda_{\rho_n * u} = \rho_n * \Lambda_u$, this can be easily verified. The proof is concluded by a classical result of regularization in L^p. □

Let Ω be an open subset of $\mathbb{R}^N$. The writing $U \subset\subset \Omega$ means that U is open and $\bar{U}$ is a compact subset of Ω.
We define

$$\mathrm{W}^{s,p}_{\mathrm{loc}}(\Omega) = \{u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega) : \text{ for all } U \subset\subset \Omega,\ u\big|_U \in \mathrm{W}^{s,p}(U)\} \ .$$

Now, we give some elementary and useful results in fractionnal Sobolev spaces. Correspondant results are well-known for $\mathrm{W}^{1,p}$ (see [1] or [4] for example).

Lemma 1.2. *Let Ω be open in $\mathbb{R}^N$ and $u \in \mathrm{W}^{s,p}_{loc}(\Omega)$. If $\phi \in \mathrm{C}^1(\Omega)$, then $\phi u \in \mathrm{W}^{s,p}_{loc}(\Omega)$ and moreover, for all $U \subset\subset \Omega$,*

$$\|\phi u\|_{s,p,U} \le \gamma \ \|u\|_{s,p,U} \ , \quad (6)$$

where γ depend only over N, s, p, U and ϕ.

Proof. Let $U \subset\subset \Omega$. Using product's rule, elementary majorations and Minkowski's inequality,

$$\|\Lambda_{\phi u}\|_{p,U\times U} \le \|\phi\|_{\infty,U} \ \|u\|_{s,p,U} + C \ \|\nabla\phi\|_{\infty,U} \ \|u\|_{p,U} \ ,$$

where C depends only of N, s and p and result from an integration in polar coordinates. □

In the same spirit, we give the trivial following result.

Lemma 1.3. *Let Ω be open in $\mathbb{R}^N$. $\mathrm{W}^{s,p}(\Omega) \cap \mathrm{L}^\infty$ is an algebra. Moreover, the product map is continuous.*

Now, we present a lemma of change of variables.

Lemma 1.4. *Let $G : \Omega \to \omega$ be a diffeomorphism of class C^1 between two open set of $\mathbb{R}^N$. If $u \in \mathrm{W}^{s,p}_{loc}(\omega)$, then $u \circ G \in \mathrm{W}^{s,p}_{loc}(\Omega)$, and moreover, for all $U \subset\subset \Omega$,*

$$\|u \circ G\|_{s,p,U} \le \gamma \ \|u\|_{s,p,G(U)} \ ,$$

where γ depend only over N, s, p, U and G.

Proof. Let $U \subset\subset \Omega$ and set $V = G(U)$. Since G is lipschitzian on U and G^{-1} is C^1 on $\overline{V}$,

$$\begin{aligned} & \int_U \int_U \frac{|u(G(x)) - u(G(y))|^p}{|x-y|^{N+sp}} dxdy \\ = \ & \int_V \int_V \frac{|u(x) - u(y)|^p}{|G^{-1}(x) - G^{-1}(y)|^{N+sp}} \ J_{G^{-1}}(x) J_{G^{-1}}(y) \ dxdy \\ \le \ & C \int_V \int_V \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy \ . \end{aligned}$$

□

Here is the same kind of result concerning left-composition. The proof is straightforward.

Lemma 1.5. *Let Ω be open in $\mathbb{R}^N$ and $u \in \mathrm{W}^{s,p}(\Omega)$. If L is globally lipschitzian on $\mathbb{R}$, then $L \circ u \in \mathrm{W}^{s,p}(\Omega)$.*

2. An Extension theorem

We begin with some notations. Let ω be open in $\mathbb{R}^{N-1}$ and $0 < \delta \le \infty$. We consider open sets of $\mathbb{R}^N$ of the form $Q = \omega \times]-\delta, \delta[$, $Q^+ = \omega \times]0, \delta[$ and $Q^- = \omega \times]-\delta, 0[$. For all $(x', x_N) \in Q$, set $\sigma(x', x_N) = (x', -x_N)$. If u is defined on Q^+, one define on Q the extension $\bar{u}$ by reflection of u by $\bar{u}(x) = u(x)$ when $x \in Q^+$ and $\bar{u}(x) = u(\sigma(x))$ when $x \in Q^-$.
The next result is easy to prove.

Lemma 2.1. *If $u \in \mathrm{W}^{s,p}(Q_+)$, then $\bar{u} \in \mathrm{W}^{s,p}(Q)$ and moreover*

$$\|\bar{u}\|_{p,Q} \le 2^{1/p} \|u\|_{p,Q_+} \ , \ |||\bar{u}|||_{s,p,Q} \le 4^{1/p} |||u|||_{s,p,Q_+} \ .$$

Proof. First inequality is trivial. Writing

$$|||\bar{u}|||^p_{s,p,Q} = \Big[\int_{Q^+} \int_{Q^+} + \int_{Q^-} \int_{Q^-} + \int_{Q^+} \int_{Q^-} + \int_{Q^-} \int_{Q^+} \Big] \frac{|\bar{u}(x) - \bar{u}(y)|^p}{|x-y|^{sp+N}} dxdy \ ,$$

it is easy to observe that one only has to study the two last terms of the r.h.s. This can be easily done as follow, first write

$$\int_{Q_-} \int_{Q_+} \frac{|u(\sigma(x)) - u(y)|^p}{|x-y|^{sp+N}} dxdy = \int_{Q_+} \int_{Q_+} \frac{|u(x) - u(y))|^p}{|\sigma(x) - y|^{sp+N}} dxdy \ ,$$

and then notice that for all $x, y \in Q^+$, $|x-y| \le |\sigma(x) - y|$. □

If u is defined on an open subset of $\mathbb{R}^N$ then the extension of u by 0 on $\mathbb{R}^N \setminus \Omega$ is denoted by $\tilde{u}$.

Lemma 2.2. *Let Ω be open in $\mathbb{R}^N$, $u \in \mathrm{W}^{s,p}(\Omega)$ and $K \subset \Omega$ be compact. If* supp $u \subset K$*, then $\tilde{u} \in \mathrm{W}^{s,p}(\mathbb{R}^N)$, and moreover*

$$\|\tilde{u}\|_{s,p,\mathbb{R}^N} \leq \gamma \ \|u\|_{s,p,\Omega} \ ,$$

where γ depend only over N, s, p, K and Ω.

Proof. Let $\delta_K > 0$ be such that $K + B(0, \delta_K) \subset \Omega$ and write

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}} dxdy = \Big[\int_\Omega\int_\Omega + \int_{\mathbb{R}^N\setminus\Omega}\int_{\mathbb{R}^N\setminus\Omega} +2\int_{\mathbb{R}^N\setminus\Omega}\int_\Omega\Big] \frac{|\tilde{u}(x)-\tilde{u}(y)|^p}{|x-y|^{N+sp}} dxdy\ .$$

The first term is $|||u|||^p_{s,p,\Omega}$, the second one is equal to 0 and the last term is computed using polar coordinates so that

$$|||\tilde{u}|||_{s,p,\mathbb{R}^N} \leq |||u|||_{s,p,\Omega} + \big(\frac{2\sigma_N}{sp}\big)^{1/p} \delta_K^{-s} \|u\|_{p,\Omega} \ ,$$

which ends the proof. □

Now, we turn to the following fundamental theorem.

Theorem 2.1. *Let Ω be an C^1-open subset of $\mathbb{R}^N$ with bounded boundary $\partial\Omega$ or the product of N open intervals. There exists a bounded linear operator*

$$P : \mathrm{W}^{s,p}(\Omega) \to \mathrm{W}^{s,p}(\mathbb{R}^N)\ ,$$

such that $Pu\big|_\Omega = u$.

Before beginning the proof of this theorem, we introduce some notations and remarks. In the case where Ω is a C^1-open subset of $\mathbb{R}^N$ with bounded boundary, we may consider a finite covering of $\partial\Omega$ by open bounded sets $U_1, \ldots, U_k$ of $\mathbb{R}^N$ such that to each U_j, we may associate an open bounded subset ω_j of $\mathbb{R}^{N-1}$, a real number $\delta_j > 0$, and a C^1-diffeomorphism $G_j : Q_j \to U_j$ where $Q_j = \omega_j \times]-\delta_j, \delta_j[$ such that $G_j \in \mathrm{C}^1(\overline{Q_j}), G_j^{-1} \in \mathrm{C}^1(\overline{U_j})$ and $G_j(Q_j^+) = U_j \cap \Omega, G_j(Q_j^0) = U_j \cap \partial\Omega$ where $Q_j^+ = \omega_j \times]0, \delta_j[$ and $Q_j^0 = \omega_j \times \{0\}$.
A classical result of partition of unity (see [4] for example) allow us to consider positive mapping $\psi_0, \ldots, \psi_k$ with $\psi_0 \in \mathcal{D}(\mathbb{R}^N \setminus \partial\Omega)$, $\psi_j \in \mathcal{D}(U_j)$ where $1 \leq j \leq k$ and $\sum_{j=0}^k \widetilde{\psi_j} = 1$.

Theorem 2.1 is established by adapting the corresponding proof in $\mathrm{W}^{1,p}$.

Proof. Assume that Ω is of class C^1 with bounded boundary. Let $u \in \mathrm{W}^{s,p}(\Omega)$ and $1 \leq j \leq k$. Set $G_j^+ = G_j\big|_{Q_j^+}$. By lemma's 1.4 and 2.1, $v_j = \overline{u \circ G_j^+} \circ G_j^{-1} \in \mathrm{W}^{s,p}(U_j)$. Furthermore, $\|v_j\|_{s,p,U_j} \leq \gamma_1 \ \|u\|_{s,p,\Omega}$. By lemma's 1.2 and 2.2, $u_j = \widetilde{\psi_j v_j} \in \mathrm{W}^{s,p}(\mathbb{R}^N)$ and $\|u_j\|_{s,p} \leq \gamma_2 \ \|u\|_{s,p,\Omega}$. Set $u_0 = \widetilde{\psi_0}$. The operator P defined by $Pu = \sum_{j=0}^k u_j$ is a suitable extension. In the case where Ω is the product of N open intervals it suffices to use a finite number of extension by reflection and a truncature. □

An easy consequence is the following.

Corollary 2.1. *Let Ω be an C^1-open subset of $\mathbb{R}^N$ with bounded boundary $\partial\Omega$ or the product of N open intervals. $\mathcal{D}(\overline{\Omega})$ is dense in $\mathrm{W}^{s,p}(\Omega)$.*

3. Traces in Sobolev Spaces

Let Ω be a C^1-open subset of $\mathbb{R}^N$ with bounded boundary. Using the notations introduced above, we set, for all $1 \le j \le k$, $G_j^0 = G_j\big|_{Q_j^0}$. Remember that the integral of u on $\partial\Omega$ is defined by

$$\int_{\partial\Omega} u(x)d\sigma(x) = \sum_{j=1}^{k} \int_{\omega_j} (\psi_j u) \circ G_j^0(x') J_{G_j^0}(x')dx' \ .$$

It is possible to show that this definition is independant of the choice of the ω_j, ψ_j and G_j. Set

$$\mathrm{W}^{s,p}(\partial\Omega) = \{u \in \mathrm{L}^p(\partial\Omega) : \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} d\sigma(x)d\sigma(y) < \infty\} \ .$$

or equivalently,

$$\mathrm{W}^{s,p}(\partial\Omega) = \{u \in \mathrm{L}^p(\partial\Omega) :$$
$$\sum_{j=1}^{k} \int_{\omega_j}\int_{\omega_j} \psi_j(G_j^0(x'))\psi_j(G_j^0(y'))\frac{|u(G_j^0(x'))-u(G_j^0(y'))|^p}{|G_j^0(x')-G_j^0(y')|^{N+sp}} \ J_{G_j^0}(x')J_{G_j^0}(y') \ dx'dy' < \infty\} \ .$$

Proofs concerning previous results about $\mathrm{W}^{s,p}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$ may easy be adapted in the case of $\mathrm{W}^{s,p}(\partial\Omega)$ using local coordinates.

Lemma 3.1. *Let Ω be an C^1-open subset of $\mathbb{R}^N$ with bounded boundary $\partial\Omega$, $K \subset \partial\Omega$ be compact and $\varphi \in \mathrm{W}^{s,p}(V)$ where V is an open subset of $\partial\Omega$. If* supp $\varphi \subset K$*, then*[1] *$\tilde{\varphi} \in \mathrm{W}^{s,p}(\partial\Omega)$, and moreover*

$$\|\tilde{\varphi}\|_{s,p,\partial\Omega} \le \gamma \ \|\varphi\|_{s,p,V} \ ,$$

where γ depend only over N, s, p, K and U.

The aim of this section is to prove the next theorem.

Theorem 3.1. *Let $1 < p < \infty$ and Ω be an C^1-open subset of $\mathbb{R}^N$ with bounded boundary $\partial\Omega$. There is an unique bounded linear mapping*

$$\mathrm{tr} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1-1/p,p}(\partial\Omega) \ ,$$

which coincide with the restriction operator on $\mathcal{D}(\overline{\Omega})$. Moreover, this application is surjective.

First, we show that this result is a corollary of the next theorem.

Theorem 3.2. *Let $1 < p < \infty$. There is an unique bounded linear mapping*

$$\mathrm{tr} : \mathrm{W}^{1,p}(\mathbb{R}^N_+) \to \mathrm{W}^{1-1/p,p}(\mathbb{R}^{N-1}) \ ,$$

which coincide with the restriction operator on $\mathcal{D}(\overline{\mathbb{R}^N_+})$. Moreover, this application is surjective.

This trace application is defined by using a density argument. Notice that if $u \in \mathrm{C}(\overline{\mathbb{R}^N_+}) \cap \mathrm{W}^{1,p}(\mathbb{R}^N_+)$, then the trace of u is simply the restriction to $\mathbb{R}^{N-1}$. Indeed, let $(u_n) \subset \mathcal{D}(\overline{\mathbb{R}^N_+})$ with $u_n \to u$ in $\mathrm{W}^{1,p}$, then $u_n\big|_{\mathbb{R}^{N-1}} = \mathrm{tr}\ u_n \to \mathrm{tr}\ u$ in L^p, so that, up to a subsequence, $\mathrm{tr}\ u = u\big|_{\mathbb{R}^{N-1}}$.

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[1]Where $\tilde{\varphi}$ denotes the extension of φ by 0 on $\partial\Omega \setminus V$.

Using theorem 3.2, we may define a trace operator

$$\mathrm{tr} : \mathrm{W}^{1,p}(\omega\times]0,\delta[) \to \mathrm{W}^{1-1/p,p}(\omega) : v \to \mathrm{tr}\ v = (\mathrm{tr}\ Pv)\big|_{\omega}\ , \tag{7}$$

where ω is a bounded open set of $\mathbb{R}^{N-1}$ and $\delta > 0$. This definition don't depend of the choice of the continous extension operator P. Indeed, let P_1 and P_2 be two continous extension operator and $(v_n), (w_n) \subset \mathcal{D}(\mathbb{R}^N)$ be such that $v_n \to P_1 v$ and $w_n \to P_2 v$ in L^p. By eventually passing to subsequences, $v_n - w_n \to 0$ a.e. on $\omega\times]0,\delta[$ so that $v_n(\cdot,0) - w_n(\cdot,0) \to 0$ a.e., hence tr $P_1 v =$ tr $P_2 v$ on ω.
Notice that, by construction, the trace of the restriction is the restriction of the trace. Using theorem 2.1, it is not difficult to observe that theorem 3.2 may be adapted to the trace operator defined by (7).

Now, we present the proof of theorem 3.1.

Proof. Existence. The trace application is defined as follow : if $u \in \mathrm{W}^{1,p}(\Omega)$, then

$$\mathrm{tr}\ u = \sum_{j=1}^{k} \widetilde{\xi_j}\ , \tag{8}$$

where $\xi_j = \mathrm{tr}\ (\phi_j u \circ G_j) \circ (G_j^0)^{-1}$ and $\phi_j = \psi_j\big|_{U_j\cap\Omega}$. Using classical results in $\mathrm{W}^{1,p}(\Omega)$, previous remarks, a variant of lemma 1.4, and also lemma 3.1, one clearly see that the map $\mathrm{tr} : \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1-1/p,p}(\partial\Omega)$ is linear and continuous and that tr $u = u\big|_{\partial\Omega}$, for all $u \in \mathcal{D}(\overline{\Omega})$. The first part of the proof is completed by some remarks.

Let U be a bounded open set of $\mathbb{R}^N$ be such that $U \cap \partial\Omega \neq \emptyset$, ω be a bounded open set of $\mathbb{R}^{N-1}$, $\delta > 0$ and $G^+ : \omega\times]0,\delta[\to U \cap \Omega$ be a C^1-diffeomorphism such that $G^+ \in \mathrm{C}^1(\overline{Q^+})$ where $Q^+ = \omega\times]0,\delta[$, and $(G^+)^{-1} \in \mathrm{C}^1(U \cap \overline{\Omega})$. The beginning of the proof suggest to define a trace operator

$$\mathrm{tr} : \mathrm{W}^{1,p}(U\cap\Omega) \to \mathrm{W}^{1-1/p,p}(U\cap\partial\Omega) : u \to (\mathrm{tr}\ (u\circ G^+))\circ(G^+)^{-1}\Big|_{U\cap\partial\Omega}\ . \tag{9}$$

By construction, if $v \in \mathrm{W}^{1,p}(\omega\times]0,\delta[)$, then

$$\mathrm{tr}\ (v\circ(G^+)^{-1}) = (\mathrm{tr}\ v)\circ(\mathrm{tr}\ (G^+)^{-1}) = (\mathrm{tr}\ v)\circ(G^+)^{-1}\Big|_{U\cap\partial\Omega}\ ,$$

i.e. the trace of a composition is equal to the composition of the traces. Again, theorem 3.2 may easily be adapted to the trace operator defined in (9). Furthermore, the trace of the restriction is the restriction of the trace. More precisely, if $u \in \mathrm{W}^{1,p}(\Omega)$, then

$$\mathrm{tr}\ (u\big|_{U\cap\Omega}) = (\mathrm{tr}\ u)\big|_{U\cap\partial\Omega}\ .$$

It results from the equality of two continuous mapping in u on a dense subset of $\mathrm{W}^{1,p}(\Omega)$.

If $\phi \in \mathcal{D}(\overline{\Omega})$ and $u \in \mathrm{W}^{1,p}(\Omega)$, it is easy to show, by using a density argument[2], that tr $\phi u = (\mathrm{tr}\ \phi)(\mathrm{tr}\ u)$. This stay true for the trace operators defined in (7) and (9).

Surjectivity. Using previous remarks, establishing the surjectivity become more obvious. Let $\varphi \in \mathrm{W}^{1-\frac{1}{p},p}(\partial\Omega)$ and $1 \leq j \leq k$. Set, for all $1 \leq j \leq k$, $\varphi_j =$

[2]Indeed, let $(u_n) \subset \mathcal{D}(\overline{\Omega})$ be such that $u_n \to u$ in $\mathrm{W}^{1,p}$ so that by definition tr $u_n \to$ tr u. One immediately verify that $\phi u_n \to \phi u$ in $\mathrm{W}^{1,p}$, hence tr $(\phi u_n) = (\mathrm{tr}\ \phi)(\mathrm{tr}\ u_n) \to (\mathrm{tr}\ \phi)(\mathrm{tr}\ u)$.

$\varphi\big|_{U_j\cap\partial\Omega}$. Using a variant of lemma 1.4, and by an estimate of the jacobian $J_{G_j^0}$ on Q_j^0, $\varphi_j \circ G_j^0 \in \mathrm{W}^{1-1/p,p}(\omega_j)$, so that one may choose $v_j \in \mathrm{W}^{1,p}(Q_j^+)$ such that tr $v_j = \varphi_j \circ G_j^0$. By the theorem of change of variables in Sobolev spaces, $u_j = v_j \circ (G_j^+)^{-1} \in \mathrm{W}^{1,p}(U_j \cap \Omega)$. Furthermore, tr $u_j = \varphi_j$. Set $u = \sum_{j=1}^k \widetilde{u_j} \in \mathrm{W}^{1,p}(\Omega)$. Since tr $(\phi_j u_j) = (\text{tr } \phi_j)$ $(\text{tr } u_j) = \phi_j\big|_{\partial\Omega\cap U_j}\varphi_j$, inserting u in the r.h.s of (8), we get

$$\text{tr } u = \sum_{j=1}^k \widetilde{\text{tr } \phi_j u_j} = \sum_{j=1}^k \widetilde{\phi_j}\big|_{\partial\Omega\cap U_j}\varphi = \varphi \ ,$$

which ends the proof. □

Now, we give the proof of theorem 3.2 which is considered [2]. A continuous version of Minkowski's inequality is needed.

Lemma 3.2. *Let $\{X, \mathcal{A}, \mu\}$ and $\{Y, \mathcal{B}, \nu\}$ be two measured spaces and $1 \leq p < \infty$. If $u \in \mathrm{L}^p(X \times Y)$, then*

$$\Big(\int_X \Big|\int_Y u(x,y)d\nu\Big|^p d\mu\Big)^{1/p} \leq \int_Y \|u(\cdot,y)\|_{p,X} d\nu \ . \tag{10}$$

The proof of this lemma rely on an argument of duality, see [2]. Here is the proof of theorem 3.2.

Proof. Existence . Let u be in $\mathcal{D}(\overline{\mathbb{R}^N_+})$. By the fundamental theorem of calculus and using Holder's inequality,

$$\|u(\cdot,0)\|^p_{p,\mathbb{R}^{N-1}} \leq p\ \|u\|^{p-1}_{p,\mathbb{R}^N_+}\big\|\partial_N u\big\|_{p,\mathbb{R}^N_+} \leq p\ \|u\|^p_{1,p,\mathbb{R}^N_+} \ . \tag{11}$$

We are now looking to the same kind of majoration concerning the semi-norm $|\|u(\cdot,0)|\|_{1-\frac{1}{p},p}$. For all $x, y \in \mathbb{R}^{N-1}$, set $z = (\frac{x+y}{2}, \lambda|\xi|)$ where $\xi = \frac{x-y}{2}$ and $\lambda > 0$ is independant of x and y and will be fixed later. First, write

$$\begin{aligned}
|u(z) - u(x,0)| &= \Big|\int_0^1 \frac{d}{d\rho}u(x-\rho\xi, \lambda\rho|\xi|)d\rho\Big| \\
&= \Big|\int_0^1 \Big(\sum_{i=1}^{N-1}\partial_i u(x-\rho\xi,\lambda\rho|\xi|)(-\xi_i) + \lambda|\xi|\partial_N u(x-\rho\xi,\lambda\rho|\xi|)\Big)d\rho\Big| \\
&\leq |\xi|\int_0^1 |\nabla_{N-1}u(x-\rho\xi,\lambda\rho|\xi|)|d\rho + \lambda|\xi|\int_0^1 |\partial_N u(x-\rho\xi,\lambda\rho|\xi|)|d\rho \ .
\end{aligned}$$

Similarly,

$$|u(z) - u(y,0)| \leq |\xi|\int_0^1 |\nabla_{N-1}u(y+\rho\xi,\lambda\rho|\xi|)|d\rho + \lambda|\xi|\int_0^1 |\partial_N u(y+\rho\xi,\lambda\rho|\xi|)|d\rho \ , \tag{12}$$

and then, using triangular inequality

$$\begin{aligned}
|\Lambda_{u(\cdot,0)}| &= \frac{|u(x,0)-u(y,0)|}{|x-y|^{1+(N-2)/p}} \\
&\leq \Big(|\xi|\int_0^1 \frac{|\nabla_{N-1}u(x-\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{1+(N-2)/p}}d\rho + \lambda|\xi|\int_0^1 \frac{|\partial_N u(x-\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{1+(N-2)/p}}d\rho + \dots\Big) \\
&\leq \Big(\int_0^1 \frac{1}{2}\frac{|\nabla_{N-1}u(x-\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{(N-2)/p}}d\rho + \int_0^1 \frac{\lambda}{2}\frac{|\partial_N u(x-\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{(N-2)/p}}d\rho + \dots\Big) \ ,
\end{aligned}$$

where the two terms coming for (12) are explicitly omitted. By Minkowski's inequality,

$$\begin{aligned}|||u(\cdot,0)|||_{1-\frac{1}{p},p,\mathbb{R}^{N-1}} &\leq \Big[\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}}\Big(\int_0^1 \frac{1}{2}\frac{|\nabla_{N-1}u(x-\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{(N-2)/p}}d\rho\Big)^p dxdy\Big]^{1/p}\\ &+ \Big[\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}}\Big(\int_0^1 \frac{\lambda}{2}\frac{|\partial_N u(x-\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{(N-2)/p}}d\rho\Big)^p dxdy\Big]^{1/p} + \dots ,\end{aligned}$$

therefore, by lemma 3.2 and symmetry,

$$\begin{aligned}|||u(\cdot,0)|||_{1-\frac{1}{p},p,\mathbb{R}^{N-1}} &\leq \int_0^1\Big(\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}}\frac{|\nabla_{N-1}u(x-\rho\xi,\lambda\rho|\xi|)|^p}{|x-y|^{N-2}}dxdy\Big)^{1/p}d\rho\\ &+\int_0^1\lambda\Big(\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}}\frac{|\partial_N u(x-\rho\xi,\lambda\rho|\xi|)|^p}{|x-y|^{N-2}}dxdy\Big)^{1/p}\ d\rho .\end{aligned}$$

Using natural changes of variables,

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}}dy\int_{\mathbb{R}^{N-1}}\frac{|\nabla_{N-1}u(x-\rho\xi,\lambda\rho|\xi|)|^p}{|x-y|^{N-2}}dx\\ =\ & 2\int_{\mathbb{R}^{N-1}}d\xi\int_{\mathbb{R}^{N-1}}\frac{|\nabla_{N-1}u(x-\rho\xi,\lambda\rho|\xi|)|^p}{|\xi|^{N-2}}dx\\ =\ & 2\int_{S^{N-1}}d\sigma\int_0^\infty dr\int_{\mathbb{R}^{N-1}}|\nabla_{N-1}u(x-\rho r\sigma,\lambda\rho r)|^p dx\\ =\ & \frac{2\sigma_N}{\lambda\rho}\int_0^\infty dx'_N\int_{\mathbb{R}^{N-1}}|\nabla_{N-1}u(x',x'_N)|^p dx'\\ =\ & \frac{2\sigma_N}{\lambda\rho}\|\nabla_{N-1}u\|^p_{p,\mathbb{R}^N_+} .\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}}\frac{|\partial_N u(x-\rho\xi,\lambda\rho|\xi|)|^p}{|x-y|^{N-2}}dxdy=\frac{2\sigma_N}{\lambda\rho}\|\partial_N u\|^p_{p,\mathbb{R}^N_+} ,$$

hence,

$$\begin{aligned}&|||u(\cdot,0)|||_{1-\frac{1}{p},p,\mathbb{R}^{N-1}}\\ \leq\ & \Big((2\sigma_N)^{1/p}\int_0^1\rho^{-1/p}d\rho\Big)\big[\lambda^{-1/p}\ \|\nabla_{N-1}u\|_{p,\mathbb{R}^N_+}+\lambda^{1-1/p}\ \|\partial_N u\|_{p,\mathbb{R}^N_+}\big]\\ =\ & (2\sigma_N)^{1/p}\frac{p}{p-1}\big[\lambda^{-1/p}\ \|\nabla_{N-1}u\|_{p,\mathbb{R}^N_+)}+\lambda^{1-1/p}\ \|\partial_N u\|_{p,\mathbb{R}^N_+}\big] .\end{aligned}$$

By choosing $\lambda=\frac{p}{p-1}\frac{\|\nabla_{N-1}u\|_p}{\|\partial_N u\|_p}$, which minimize the r.h.s. of the previous inequality,

$$\begin{aligned}|||u(\cdot,0)|||_{1-\frac{1}{p},p,\mathbb{R}^{N-1}} &\leq \Big(2\sigma_N(p-1)\Big)^{1/p}\Big(\frac{p-1}{p}\Big)^2\|\nabla_{N-1}u\|^{1-1/p}_{p,\mathbb{R}^N_+}\|\partial_N u\|^{1/p}_{p,\mathbb{R}^N_+}\\ &\leq \Big(2\sigma_N(p-1)\Big)^{1/p}\Big(\frac{p-1}{p}\Big)^2\|u\|_{1,p,\mathbb{R}^N_+} .\end{aligned}$$

The first part of the proof is concluded by density.

Surjectivity . Let φ be in $\mathrm{W}^{1-\frac{1}{p},p}(\mathbb{R}^{N-1})$. We denote by v the Poisson's integral of φ. First, we prove the two following inequalities

$$\|v(\cdot,t)\|_{p,\mathbb{R}^{N-1}} \leq \|\varphi\|_{p,\mathbb{R}^{N-1}} \tag{13}$$

$$\|\nabla v\|_{p,\mathbb{R}^N_+} \leq \gamma\ |||\varphi|||_{1-1/p,p,\mathbb{R}^{N-1}} , \tag{14}$$

where γ only depend of N and p. The first inequality is trivial, it relies on elementary properties of Poisson's kernels. The second one requires more work. Again by elementary properties of Poisson's kernels, for all $\alpha \in \mathbb{N}^N$ with $|\alpha| = 1$,

$$\partial^\alpha v(x,t) = \int_{\mathbb{R}^{N-1}} \partial^\alpha H(x-y,t)\varphi(y)dy = \int_{\mathbb{R}^{N-1}} \partial^\alpha H(x-y,t)(\varphi(y)-\varphi(x))dy \ .$$

Using classical majorations on the partial derivatives of H and polar coordinates, one thus finds

$$\begin{aligned} |\nabla v(x,t)| &\leq C\int_{\mathbb{R}^{N-1}} \frac{|\varphi(y)-\varphi(x)|}{(|x-y|+t)^N}dy \\ &\leq C\int_{\mathbb{R}^{N-1}} \frac{|\varphi(x+tz)-\varphi(x)|}{t(|z|+1)^N}dz \\ &\leq C\int_{S^{N-1}} d\sigma \int_0^\infty \frac{r^{N-2}}{(1+r)^N}\frac{|\varphi(x+tr\sigma)-\varphi(x)|}{t}dr \ , \end{aligned}$$

where C depend only over N. Then by continuous version of Minkowski's inquality (lemma 3.2) and using some natural changes of variables,

$$\begin{aligned} \|\nabla v\|_{p,\mathbb{R}^N_+} &\leq C\Big[\int_{\mathbb{R}^N_+} dxdt\Big(\int_{S^{N-1}} d\sigma \int_0^\infty \frac{r^{N-2}}{(1+r)^N}\frac{|\varphi(x+tr\sigma)-\varphi(x)|}{t}dr\Big)^p\Big]^{1/p} \\ &\leq C\int_{S^{N-1}} d\sigma \int_0^\infty \frac{r^{N-2}}{(1+r)^N}dr\Big(\int_{\mathbb{R}^N_+} \frac{|\varphi(x+tr\sigma)-\varphi(x)|^p}{t^p}dxdt\Big)^{1/p} \\ &\leq C\int_{S^{N-1}} d\sigma \int_0^\infty \frac{r^{N-1-1/p}}{(1+r)^N}dr\Big(\int_{\mathbb{R}^N_+} \frac{|\varphi(x+\rho\sigma)-\varphi(x)|^p}{\rho^p}dxd\rho\Big)^{1/p} \\ &\leq C'\int_{S^{N-1}} d\sigma\Big(\int_{\mathbb{R}^N_+} \frac{|\varphi(x+\rho\sigma)-\varphi(x)|^p}{\rho^p}dxd\rho\Big)^{1/p} \\ &\leq \gamma\Big(\int_{S^{N-1}} d\sigma\int_{\mathbb{R}^N_+} \frac{|\varphi(x+\rho\sigma)-\varphi(x)|^p}{\rho^p}dxd\rho\Big)^{1/p} \\ &= \gamma\Big(\int_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+p-2}}dx\ dy\Big)^{1/p} = \gamma\ |||\varphi|||_{1-1/p,p,\mathbb{R}^{N-1}} \ , \end{aligned}$$

where the last inequality is obtained using Jensen's inequality applied to the convex map $x \to -x^{1/p}$ and by taking care to normalize the mesure on S^{N-1}. This establishes (14). It becomes obvious to finish the proof. Taking $u(x,t) = e^{-t/p}v(x,t)$, we prove that tr $u = \varphi$. By density (see theorem 1.2), consider a sequence $(\varphi_n) \subset \mathcal{D}(\mathbb{R}^{N-1})$ such that $\varphi_n \to \varphi$ in $\mathrm{W}^{1-1/p,p}$. Let $(u_n) \subset \mathrm{C}^\infty(\overline{\mathbb{R}^N_+}) \cap \mathrm{W}^{1,p}(\mathbb{R}^N_+)$ be the associated sequence given by $u_n = e^{-t/p}\ v_n$ where $v_n(x,t)$ is the Poisson's integral (or the harmonic extension to $\mathbb{R}^N_+$) of φ_n . Using inequalities 13 and 14, one easily prove that $u_n \to u$ in $\mathrm{W}^{1,p}$. Since tr $v_n =$ tr $u_n = \varphi_n$, the proof is concluded. □

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