Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 53(1), 2026, Pages 220–240, DOI: 10.52846/ami.v53i1.2164
ISSN: 1223-6934

A Stochastic Prey-Predator Model Under Harvesting: Theoretical and Numerical Analysis

Lamia Bousmaha, Karima Smail, Omar Kebiri, Ali Moussaoui, and Abdeldjebar Kandouci

Abstract. A stochastic fish prey predator model with functional response is proposed and investigated. We show there is a unique positive solution to the model with positive initial value and we show that the positive solution to the stochastic system is stochastically bounded. Besides, a condition for the system to be extinct is given and persistent conditions are established. We further investigate the stability of our system. Theoretical results are illustrated using numerical simulations.

2020 *Mathematics Subject Classification.* 37N25, 60H10, 92D25.
Key words and phrases. Prey-predator fishery, Stochastic model, existence of positive global solution, Persistent in mean, Extinction, Stability.

1. Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [11]. There are many factors which affect population dynamics of biological and mathematical models. A crucial element of all models is the so-called ”functional response”, which is the function representing the prey consumption per unit time. Recently, in [21], authors proposed a new response functional in order to explain the influence of changing water level fluctuations in an artificial lake on fish predator-prey dynamics. In the studied lake, two interdependent species are considered; the pike (brochet in French) which is the most important predator and the roach (gardon in French) which is the prey.

When a predator attacks a prey, it has access to a certain quantity of food depending on the water level. When water level is low the predator is more in contact with the prey. Let $b(t)$ be the accessibility function for the prey. The function b is annual periodic and continuous, that is, b is 1-periodic. The minimum value b_1 is reached in spring, and the maximum value b_2 is attained during autumn. The predator needs a quantity γ for his food, but he has access to a quantity

$$g(t, x, y) = \frac{b(t)x}{y + D},$$

Received May 17, 2025. Accepted December 26, 2025.
The second and third named authors acknowledge the funding of the ERASMUS Plus KA171 project.

which depends on the water level, where x denoting the prey's density, y that of the predator and D measures other causes of mortality outside of predation. Thus, if

$$g(t,x,y) \geq \gamma,$$

then the predator will be satisfied with the quantity γ for his food. Otherwise, if

$$g(t,x,y) < \gamma,$$

the predator will content himself with

$$g(t,x,y) = \frac{b(t)x}{y+D}.$$

Consequently, the quantity of food received per predator and per unit of time is

$$\min\left(\frac{b(t)x}{y+D},\gamma\right).$$

The authors in [21] studied the non-autonomous prey-predator model. The authors have established sufficient conditions for the existence of positive periodic solutions of the prey-predator system. Such a solution describes an equilibrium situation consistent with the variability of environmental conditions, such that both populations survive. (See [21],for more details). The authors in [15] studied the following autonomous prey-predator model

$$\begin{cases} \dot{x}(t) = ax(t)\left(1 - \dfrac{x(t)}{K}\right) - \min\left(\dfrac{\bar{b}x(t)}{y(t)+D},\gamma\right)y(t) - mux(t) \\ \dot{y}(t) = -qy(t) + e\min\left(\dfrac{\bar{b}x(t)}{y(t)+D},\gamma\right)y(t), \end{cases} \tag{1}$$

where the prey grow logistically with carrying capacity K and intrinsic growth rate a. m is the catchability coefficient of the prey species, and u denotes the effort devoted to the harvesting. Using as predation rate, the mean function $\bar{b} = \int_0^1 b(t)dt$.

On the other hand, population systems are often affected by environmental noise, and hence stochastic differential equation models play a significant role in various branches of applied sciences including biology and population dynamics, as they provide some additional degree of realism compared to their deterministic counterpart ([9],[19]). In reality, due to continuous fluctuations in the environment (e.g. variation in intensity of sunlight, temperature, water level, etc.), parameters involved in models are not absolute constants, but they always fluctuate around some average value. As a result the population density never attains a fixed value with the advancement of time but rather exhibits continuous oscillation around some average values. Based upon these factors, stochastic population models have received more and more attention. (see e.g. [6], [7], [14], [20], [23], [24]).

In [7], considering that fluctuations in the environment would manifest themselves mainly as fluctuations in the intrinsic growth rate of the prey population and in the death rate of the predator population. In the present work, we incorporate stochastic perturbations into the first equation of the deterministic system (1) such that the intensity of the noise increases or decreases with the size of the prey population. Additionally, the predator's death rate is perturbed according to $q \to q + \beta dB^2(t)$, where B^1 and B^2 are dependent Brownian motions, $\alpha x(t)$ and $\beta y(t)$ represent the diffusion coefficients.. Therefore, the corresponding autonomous stochastic system to (1) took the following form:

$$\begin{cases} dx(t) = x(t)\left[a\left(1-\dfrac{x(t)}{K}\right) - \min\left(\dfrac{\bar{b}x(t)}{y(t)+D},\gamma\right)\dfrac{y(t)}{x(t)} - mu\right]dt + \alpha x(t)dB^1(t) \\ dy(t) = y(t)\left[-q + e\min\left(\dfrac{\bar{b}x(t)}{y(t)+D},\gamma\right)\right]dt - \beta y(t)dB^2(t). \end{cases} \tag{2}$$

For system (1), authors in [21] showed that the system possessed three equilibrium: the trivial equilibrium $P^0 = (0,0)$, the predator free equilibrum points $P^1 = (\bar{x}, 0)$ where $\bar{x} = \frac{K}{a}(a - mu)$ and the steady state of co-existence (interior equilibrium point) $P^*(x^*, y^*)$ where $y^* = \frac{1}{2}\left(-B + \sqrt{B^2 - 4C}\right)$, $x^* = \frac{q}{e\bar{b}}(y^* + D)$, and

$$B = 2D - \frac{e\bar{b}K(a - mu - \bar{b})}{aq} = D - m_2, \quad C = D^2 - \frac{e\bar{b}KD(a - mu)}{aq}.$$

They showed that the co-existing equilibrium point P^* is globally asymptotically stable under certain conditions.

We aim to consider the dynamical properties of the stochastic model (2). In this paper, we show some properties of the stochastic differential equation (2) including: the global existence, uniqueness and boundedness of positive solution. Furthermore, we obtain an interesting result: under some conditions, the stochastic system (2) is persistent in mean.

The organization of this paper is as follows. In Section 2, by Itô's formula and the comparison theorem of stochastic equations, we show that stochastic system (2) has a unique a.s. a global positive solution (x, y) with any initial value $x(0) = x_0 > 0$, $y(0) = y_0 > 0$. To a population system, the stochastic boundedness is one of most important topics because boundedness of a system guarantees its validity in these types of dynamics. So we show that both the prey population and the predator population of system (2) are bounded in mean. Then, in Section 3, we establish some sufficient conditions for the persistence and extinction of both species. We further investigate the stability of our system in Section 4. The numerical simulations presented in Section 5 are used to verify the theoretical results. Finally we close the paper with conclusions in the last section.

2. Properties of the solution

Throughout this paper, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

Since stochastic system (2) describes population dynamics, it is necessary for the solution of the system to be positive and not to explode to infinity in a finite time. Moreover, in order for a stochastic differential equation to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of equation are generally required to satisfy the linear growth condition and local Lipschitz condition (e.g. [1], [17], [23]). We will show there is a unique positive solution with positive initial value of system (2), then a stochastic ultimate boundedness is studied.

2.1. Positive and global solution.

Theorem 2.1. *For any initial value,* $(x_0, y_0) \in \mathbb{R}^2_+$ *there exists a unique solution* $(x(t), y(t))_{t\geq 0}$ *of the system, and the solution will remain in* $\mathbb{R}^2_+$ *a.s.*

Proof. Since the coefficients of (2) are locally Lipschitz and satisfy the linear growth condition, uniqueness of the solution until an explosion time τ_e is guaranteed for any positive initial condition i.e. (there is a unique solution $(x(t), y(t))$ on $t \in [0, \tau_e)$).
Let us now prove global existence of the solution, to show that $\tau_e = \infty$. Let $n_0 > 0$ be sufficiently large such that $(x_0, y_0) \in D_{n_0} = \left[\frac{1}{n_0}, n_0\right] \times \left[\frac{1}{n_0}, n_0\right]$. For each integer $n > n_0$, we define the following stopping time:

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{x(t), y(t)\} \leq \frac{1}{n} \text{ or } \max\{x(t), y(t)\} \geq n\right\}.$$

We let $\inf \emptyset = \infty$. From the definition of stopping time, it is easy to see that τ_n is increasing as $n \to \infty$. Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, then $\tau_\infty \leq \tau_e$ almost surely.

Now left to show $\tau_\infty = \infty$. We assume that this is false. If the statement is false, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that:

$$\mathbb{P}(\tau_\infty \leq T) > \varepsilon. \tag{3}$$

Consequently, by denoting $\Omega_n = \{\tau_n \leq T\}$, then is an integer such that, for all $n_1 \geq n_0$,

$$\mathbb{P}(\tau_n \leq T) \geq \varepsilon, \ \forall n \geq n_1. \tag{4}$$

Define a function $V : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ by: $V(x, y) = (x - 1 - lnx) + (y - 1 - lny)$ which is non-negative.
Applying Itô's formula to our model (2), we obtain:

$$dV(x(t), y(t)) = \mathcal{L}V(x(t), y(t))dt + \alpha(x(t) - 1)dB^1(t) - \beta(y(t) - 1)dB^2(t),$$

where

$$\begin{aligned}
\mathcal{L}V(x(t), y(t)) &= \left(1 - \frac{1}{x(t)}\right)\left(ax(t)\left(1 - \frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}x(t)}{y(t) + D}, \gamma\right)y(t) - mux(t)\right) \\
&\quad + \frac{\alpha^2}{2} + \left(1 - \frac{1}{y(t)}\right)\left(- qy(t) + ey(t)\min\left(\frac{\bar{b}x(t)}{y(t) + D}, \gamma\right)\right) + \frac{\beta^2}{2} \\
&= (x(t) - 1)\left(a\left(1 - \frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}x(t)}{y(t) + D}, \gamma\right)\frac{y(t)}{x(t)} - mu\right) \\
&\quad + (y(t) - 1)\left(- q + e\min\left(\frac{\bar{b}x(t)}{y(t) + D}, \gamma\right)\right) + \frac{\alpha^2 + \beta^2}{2}.
\end{aligned} \tag{5}$$

It is easy to show that $\mathcal{L}V(x, y)$ is bounded above, say by N. So we have,

$$dV(x(t), y(t)) \leq Ndt + \alpha(x(t) - 1)dB^1(t) - \beta(y(t) - 1)dB^2(t). \tag{6}$$

Integrating both sides of (6) from 0 to $\tau_n \wedge T$ and then taking the expectation, one gets:

$$\mathbb{E}V(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V(x_0, y_0) + N\mathbb{E}(\tau_n \wedge T)),$$

therefore

$$\mathbb{E}V(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq V(x_0, y_0) + NT. \tag{7}$$

Let $\Omega_n = \{\tau_n \leq T\}$ for $n > n_1$. From (4), we obtain $\mathbb{P}(\Omega_n) > \varepsilon$. For each $w \in \Omega_n$, $x(\tau_n)$ or $y(\tau_n)$ equals either n or $\frac{1}{n}$. So $V(x(\tau_n), y(\tau_n))$ is not less than either

$$n - 1 - \ln n \text{ or } \frac{1}{n} - 1 - \ln\frac{1}{n} = \frac{1}{n} - 1 + \ln n,$$

thus,

$$V(x(\tau_n), y(\tau_n)) \geq (n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right),$$

holds. According (7), we have

$$V(x_0, y_0) + NT \geq \mathbb{E}\Big[I_{\Omega_n} V(x(\tau_n), y(\tau_n))\Big] \geq \varepsilon\,(n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right),$$

in which I_{Ω_n} denotes the indicator function of Ω_n. Letting $n \to \infty$, then we obtain

$$\infty > V(x_0, y_0) + NT = \infty,$$

which leads to a contradiction, so we must have $\tau_\infty = \infty$ almost surely, which implies that $\tau_e = \infty$. □

2.2. Stochastically Ultimate Boundedness. Theorem 2.1 shows that the solution of system (2) remains in the positive cone $\mathbb{R}^2_+$. However, this non-explosion property in a population dynamical system is often not good enough. Therefore, the property of ultimate boundedness is more desired. First, we recall the definition of stochastically ultimate boundedness.

Definition 2.1. The solution $(w(t))_{t\geq 0} = (x(t), y(t))_{t\geq 0}$ of system (2) is said to be stochastically ultimately bounded if, for any $\varepsilon \in (0, 1)$, there is a constant $H := H(\varepsilon)$ such that for any initial value $w_0 = (x_0, y_0) \in \mathbb{R}^2_+$,

$$\limsup_{t\longrightarrow\infty} \mathbb{P}\{||w(t)||_2 > H\} < \varepsilon.$$

Theorem 2.2. *The solutions of system (2) are stochastically ultimately bounded for any initial value $w_0 = (x_0, y_0) \in \mathbb{R}^2_+$.*

Proof. By Theorem 2.1, the solution $(x(t), y(t))$ remains in $\mathbb{R}^2_+$ for all $t \geq 0$. Define the function $V_1(t, x) = e^t x^\theta$ for $\theta > 0$.
By Itô's formula, we have

$$\begin{aligned}
\mathcal{L}V_1(t, x(t), y(t)) &= e^t x(t)^\theta \Bigg[1 + \left(a\left(1 - \frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}x(t)}{y(t) + D}, \gamma\right)\frac{y(t)}{x(t)} - mu\right)\theta \\
&\qquad + \frac{(\theta - 1)\theta}{2}\alpha^2\Bigg] \\
&\leq e^t\left[\left(1 + a\theta + \frac{(\theta - 1)\theta}{2}\alpha^2\right)x(t)^\theta - \frac{a\theta}{K}x(t)^{\theta+1}\right] \\
&\leq M_1(\theta)e^t.
\end{aligned} \tag{8}$$

Integrating both sides of (8) from 0 to t and then taking expectations, we get

$$e^t\mathbb{E}(x^\theta(t)) - \mathbb{E}(x_0^\theta) \leq M_1(\theta)e^t.$$

Hence, we have

$$\limsup_{t\longrightarrow\infty} \mathbb{E}(x^\theta(t)) \leq M_1(\theta) < +\infty.$$

Similarly, defining the function $V_2(t,y) = e^t y^\theta$ for $\theta > 0$ and applying the Itô's formula, we get

$$\begin{aligned}
\mathcal{L}V_2(t,x(t),y(t)) &= e^t y(t)^\theta \Bigg[1 + \Big(-q + e\min\Big(\tfrac{\bar{b}x(t)}{y(t)+D},\gamma\Big)\Big)\theta + \tfrac{(\theta-1)\theta}{2}\beta^2\Bigg] \\
&\le e^t y(t)^\theta \Bigg[1 + e\gamma\theta - q\theta + \frac{(\theta-1)\theta}{2}\beta^2\Bigg] \\
&\le M_2(\theta)e^t.
\end{aligned} \tag{9}$$

Then, $e^t\mathbb{E}(y^\theta(t)) - E(y_0^\theta) \le M_2(\theta)e^t$. So we have $\limsup\limits_{t\longrightarrow\infty}\mathbb{E}(y^\theta(t)) \le M_2(\theta) < +\infty$. For $w(t) = (x(t),y(t)) \in \mathbb{R}_+^2$, we obtain

$$\begin{aligned}
||w(t)||^\theta &\le \left(2\max\left\{x^2(t),y^2(t)\right\}\right)^{\frac{\theta}{2}} \\
&\le 2^{\frac{\theta}{2}}(x^\theta(t),y^\theta(t)).
\end{aligned}$$

Consequently,

$$\limsup_{t\longrightarrow\infty}\mathbb{E}||w(t)||^\rho \le M_3(\theta) < +\infty,$$

where $M_3(\theta) = 2^{\frac{\theta}{2}}(M_1(\theta) + M_2(\theta))$. By Chebyshev's inequality, we get that all solutions are stochastically bounded. □

3. The long-time behavior of the system

In ecology, it is critical to discuss persistence and extinction. This section looks at the solution behavior of system (16). We first establish the conditions for persistence, then those for extinction.

3.1. The equivalent system. The system given by (2), is quite complex and difficult to study directly. To make the analysis and simulations easier, we follow a similar strategy to the one used by the authors in the deterministic case in [15]. For this reason, we propose an equivalent system that keeps the main features of our stochastic model but is simpler to handle.

Proposition 3.1. *If $b < \frac{\gamma(y_0+D)}{x_0}$, we have for all $t \ge 0$, $\bar{b}x(t) < \gamma(y(t)+D)$ a.s.*

Proof. Let $f(x(t),y(t)) = bx(t) - \gamma(y(t)+D)$ and $\tau^+ = \inf\{t > 0, f(t) > 0\}$. Let $n_0 > 0$ be sufficiently large such that $f(0) \in \left[-n_0, -\frac{1}{n_0}\right]$. For each integer $n \ge n_0$, define the stopping time as

$$\tau_n = \inf\left\{t \in [0,\tau^+), f(t) \notin \left[-n, -\frac{1}{n}\right]\right\}.$$

Obviously, τ_n is increasing as $n \to \infty$. Set $\tau_\infty = \lim\limits_{n\to\infty}\tau_n$, hence $\tau_\infty \le \tau^+$ almost surely.

Now, we want to show that $\tau_\infty = \infty$. We assume that this is false. If the statement is false, then there are a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ such that

$$\mathbb{P}(\tau_\infty \le T) > \varepsilon. \tag{10}$$

Thus, there is an integer $n_1 \ge n_0$ such that

$$\mathbb{P}\{\tau_n \le T\} \ge \varepsilon,\ \forall n \ge n_1. \tag{11}$$

Define a function $\widetilde{V} : \mathbb{R}^2_- \longrightarrow \mathbb{R}$ by:

$$\widetilde{V}(x(t), y(t)) = (\bar{b}x(t) - \gamma(y(t) + D)) - 1 - \ln(-\bar{b}x(t) + \gamma(y(t) + D)),$$

for $f(x(t), y(t)) = \bar{b}x(t) - \gamma(y(t) + D)$, the form of $\widetilde{V}$ will be $\widetilde{V} = (z - 1 - ln(-z))$ which is non-negative.

$$\begin{aligned}
&d\widetilde{V}(x(t), y(t)) \\
&\quad = \sigma_t \widetilde{V} + \sigma_x \widetilde{V} dx(t) + \sigma_y \widetilde{V} dy(t) + \tfrac{1}{2}\sigma_{xx}\widetilde{V} < x(t), x(t) > \\
&\qquad + \sigma_{xy}\widetilde{V} < x(t), y(t) > + \tfrac{1}{2}\sigma_{yy} < y(t), y(t) > \\
&\quad = \left(\bar{b} + \frac{\bar{b}}{\gamma(y(t)+D) - \bar{b}x(t)}\right) dx + \left(-\gamma - \frac{\gamma}{\gamma(y(t)+D) - \bar{b}x(t)}\right) dy \\
&\qquad + \left(\tfrac{\bar{b}}{2} + \tfrac{\bar{b}}{(\gamma(y(t)+D) - \bar{b}x(t))^2}\right) \alpha^2 x^2(t) dt + \left(-\frac{\gamma}{2} \cdot \frac{-\gamma}{\gamma(y(t)+D) - \bar{b}x(t))^2}\right) \beta^2 y^2(t) dt \\
&\qquad + \bar{b}\left(\tfrac{-\gamma}{\gamma(y(t)+D) - \bar{b}x(t))^2}\right) \alpha\beta x(t) y(t) dt \\
&\quad = \left(\bar{b} - \frac{\bar{b}}{f(t)}\right)\left[a\left(1 - \frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}x(t)}{y(t)+\gamma}, \gamma\right)\frac{y(t)}{x(t)} - mu\right] x(t) \\
&\qquad - \left(\gamma - \tfrac{\gamma}{f(t)}\right)\left[-d + e\min\left(\tfrac{\bar{b}x(t)}{y(t)+D}, \gamma\right)\right] y(t) + \tfrac{\bar{b}^2}{2f(t)^2}\alpha^2 x^2(t) \\
&\qquad + \frac{\gamma^2}{2f(t)^2}\beta^2 y^2(t) dt + \left(b - \frac{b}{f(t)}\right)\alpha x(t) dB^1(t) + \left(\gamma - \frac{\gamma}{f(t)}\right)\beta y(t) dB^2(t).
\end{aligned} \tag{12}$$

$$d\widetilde{V}(x(t), y(t)) = \mathcal{L}V(x(t), y(t))dt + \alpha\left(\bar{b} - \frac{\bar{b}}{f(t)}\right) x(t) dB^1(t) + \beta\left(\frac{\gamma}{f(t)} - \gamma\right) y(t) dB^2(t),$$

where

$$\begin{aligned}
\mathcal{L}\widetilde{V}(x(t), y(t)) &= \left(\bar{b} - \tfrac{\bar{b}}{f(t)}\right)\left[x(t)\left(1 - \tfrac{x(t)}{K}\right) - x(t)\min\left(\tfrac{\bar{b}x(t)}{y(t)+D}, \gamma\right)\tfrac{y(t)}{x(t)} - mu\right] \\
&\quad + \left(1 - \frac{1}{y}\right)\left[-qy(t) + ey(t)\min\left(\frac{\bar{b}x(t)}{y(t)+D}, \gamma\right)\right] + \frac{\beta^2}{2} \\
&= (x - 1)\left(a\left(1 - \frac{x(t)}{K}\right) - \min\left(\frac{\bar{b}x(t)}{y(t)+D}, \gamma\right)\frac{y(t)}{x(t)} - mu(t)\right) \\
&\quad + (y - 1)\left(-q + e\min\left(\frac{\bar{b}x(t)}{y(t)+D}, \gamma\right)\right) + \frac{\alpha^2 + \beta^2}{2} \\
&\leq \tilde{P},
\end{aligned} \tag{13}$$

with $\tilde{P}$ is a positive constant. So we have

$$d\widetilde{V}(x(t), y(t)) \leq \widetilde{P}dt + \alpha(x(t) - 1)dB^1(t) + \beta(y(t) - 1)dB^2(t). \tag{14}$$

Integrating both sides of (14) from 0 to $\tau_n \wedge T$ and then taking the expectation, one gets

$$\mathbb{E}\widetilde{V}(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq \widetilde{V}(x_0, y_0) + N\mathbb{E}(\tau_n \wedge T)),$$

therefore

$$\mathbb{E}\widetilde{V}(x(\tau_n \wedge T), y(\tau_n \wedge T)) \leq \widetilde{V}(x_0, y_0) + NT. \tag{15}$$

Let $\Omega_n = \{\tau_n \leq T\}$ for $n > n_1$. From (11), we obtain $\mathbb{P}(\Omega_n) > \varepsilon$. For each $w \in \Omega_n$, $x(\tau_n)$ or $y(\tau_n)$ equals either n or $\frac{1}{n}$. So $V(x(\tau_n), y(\tau_n))$ is not less than either

$$n - 1 - \ln n \text{ or } \frac{1}{n} - 1 - \ln\frac{1}{n} = \frac{1}{n} - 1 + \ln n.$$

Thus

$$\widetilde{V}(x(\tau_n), y(\tau_n)) \geq (n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right)$$

holds. According (15), we have

$$V(x_0, y_0) + NT \geq \mathbb{E}\Big[I_{\Omega_n} V(x(\tau_n), y(\tau_n))\Big] \geq \varepsilon(n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 + \ln n\right),$$

in which I_{Ω_n} denotes the indicator function of Ω_n. Letting $n \to \infty$, then we obtain

$$\infty > V(x(0), y(0)) + NT = \infty,$$

which leads to a contradiction, so we must have $\tau_\infty = \infty$ almost surely. □

Hence, system (2) is reduced to the simple form

$$\begin{cases} dx(t) = x(t)\left[a\left(1 - \dfrac{x(t)}{K}\right) - \dfrac{\bar{b}y(t)}{y(t) + D} - mu\right]dt + \alpha x(t) dB^1(t) \\ dy(t) = y(t)\left[-q + e\dfrac{\bar{b}x(t)}{y(t) + D}\right]dt - \beta y(t) dB^2(t). \end{cases} \tag{16}$$

3.2. Persistence. We first present persistence in mean proposed in [16]

Definition 3.1. The system is said to be persistent in mean if:

$$\liminf_{t \longrightarrow +\infty} \frac{1}{t}\int_0^t x(s)ds > 0 \ , \ \liminf_{t \longrightarrow +\infty} \frac{1}{t}\int_0^t y(s)ds > 0.$$

In this part, we always assume:
Assumption 1. $a - \bar{b} - mu - \frac{\alpha^2}{2} > 0$.
Assumption 2. $e\bar{b} - D\left(q + \frac{\beta^2}{2}\right) > 0$.

Theorem 3.2. *Suppose that Assumption 1 holds, for any $x_0 > 0$, the first component x of the solution to (16) satisfies*

$$\liminf_{t \longrightarrow +\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{\left(a - mu - \bar{b} - \frac{\alpha^2}{2}\right)K}{a}. \tag{17}$$

Before moving to prove Theorem 3.2, we first present the following lemma proposed in [6], [7].

Lemma 3.3. *Consider a one-dimensional stochastic differential equation*

$$dX(t) = X(t)\left[(a - bX(t))dt + \sigma dB(t)\right], \tag{18}$$

where parameters a, b and σ are positive, $a > \frac{\sigma^2}{2}$ and B is a standard Brownian motion.
Suppose $a > \frac{\sigma^2}{2}$ and X is the solution of system (18) with any initial value $X_0 > 0$. Then

$$\lim_{t \to \infty} \frac{\log X(t)}{t} = 0 \quad a.s. \tag{19}$$

and

$$\lim_{t \to \infty} \frac{1}{t}\int_0^t X(s)ds = \frac{a - \frac{\sigma^2}{2}}{b} \quad a.s. \tag{20}$$

Now, we proof Theorem 3.2.

Proof. Since the first component x of the solution to (16) is positive, we have

$$dx(t) \leq x(t)\left(a - mu - \frac{a}{K}x(t)\right)dt + \alpha x(t)dB^1(t).$$

Let

$$\Phi(t) = \frac{e^{(a-mu-\frac{\alpha^2}{2})t+\alpha B^1(t)}}{\frac{1}{x_0} + \frac{a}{K}\int_0^t e^{(a-mu-\frac{\alpha^2}{2})s+\alpha B^1(s)}ds}.$$

Then $\Phi(t)$ is the unique solution of equation

$$\begin{cases} d\Phi(t) = \Phi(t)\left(a - mu - \frac{a}{K}\Phi(t)\right)dt + \alpha\Phi(t)dB^1(t), \\ \Phi(0) = x_0 \end{cases} \tag{21}$$

and by the comparison theorem of stochastic equations we get

$$x(t) \leq \Phi(t) \quad \text{a.s.}$$

On the other hand, we have

$$dx(t) \geq x(t)\left(a - mu - \bar{b} - \frac{a}{K}x(t)\right)dt + \alpha x(t)dB^1(t).$$

Similarly,

$$\phi(t) = \frac{e^{(a-mu-\bar{b}-\frac{\alpha^2}{2})t+\alpha B^1(t)}}{\frac{1}{x_0} + \frac{a}{K}\int_0^t e^{(a-mu-\bar{b}-\frac{\alpha^2}{2})s+\alpha B^1(s)}ds}$$

is the unique solution of equation

$$\begin{cases} d\phi(t) = \phi(t)\left(a - mu - \bar{b} - \frac{a}{K}\phi(t)\right)dt + \alpha\phi(t)dB^1(t), \\ \phi(0) = x_0. \end{cases} \tag{22}$$

and

$$x(t) \geq \phi(t) \quad \text{a.s.}$$

Consequently,

$$\phi(t) \leq x(t) \leq \Phi(t) \quad \text{a.s.} \tag{23}$$

Since Φ and ϕ are solutions of systems (21) and (22), respectively. From Lemma 3.3 given above, when $a - \bar{b} - mu - \frac{\alpha^2}{2} > 0$, we get properties of solutions Φ and ϕ:

$$\lim_{t\to\infty} \frac{\log\Phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi(s)ds = \frac{\left(a - mu - \frac{\alpha^2}{2}\right)K}{a} \quad \text{a.s,}$$

and

$$\lim_{t\to\infty} \frac{\log\phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t \phi(s)ds = \frac{\left(a - mu - \bar{b} - \frac{\alpha^2}{2}\right)K}{a} \quad \text{a.s.}$$

The inequalities above, together with (23), imply

$$\lim_{t\to\infty} \frac{\log x(t)}{t} = 0 \quad \text{a.s,}$$

and

$$\frac{\left(a-mu-\bar{b}-\frac{\alpha^2}{2}\right)K}{a} \le \liminf_{t\longrightarrow+\infty}\frac{1}{t}\int_0^t x(s)ds$$
$$\le \limsup_{t\longrightarrow+\infty}\frac{1}{t}\int_0^t x(s)ds \le \frac{\left(a-mu-\frac{\alpha^2}{2}\right)K}{a} \quad \text{a.s.}$$

□

Theorem 3.4. *Suppose that Assumption 2 holds, and the second component y of the solution to (16) is positive with initial value $y_0>0$, then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{x(s)}{y(s)+D}ds = \frac{q+\frac{\beta^2}{2}}{e\bar{b}} \quad a.s. \tag{24}$$

Moreover, we can see

$$\liminf_{t\longrightarrow+\infty}\frac{1}{t}\int_0^t y(s)ds \ge \frac{e\bar{b}-D\left(q+\frac{\beta^2}{2}\right)}{e\bar{b}}.$$

Before setting the proof of Theorem 3.4, we need the following lemma.

Lemma 3.5. *Under Assumption 2, for any initial value $y_0>0$, the second component y of the solution to (16) satisfies*

$$\lim_{t\to\infty}\frac{\log y(t)}{t} = 0 \quad a.s.$$

Proof.

$$dy(t) \le \left(-qy(t)+\frac{e\bar{b}}{D}\Phi(t)\right)dt-\beta y(t)dB^2(t). \tag{25}$$

Let $\Psi(t)$ be the solution of

$$\begin{cases} d\Psi(t)=\left(-q\Psi(t)+\frac{e\bar{b}}{D}\Phi(t)\right)dt-\beta\Psi(t)dB^2(t),\\ \Psi(0)=y_0. \end{cases} \tag{26}$$

Then, by the comparison theorem of stochastic equations, we have $y(t)\le\Psi(t)$ a.s. Integrating (26) from T to t $(t>T)$, where T satisfies $\frac{1}{2}e^{(a-mu-\frac{\alpha^2}{2})T}\ge 1$, yields

$$\Psi(t)=\Psi(T)e^{-(q+\frac{\beta^2}{2})(t-T)-\beta(B^2(t)-B^2(T))}+\frac{e\bar{b}}{D}\int_T^t \Phi(s)e^{-(q+\frac{\beta^2}{2})(t-s)-\beta(B^2(t)-B^2(s))}ds. \tag{27}$$

If $s\ge T$, then $\frac{1}{2}e^{(a-mu-\frac{\alpha^2}{2})s}\ge 1$ and

$$\Phi(s)=\frac{e^{(a-mu-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{\frac{1}{x_0}+\frac{a}{K}\int_0^s e^{(a-mu-\frac{\alpha^2}{2})v+\alpha B^1(v)}dv}$$
$$\le \frac{e^{(a-mu-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{\frac{a}{K}\int_0^s e^{(a-mu-\frac{\alpha^2}{2})v+\alpha B^1(v)}dv}$$

$$
\begin{aligned}
&\leq \frac{e^{(a-mu-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{\frac{a}{K}e^{\alpha \min\limits_{0\leq v\leq s} B^1(v)}\int_0^s e^{(a-mu-\frac{\alpha^2}{2})v}dv} \\
&= \frac{\left(a-mu-\frac{\alpha^2}{2}\right)K}{a}\frac{e^{(a-mu-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{e^{\alpha \min\limits_{0\leq v\leq s} B^1(v)}\left[e^{(a-mu-\frac{\alpha^2}{2})s}-1\right]} \\
&\leq \frac{2\left(a-mu-\frac{\alpha^2}{2}\right)K}{a}\frac{e^{(a-mu-\frac{\alpha^2}{2})s+\alpha B^1(s)}}{e^{\alpha \min\limits_{0\leq v\leq s} B^1(v)}e^{(a-mu-\frac{\alpha^2}{2})s}} \\
&= \frac{2\left(a-mu-\frac{\alpha^2}{2}\right)K}{a}e^{\alpha(B^1(s)-\min\limits_{0\leq v\leq s} B^1(v))}.
\end{aligned}
$$

Replacing this in (27), we obtain

$$
\begin{aligned}
\Psi(t) &\leq \Psi(T)e^{-(q+\frac{\beta^2}{2})(t-T)-\beta(B^2(t)-B^2(T))} \\
&+ \frac{2e\bar{b}\left(a-mu-\frac{\alpha^2}{2}\right)K}{aD}\int_T^t e^{\alpha(B^1(s)-\min\limits_{0\leq v\leq s} B^1(v))}e^{-(q+\frac{\beta^2}{2})(t-s)-\beta(B^2(t)-B^2(s))}ds \\
&\leq \Psi(T)e^{-(q+\frac{\beta^2}{2})(t-T)-\beta(B^2(t)-B^2(T))} \\
&+ \frac{2e\bar{b}\left(a-mu-\frac{\alpha^2}{2}\right)K}{aD}e^{\alpha(\max\limits_{0\leq s\leq t} B^1(s)-\min\limits_{0\leq s\leq t} B^1(s))+\beta(\max\limits_{0\leq s\leq t} B^2(s)-B^2(t))} \\
&\times \int_T^t e^{-(q+\frac{\beta^2}{2})(t-s)}ds \\
&= \Psi(T)e^{-(q+\frac{\beta^2}{2})(t-T)-\beta(B^2(t)-B^2(T))} \\
&+ \frac{2e\bar{b}\left(a-mu-\frac{\alpha^2}{2}\right)K}{aD\left(q+\frac{\beta^2}{2}\right)}e^{\alpha(\max\limits_{0\leq s\leq t} B^1(s)-\min\limits_{0\leq s\leq t} B^1(s))+\beta(\max\limits_{0\leq s\leq t} B^2(s)-B^2(t))}\left[1-e^{-(q+\frac{\beta^2}{2})(t-T)}\right] \\
&\leq e^{\alpha(\max\limits_{0\leq s\leq t} B^1(s)-\min\limits_{0\leq s\leq t} B^1(s))+\beta(\max\limits_{0\leq s\leq t} B^2(s)-B^2(t))}\left[\Psi(T)+\frac{2e\bar{b}\left(a-mu-\frac{\alpha^2}{2}\right)K}{aD\left(q+\frac{\beta^2}{2}\right)}\right] \\
&:= K_1 e^{\alpha(\max\limits_{0\leq s\leq t} B^1(s)-\min\limits_{0\leq s\leq t} B^1(s))+\beta(\max\limits_{0\leq s\leq t} B^2(s)-B^2(t))},
\end{aligned}
$$

where $K_1 = \Psi(T) + \frac{2e\bar{b}\left(a-mu-\frac{\alpha^2}{2}\right)K}{aD\left(q+\frac{\beta^2}{2}\right)}$ is a constant. Therefore, we get

$$
\frac{\log\Psi(t)}{t} \leq \alpha\frac{\max\limits_{0\leq s\leq t} B^1(s)}{t} - \alpha\frac{\min\limits_{0\leq s\leq t} B^1(s)}{t} + \beta\frac{\max\limits_{0\leq s\leq t} B^2(s)}{t} - \beta\frac{B^2(t)}{t} + \frac{\log K_1}{t}. \quad (28)
$$

The distributions of $\max_{0\leq s\leq t} B^1(s)$ and $\max_{0\leq s\leq t} B^2(s)$ are the same as $|B^1(t)|$ and $|B^2(t)|$ respectively, and $\min_{0\leq s\leq t} B^1(s)$ has the same distribution as $-\max_{0\leq s\leq t} B^1(s)$. Letting $t\to\infty$, by the strong law of large numbers, (28) implies

$$
\limsup_{t\to\infty}\frac{\log\Psi(t)}{t} \leq 0 \quad \text{a.s,}
$$

as a result,

$$
\limsup_{t\to\infty}\frac{\log y(t)}{t} \leq 0 \quad \text{a.s.} \quad (29)
$$

On the other hand, we have

$$\begin{aligned} dy(t) &= y(t)\left(-q + e\bar{b}\frac{x(t)}{y(t)+D}\right)dt - \beta y(t)dB^2(t) \\ &\geq y(t)\left(-q + \frac{e\bar{b}}{D} - \frac{e\bar{b}}{D}y(t)\right)dt - \beta y(t)dB^2(t). \end{aligned}$$

Let

$$\begin{cases} d\psi(t) = \psi(t)\left(-q + \frac{e\bar{b}}{D} - \frac{e\bar{b}}{D}\psi(t)\right)dt - \beta\psi(t)dB^2(t), \\ \psi(0) = y_0. \end{cases}$$

According to Lemma 3.3, if $\beta < 0$ and Assumption 2 holds we have

$$\lim_{t\to\infty} \frac{\log\psi(t)}{t} = 0 \quad a.s,$$

as a result

$$\liminf_{t\to\infty} \frac{\log\psi(t)}{t} \geq 0 \quad a.s. \tag{30}$$

Therefore, (29) and (30) imply

$$\lim_{t\to\infty} \frac{\log y(t)}{t} = 0 \quad a.s. \tag{31}$$

□

Now, we can prove Theorem 3.4.

Proof. Denote $V(y) = \log y$. By Itô's formula, we obtain

$$d\log y(t) = \left(-q - \frac{\beta^2}{2} + \frac{e\bar{b}x(t)}{y(t)+D}\right)dt - \beta dB^2(t).$$

Integrating it from 0 to t, yields

$$\log y(t) - \log y(0) = \left(-q - \frac{\beta^2}{2}\right)t + \int_0^t \frac{e\bar{b}x(s)}{y(s)+D}ds - \beta B^2(t).$$

and

$$\frac{\log y(t) - \log y(0)}{t} = -q - \frac{\beta^2}{2} + \frac{1}{t}\int_0^t \frac{e\bar{b}x(s)}{y(s)+D}ds - \beta\frac{B^2(t)}{t}. \tag{32}$$

Letting $t \to \infty$, (31) and (32) imply

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{x(s)}{y(s)+D}ds = \frac{q + \frac{\beta^2}{2}}{e\bar{b}} \quad a.s. \tag{33}$$

We can easily see that

$$\liminf_{t\longrightarrow+\infty} \frac{1}{t}\int_0^t y(s)ds \geq 1 - \lim_{t\longrightarrow+\infty} D\int_0^t \frac{x(s)}{y(s)+D}ds \geq 1 - D\frac{q+\frac{\beta^2}{2}}{e\bar{b}}.$$

So, finally we have

$$\liminf_{t\longrightarrow+\infty} \frac{1}{t}\int_0^t y(s)ds \geq \frac{e\bar{b} - D\left(q + \frac{\beta^2}{2}\right)}{e\bar{b}} > 0.$$

□

Remark 3.1. Under Assumptions 1, 2 and by Theorem 3.2 and Theorem 3.4, we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{K\left(a-\bar{b}-\frac{\alpha^2}{2}-mu\right)}{a} > 0 \quad a.s.$$

and

$$\liminf_{t\longrightarrow+\infty} \frac{1}{t}\int_0^t y(s)ds \geq 1 - D\frac{q+\frac{\beta^2}{2}}{e\bar{b}} > 0 \quad a.s.$$

According to Definition (3.1), we obtain the stochastic system (16) is persistent in mean.

3.3. Extinction.

Definition 3.2. A population $z(t)$ goes extinct if:

$$\lim_{t\to\infty} z(t) = 0. \quad \text{a.s.}$$

We assume that Assumptions 1 and 2 are not satisfied and examine the case where $a - mu - \bar{b} - \frac{\alpha^2}{2} < 0$.

Theorem 3.6. *Let $(x(t), y(t))_{t\geq 0}$ be the solution of system (16) with any initial values $x_0 > 0, y_0 > 0$.*
If

$$a - mu - \bar{b} - \frac{\alpha^2}{2} < 0,$$

then

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0 \quad a.s.$$

Before moving to the proof of extinction in the model, we need the following lemma, which can be found in [6, 7].

Lemma 3.7. *Consider the following stochastic equation*

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t). \tag{34}$$

Assume $(X(t))_{t\geq 0}$ is the solution of system (34). If $S(-\infty) > -\infty$ and $S(+\infty) = +\infty$, then

$$\lim_{t\to\infty} X(t) = -\infty,$$

where the scale function

$$S(u) = \int_0^u e^{-\int_0^v \frac{2\mu(t,y)}{\sigma^2(t,y)}dy} dv.$$

Now, we can move to prove Theorem 3.6.

Proof. Define $U(x(t)) = \log x(t)$ and $V(y(t)) = \log y(t)$. Applying Itô's formula, we obtain

$$\begin{aligned} dU(x(t)) &= \left(a - mu - \frac{\alpha^2}{2} - \frac{a}{K}e^{U(x(t))} - \frac{\bar{b}e^{V(y(t))}}{e^{V(y(t))}+D}\right)dt + \alpha dB^1(t) \\ &\leq \left(a - \frac{\alpha^2}{2} - mu\right)dt + \alpha dB^1(t). \end{aligned}$$

By using the stochastic comparison theorem and the properties of diffusion processes (refer to Lemma (3.7)), for $\mu(t,x) = a - mu - \frac{\alpha^2}{2}$ and $\sigma(t,x) = \alpha$, it is easy to determine that $S(-\infty) > -\infty$ and $S(+\infty) = +\infty$, leading to

$$\lim_{t\to\infty} U(x(t)) = -\infty \quad \text{a.s.}$$

which implies that

$$\lim_{t\to\infty} x(t) = 0 \quad \text{a.s.}$$

In a such case, it follows either that

$$\lim_{t\to\infty} y(t) = 0 \quad \text{a.s.} \tag{35}$$

If this were not the case, then we must have

$$\limsup_{t\to\infty} y(t) := k > 0 \quad \text{a.s.}$$

Consequently, for any arbitrarily small $\varepsilon > 0$, there exist t_0 and a subset Ω_ε such that $\mathbb{P}(\Omega_\varepsilon) \geq 1 - \varepsilon$ and

$$\frac{e\bar{b}x(t)}{y(t) + D} \leq eb\varepsilon \quad \text{for } t \geq t_0 \text{ and } \omega \in \Omega_\varepsilon.$$

Thus, we derive

$$y(t)(-qdt - \beta dB^2(t)) \leq dy(t) \leq y(t)((-q + e\bar{b}\varepsilon)dt - \beta dB^2(t)),$$

and

$$-\left(q + \frac{\beta^2}{2}\right) dt - \beta dB^2(t) \leq dV(y(t)) \leq \left(-\left(q + \frac{\beta^2}{2}\right) + e\bar{b}\varepsilon\right) dt - \beta dB^2(t).$$

By the same reasoning as above and the arbitrariness of ε, we can get

$$\lim_{t\to\infty} V(y(t)) = -\infty \quad a.s.,$$

i.e.,

$$\lim_{t\to\infty} y(t) = 0 \quad a.s.$$

There is a contradiction, hence (35) is true. □

Remark 3.2. Given the condition $a - mu - \bar{b} - \frac{\alpha^2}{2} < 0$ and the proof of Theorem 3.6, if the prey goes extinct, the predator will also die out.

What about the other case?
In response to this question, we make use of the following result.

Theorem 3.8. *Let $(x(t), y(t))_{t\geq 0}$ be the solution of system (16) with any initial values $x_0 > 0, y_0 > 0$.*
If $a - mu - \bar{b} - \frac{\alpha^2}{2} > 0$ and $e\bar{b} - D\left(q + \frac{\beta^2}{2}\right) < 0$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t x(s)ds = \frac{K\left(a - mu - \frac{\alpha^2}{2}\right)}{a}, \quad \lim_{t\to\infty} y(t) = 0 \quad a.s.$$

Proof. Consider the predator population $y(t)$, we can easily see that

$$y(t)\left(-qdt-\beta dB^2(t)\right) \leq dy(t) \leq y(t)\left(\left(-q+\frac{e\bar{b}}{D}y(t)\right)dt-\beta dB^2(t)\right),$$

by the comparison theorem of stochastic equations and if $e\bar{b}-D\left(q+\frac{\beta^2}{2}\right)<0$, then

$$\lim_{t\to\infty} y(t)=0 \quad a.s. \tag{36}$$

Now, let us analyze the behavior of the prey population $x(t)$. Clearly, we have:

$$dx(t)=x(t)\left(a-mu-\frac{a}{K}x(t)-\frac{\bar{b}y(t)}{y(t)+D}\right)dt+\alpha x(t)dB^1(t),$$

which can be rewritten as

$$dx(t)\geq x(t)\left(a-mu-\bar{b}-\frac{a}{K}x(t)\right)dt+\alpha x(t)dB_1(t),$$

if $a-mu-\bar{b}-\frac{\alpha^2}{2}>0$, then by Lemma 3.3 , we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\geq\frac{a-mu-\bar{b}-\frac{\sigma^2}{2}}{b}>0 \quad a.s.$$

This implies that there exists $T_0>0$ and a positive constant k_0 such that $x(t)>k_0$ almost surely for $t\geq T_0$. Moreover, Equation (36) tells us that for any $\varepsilon>0$, there exist $T>T_0$ and a set Ω_ε such that $\mathbb{P}(\Omega_\varepsilon)\geq 1-\varepsilon$ and

$$\frac{y(t)}{y(t)+D}\leq b\varepsilon, \quad \text{for } \omega\in\Omega_\varepsilon, t\geq T.$$

Thus, when $\omega\in\Omega_\varepsilon, t\geq T$, we get

$$dx(t)\geq x(t)\left(a-mu-\bar{b}\varepsilon-\frac{a}{K}x(t)\right)dt+\alpha x(t)dB^1(t),$$

as a result, we obtain:

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\geq\frac{K\left(a-mu-\bar{b}\varepsilon-\frac{\alpha^2}{2}\right)}{a}>0.$$

On the other hand, we have the inequality:

$$dx(t)\leq x(t)(a-mu-\frac{\alpha^2}{2})dt+\alpha x(t)dB^1(t).$$

Thus, we can deduce then

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\leq\frac{K\left(a-mu-\frac{\alpha^2}{2}\right)}{a} \quad a.s.$$

Therefore, by the arbitrariness of ε, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds=\frac{K\left(a-mu-\frac{\alpha^2}{2}\right)}{a} \quad a.s.$$

∎

4. The stability of the solution with respect to initial values.

In stochastic dynamic systems, small variations in initial conditions can influence long-term behavior due to random fluctuations. In this section we aim to show the stability of system (16). We define the norm used to assess stability, considering two solutions z and $\hat{z}$ of the same stochastic differential equation, corresponding to different initial conditions z_0 and $\hat{z}_0$ as:

$$||z-\hat{z}||^2 := \mathbb{E}\left[\sup_{t\in[0,T]} |z(t)-\hat{z}(t)|^2 e^{\lambda t}\right],$$

where $\lambda > 0$ is a given constant and this norm is equivalent to the standard supremum norm for $\lambda = 0$.

Let (x, y) and $(\hat{x}, \hat{y})$ are two solutions of the stochastic system (16) with different initial conditions (x_0, y_0) and $(\hat{x}_0, \hat{y}_0)$. Such that $x_0 \longrightarrow \hat{x}_0$ and $y_0 \longrightarrow \hat{y}_0$.

By taking the difference between the integral forms of $x(t)$ and $\hat{x}(t)$, we get

$$x(t)-\hat{x}(t) = x_0-\hat{x}_0+\int_0^t \Big[a(x(s)-\hat{x}(s))-\frac{a}{K}(x(s)-\hat{x}(s))-\bar{b}\left(\frac{x(s)y(s)}{y(s)+D}-\frac{\hat{x}(s)\hat{y}(s)}{\hat{y}(s)+D}\right)$$
$$-mu(x(s)-\hat{x}(s)\Big]\,ds+\alpha\int_0^t (x(s)-\hat{x}(s))dB^1(s).$$

Applying Itô's formula, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} |x(t)-\hat{x}(t)|^2 e^{\lambda t}\right] \le |x_0-\hat{x}_0| + \lambda\mathbb{E}\left[\int_0^t e^{\lambda s}|x(s)-\hat{x}(s)|^2 ds\right]$$
$$+\mathit{\Gamma}\;\mathbb{E}\left[\int_0^t e^{\lambda s}|x(s)-\hat{x}(s)|^2 ds\right],$$

with $\mathit{\Gamma} = \mathbb{E}(\Lambda)$ and

$$\Lambda = 2a+\frac{2a}{K}\sup_{s\in[0,T]}|x(s)-\hat{x}(s)|$$
$$+2\bar{b}\left(\frac{\max_{s\in[0,T]}\left(\sup_{s\in[0,T]} y(s), \sup_{s\in[0,T]} \hat{y}(s)\right)}{\min_{s\in[0,T]}\left(\min_{s\in[0,T]} y(s)+D, \min_{s\in[0,T]} \hat{y}(s)+D\right)}\right)+2mu+\alpha^2.$$

By taking $\lambda = -\mathit{\Gamma}$, we obtain $\mathbb{E}\left[\sup_{t\in[0,T]}|x(t)-\hat{x}(t)|^2 e^{\lambda t}\right] \le |x_0-\hat{x}_0|$ which means that $\|x-\hat{x}\| \longrightarrow 0$ when $x_0 \longrightarrow \hat{x}_0$.

Now, let us consider $y(t)$ and we proceed the same as for $x(t)$ we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} |y(t)-\hat{y}(t)|^2 e^{\theta t}\right] \le |y_0-\hat{y}_0| + \theta\mathbb{E}\left[\int_0^t e^{\lambda s}|y(s)-\hat{y}(s)|^2 ds\right]$$
$$+\xi\;\mathbb{E}\left[\int_0^t e^{\theta s}|y(s)-\hat{y}(s)|^2 ds\right],$$

with $\xi = \mathbb{E}(\gamma)$ and

$$\gamma = 2q + 2b\left(\frac{\max_{s\in[0,T]}\left(\sup_{s\in[0,T]} x(s), \sup_{s\in[0,T]} \hat{x}(s)\right)}{\min_{s\in[0,T]}\left(\min_{s\in[0,T]} y(s) + D, \min_{s\in[0,T]} \hat{y}(s) + D\right)}\right) + \beta^2,$$

and if we take $\theta = -\xi$, we obtain $\mathbb{E}\left[\sup_{t\in[0,T]} |y(t) - \hat{y}(t)|^2 e^{\theta t}\right] \leq |y_0 - \hat{y}_0|$ which means that $\|y - \hat{y}\| \longrightarrow 0$ when $y_0 \longrightarrow \hat{y}_0$.

Remark 4.1. We have established the system's stability in sense that the trajectories remain close whenever the initial conditions x_0 is close to $\hat{x}_0$ and y_0 is close $\hat{y}_0$.

5. Numerical simulation

In this section, we use Milstein method mentioned in [8] to get the simulation of the stochastic system (16) in order to confirm the results set above.
Consider the discretization of system (16)

$$x_{i+1} = x_i + \left(ax_i\left(1 - \frac{x_i}{K}\right) - \frac{\bar{b}x_iy_i}{y_i + D} - mx_iu\right)\Delta t + \alpha x_i\,\Delta B_i^1 + \frac{1}{2}\alpha^2 x_i\left(\left(\Delta B_i^1\right)^2 - \Delta t\right),$$

$$y_{i+1} = y_i + \left(-qy_i + \frac{e\bar{b}x_iy_i}{y_i + D}\right)\Delta t - \beta y_i\,\Delta B_i^2 - \frac{1}{2}\beta^2 y_i\left(\left(\Delta B_i^2\right)^2 - \Delta t\right),$$

where $\Delta B_i = B_{t_{i+1}} - B_{t_i} \sim \mathcal{N}(0, \Delta t)$ and $\Delta t = t_{i+1} - t_i$ is the time step.

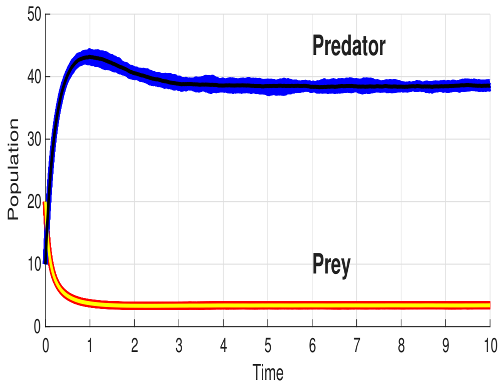

FIGURE 1. Solution of system (16) for $(x_0, y_0) = (20, 10)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 0.01$ and $\beta = -0.01$.

We have chosen suitable parameters based on those proposed in [15] for the studied model, which is influenced by water level fluctuation and harvesting in the deterministic case. We have considered a stochastic perturbation in parameters a and q to modelize the effect of environmental noise. The blue and red lines represent the solution of the stochastic system 16 while the black and yellow lines represent the solution of the deterministic one. First, we can see that the solution of the stochastic system is positive, which confirms Theorem 2.1. Comparing figure 1 and figure 2, we observe that when α and β are close to zero, the dynamics of the stochastic system is getting similar to the deterministic system's.

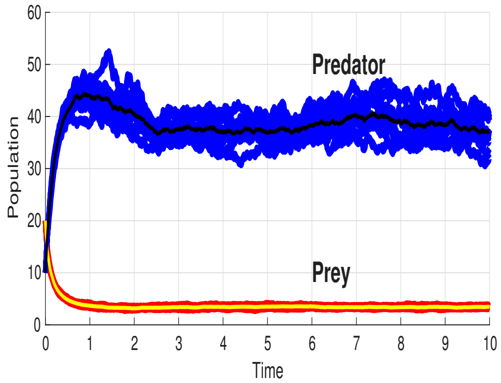

FIGURE 2. Solution of system (16) for $(x_0, y_0) = (20, 10)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 0.1$ and $\beta = -0.1$.

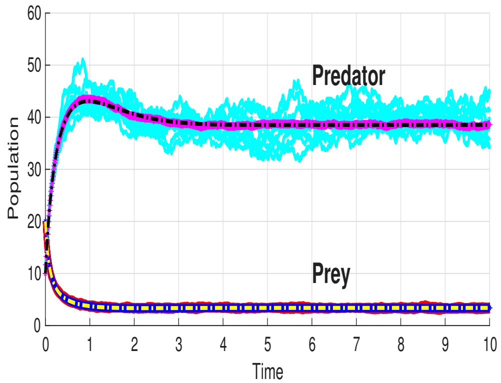

FIGURE 3. Solutions of systems (16) for $(x_0, y_0) = (20, 10)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 0.1$ and $\beta = -0.1$.

As mentioned in [15], the deterministic equivalent system possesses a coexisting equilibrium point $P^* = (x^*, y^*)$, which is globally asymptotically stable. Figure 3 represents the simulation of 10 trajectories of the solutions of the stochastic system (16). The trajectories in cyan represent the evolution of the predator population, while the red ones represent the dynamics of the prey population, and this what makes the stochastic system more realistic than the deterministic one is that in the studied lake we can't predict what exactly will happen due to the huge amount of natural changes. the stars in magenta and those in blue represent the means of x

and y, respectively. We can see that the stochastic system converges in mean to the coexisting equilibrium, so the numerical simulation confirms the persistence of both species under assumptions 1 and 2 because we have here $a-\bar{b}-mu-\frac{\alpha^2}{2}\approx 1.0888>0$ and $e\bar{b}-D\left(q+\frac{\beta^2}{2}\right)\approx 8.5450>0$, so Theorems 3.2 and 3.4 are verified numerically.

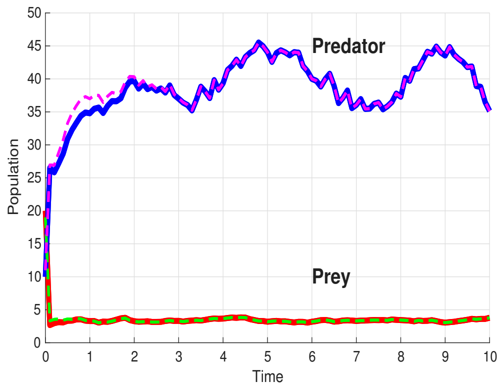

FIGURE 4. Solution of system (16) for $(x_0, y_0) = (20, 10)$ and $(x_0^{'}, y_0^{'}) = (19, 11)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 0.1$ and $\beta = -0.1$.

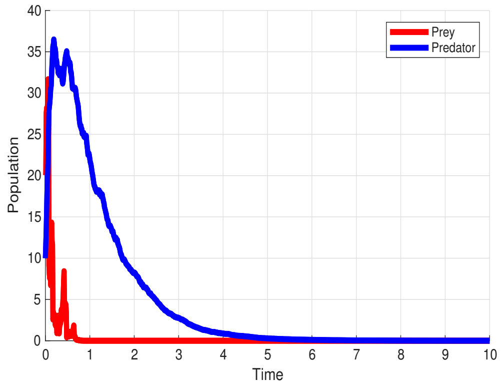

FIGURE 5. Solutions of systems (16) for $(x_0, y_0) = (20, 10)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 5$ and $\beta = -0.1$.

Figure 4 confirms the stability of the stochastic system (16). Given closed initial conditions, the trajectories are almost the same for both populations, and after calculation of the mean square error, we get $MSE_x = 0.0041$ and $MSE_y = 0.0097$, which means that a small variation in the initial population of prey and predator doesn't influence the behavior of the system.

Here in Figure 5 we verify numerically Theorem 3.6 and we get the extinction of both spaces since Assumption 1 is not satisfied, we have here $a - \bar{b} - mu - \frac{\alpha^2}{2} \approx -11.4000 < 0$.

Figure 6 represents the extinction of predator population when Assumption 2 is not satisfied because $e\bar{b} - D\left(q + \frac{\beta^2}{2}\right) \approx -119.5000 < 0$ we have here the stability in average of the prey population.

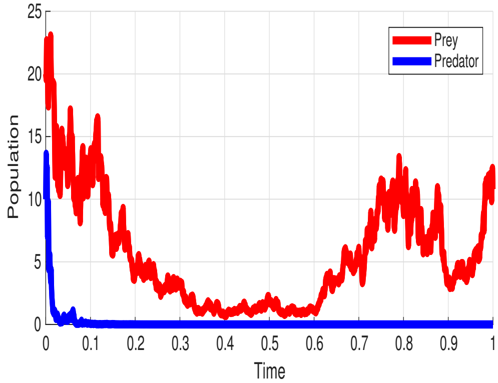

FIGURE 6. Solution of system (16) for $(x_0, y_0) = (20, 10)$, $a = 12$, $\bar{b} = 10$, $K = 20$, $e = 1.25$, $D = 4$, $m = 6$, $u = 0.15$, $q = 1$, $\Delta t = 0.001$, $\alpha = 3$ and $\beta = -8$.

6. Conclusion

In this paper, we studied a stochastic prey-predator model with functional response based on the quantity of food received by the predator, where we considered the harvesting strategy. The dynamic is studied theoretically and numerically.

According to our results, Theorems 3.2 and 3.4, when the environmental noise represented by parameters α and β has an impact, such that assumptions 1 and 2 are not satisfied, both populations will be extinct.

The numerical analysis presented in the last section, illustrates the theoretical results of the positivity of the solution, persistence and extinction of the population.

References

[1] A. Friedman, *Stochastic Differential Equations and their Applications*, Academic Press, New York, 1976.

[2] A. Moussaoui, S. M. Bouguima, A prey-predator interaction under fluctuating level water, *Math. Methods Appl. Sci.* **38** (2015), 123-137.

[3] A. Moussaoui, S. Bassaid, E.A. Dads, The impact of water level fluctuations on a delayed prey-predator model, *Nonlinear Anal. Real World Appl.* **21** (2015), 170-184.

[4] A. Moussaoui, A reaction-diffusion equations modelling the effect of fluctuating water levels on prey-predator interactions, *Appl. Math. Comput.* **268** (2015), 1110-1121.

[5] A. Moussaoui, S.M. Bouguima, Seasonal influences on a prey-predator model, *J. Appl. Math. Comput.* **50** (2016), 39–57.

[6] C. Ji, D. Jiang, N. Shi, Analysis of a predator-prey model with modified Leslie-Gower and Holling-type schemes with stochastic perturbation, *J. Math. Anal. Appl.*, **359** (2009), 482-498.

[7] C. Ji, D. Jiang, X. Li, Qualitative analysis of a stochastic ratio-dependent predator-prey system, *J. Comput. Appl. Math.*, **235** (2010), 1326-1341.
[8] D.J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Review*, **43** (2001), 525-546.
[9] E. Renshaw, *Modelling Biological Populations in Space and Time*, Cambridge University Press, Cambridge, 1991.
[10] F. Campillo, C. Lobry, Effect of population size in a predator-prey model, *Ecological Modelling* **246** (2012), 1-10.
[11] H.I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.
[12] I. Karatzas, S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, Berlin, 2012.
[13] J. Lv, K. Wang, Asymptotic properties of a stochastic predator-prey system with Holling functional response, *Commun. Nonlinear Sci. Numer. Simulat.* **16** (2011), 4037-4048.
[14] D. Jiang, N. Shi , X. Li, Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation, *J.* Math Anal Appl. **340** (2008), 588-597.
[15] K. Belkhodja, A. Moussaoui, M.A. Aziz Alaoui, Optimal harvesting and stability for a prey-predator model, *Nonlinear Analysis: Real World Applications* **39** (2017), 321-336.
[16] L. Chen, J. Chen, *Nonlinear Biological Dynamical System*, Science Press, Beijing, 1993.
[17] L. Arnold, *Stochastic Differential Equations: Theory and Applications*, Wiley, New York, 1972.
[18] M.A. Menouer, A. Moussaoui, Effects of consecutive water level fluctuations and harvesting on predator-prey interactions, *Chaos, Solitons & Fractals* **91** (2016), 434-442.
[19] M. Carletti, K. Burrage, P.M. Burrage, Numerical simulation of stochastic ordinary differential equations in biomathematical modelling, *Math. Comput. Simulation* **64** (2004), 271-277.
[20] M. Liu, K. Wang, Global stability of a nonlinear stochastic predator-prey system with Beddington DeAngelis functional response, *Commun Nonlinear Sci Numer Simulat.* **16** (2011), 1114-1121.
[21] N. Chiboub, S.M. Bouguima, A. Moussaoui, The effect of water level in a prey-predator interactions: A nonlinear analysis study, *Chaos Solitons Fractals* **45** (2012), 205-212.
[22] X. Mao , G. Marion , E. Renshaw, Environmental Brownian noise suppresses explosions in population dynamics, *Stochastic Process Appl.* **97** (2002), 95-110
[23] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, New York, 1997.
[24] Y. Li, H. Gao, Existence, uniqueness and global asymptotic stability of positive solutions of a predator-prey system with Holling II functional response with random perturbation, *Nonlinear Anal.* **68** (2008), 1694-1705.

(Lamia Bousmaha, Abdeldjebar Kandouci) Laboratory of Stochastic models, statistics and applications, University of Saida, Dr. Moulay Tahar, Algeria.
E-mail address: lamia.bousmaha@gmail.com, Kandouci1974@yahoo.fr

(Karima Smail) Laboratory of Stochastic models, statistics and applications, University of Saida, Dr. Moulay Tahar, Algeria. Department of Stochastics and its Applications, University of Cottbus, Germany.
E-mail address: smailkarima90@gmail.com

(Omar Kebiri) Department of Stochastics and its Applications, University of Cottbus, Germany. Institute of Mathematics, the free University of Berlin, Germany.
E-mail address: omar.kebiri@b-tu.de

(Ali Moussaoui) Laboratory of Nonlinear Analysis and Applied Mathematics, University of Tlemcen, Abou Bekr Belkaid, Algeria.
E-mail address: moussaoui.ali@gmail.com