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On Endomorphisms of BCH-Algebras

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Abstract. In this paper we introduce the notion of a BCH-endomorphism. It is proved that L_0 is the only non-identity BCH-endomorphism of left type, where $L_0^3(x) = L_0(x)$ for every x in a BCH-algebra $(X; *, 0)$ and L_0^2 is idempotent. Some more properties of left and right mappings of BCH-algebras are also investigated with BCH-characterization.

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1. Introduction

Y. Imai and K. Iséki introduced two classes of logical algebras: BCK-algebras and BCI-algebras [9, 10]. It is known that the class of BCI-algebras is a generalization of the class of BCK-algebras. In [5, 6], Q. P. Hu and X. Li introduced a wider class of logical algebras: BCH-algebras. They have shown that the class of BCH-algebras is further a generalization of the class of BCI-algebras. The authors of this paper introduced a class of K-algebras with extended study in [12-15]. Recently, same authors have proved in [15] that a class of K-algebras as a generalization of a family of $BCH/BCI/BCK$-algebras.

K. H. Dar introduced the notions of left and right mappings over BCK-algebras in [1] and further discussed in [2]. The notions of left and right mappings over BCI-algebras have been discussed in [3]. In this paper we introduce the notion of BCH-endomorphisms. Some more properties of left and right mappings of BCH-algebras are investigated with special focus on the left map L_0.

2. Preliminaries

In this section we cite some elementary aspects that will be used in the sequel of this paper:

Definition 2.1. [5, 6] *An algebra $(X; *, 0)$ of type (2, 0) is called a BCH-algebra if, for all x, y, $z \in X$, the following axioms hold:*

(H1) $x * x = 0$,

(H2) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,

(H3) $(x * y) * z = (x * z) * y$.

In a BCH-algebra X, the following hold as immediate deductions.

(H4) $x * 0 = x$.
(H5) $x * 0 = 0 \Longleftrightarrow x = 0$.
(H6) $0 * (x * y) = (0 * x) * (0 * y)$.
(H7) $(x * (x * y)) * y = 0$.

Definition 2.2. *A nonempty subset S of a BCH-algebra $(X; *, 0)$ is a called BCH-subalgebra if$x * y \in S$, for all x, $y \in S$.*

Definition 2.3. *A BCH-subalgebra J is called BCH- ideal if $x * y$ and $x \in J \Rightarrow y \in J$ for all x, $y \in X$.*

Example 2.1. *The subset $0 * X = \{0 * x : x \in X\}$ forms a BCH-subalgebra since $0 \in 0 * X$ and $(0 * x) * (0 * y) = 0 * (x * y) \in 0 * X$, for every $0 * x$, $0 * y \in 0 * X$. It easily follows that $0 * X$ is a BCH-ideal.*

Example 2.2. *The subset $0 * (0 * X) = \{0 * (0 * x) : x \in X\}$ of a BCH-algebra $(X; *, 0)$ is a BCH-ideal.*

3. *BCH*-endomorphisms

Definition 3.1. *A mapping $\phi : X \to X$ on a BCH-algebra $(X; *, 0)$ is called a BCH-endomorphism if $\phi(x * y) = \phi(x) * \phi(y)$, for all x, $y \in X$.*

The set $End(X)$ of all endomorphisms of X forms a semigroup, under the binary operation of their composition $(\circ)$. Each $\phi \in End(X)$ acts the following way on X:

Proposition 3.1. *If ϕ is a BCH-endomorphism of $(X; *, 0)$ then*
(i) $\phi(0) = 0$.
(ii) $\phi(0 * x) = 0 * \phi(x)$.
(iii) *If $x * y = 0$ then $\phi(x) * \phi(y) = 0$.*
(iv) *If S is a BCH-subalgebra of X then so is $\phi(S)$.*
(v) *If S is a BCH-ideal of X then so is $\phi(S)$.*
(vi) *$Ker\phi = \{x \in X : \phi(x) = 0\}$ is an ideal of X, for each ϕ in $End(X)$.*

Proof. Straightforward. □

Definition 3.2. *For each element $x \in X$ there associates a pair L_x , R_x of left and right mappings respectively, which are defined by $L_x(t) = x * t$ and $R_x(t) = t * x$, for all $t \in X$ (see [1]).*

If $L = \{L_x : x \in X\}$ and $R = \{R_x : x \in X\}$. Then L and R both are in one-to-one corresponding with BCH-algebra X where $L_x(t) = R_t(x)$ for all $x, t \in X$. The mappings of L and R compose together the following way:

Proposition 3.2. *The mappings L and R on a BCH-algebra $(X; *, 0)$ compose by the following properties.*
(a) $R_y \circ L_0 = L_{0*y}$.
(b) $R_x \circ R_y = R_y \circ R_x$.

(c) $L_0 \circ R_y = R_{0*y} \circ L_0 = L_{0*(0*y)}$.
(d) $L_x \circ R_0 = L_x = R_0 \circ L_x$.
(e) $R_y \circ L_x = L_{x*y}$.
(f) $L_0 \circ L_x = L_{0*x} \circ L_0$.

Proof. Routine. □

Remark 3.1. *It is an important to note that L_0 is an endomorphism of BCH-algebras $(X; *, 0)$ with its powers by the following Cayley table:*

$\circ$	I	L_0	L_0^2
I	I	L_0	L_0^2
L_0	L_0	L_0^2	L_0
L_0^2	L_0^2	L_0	L_0^2

Proposition 3.3. *In a BCH-algebra $(X; *, 0)$, for all x, y*

$$L_0 \circ (L_x \circ R_y) = L_{0*x} \circ L_{0*(0*y)}.$$

Proof.

$$\begin{aligned} L_0 \circ (L_x \circ R_y) &= (L_0 \circ L_x) \circ R_y \\ &= (L_{0*x} \circ L_0) \circ R_y \\ &= L_{0*x} \circ (L_0 \circ R_y) \\ &= L_{0*x} \circ L_{0*(0*y)}. \end{aligned}$$

□

Corollary 3.1. *For x, $y \in X$,*
(a) $L_0 \circ (R_y \circ L_x) = (R_{0*y} \circ L_{0*x}) \circ L_0 = L_0 \circ L_{x*y}$.
(b) $L_0 \circ (L_y \circ L_x) = L_0^2 \circ L_{x*y}$.

Theorem 3.1. *L_0 is the only BCH-endomorphism of X in L.*

Proof. Let x, $y \in X$. Then

$$\begin{aligned} L_0(x) * L_0(y) &= L_{(x*y)*(x*y)}(x) * L_0(y) \\ &= (((x*y)*x)*(x*y)) * L_0(y) \\ &= ((0*y)*(x*y))*(0*y) \\ &= 0*(x*y) \\ &= L_0(x*y), \end{aligned}$$

which proves the axiom (6) of BCH-algebra that L_0 in L is an endomorphism of X. L_0 is unique BCH-endomorphism since , for non-zero x in X, L_x is not a BCH-endomorphism by the contradiction, $x = L_x(0) = L_x(0*0) = L_x(0) * L_x(0) = 0$. □

Corollary 3.2. *L_0 is a central BCH-endomorphism.*

Proof. Let ϕ be an arbitrary endomorphism of $(X;*,0)$. Then

$$\phi \circ L_0(x) = \phi(0*x) = 0*\phi(x) = L_0 \circ \phi(x),$$

for all $x \in X$. Hence $\phi \circ L_0 = L_0 \circ \phi$. □

In order to exhibit action of L_0 on a BCH-algebra $(X;*,0)$, we define $0*x = 0^1*x$, $0*(0*x) = 0^2*x$, $0*(0*(0*x)) = 0^3*x$, $0*(0*(0*\cdots(n-times)*x) = 0^n*x$ for any positive integer n. If $x \in X$, we observe that:

(a) $0*(0^k*x) = 0^{k+1}*x$.

(b) $0^l*(0^m*x) = 0^{l+m}*x$.

(c) $(0^l*x)*(0^l*y) = 0^l*(x*y)$ for positive integers l, m, k and x, $y \in X$.

We exhibit that:

Proposition 3.4. [4] *In a BCH-algebra $(X;*,0)$,*

(d) $0^3*x = 0*x$ *for all* $x \in X$.

Proof. Since

$$\begin{aligned} (0*x)*(0*x) &= (0^2*x)*x = 0 \\ therefore,\ 0^3*x &= 0*(0^2*x) \\ &= ((0^2*x)*x)*(0^2*x) \\ &= 0*x\ [by\ H_3]. \end{aligned}$$

□

Corollary 3.3. *L_0 is a periodic map of period 2.*

Corollary 3.4. *L_0^2 is identity on $L_0(X) = \{0*x : x \in X\}$.*

Corollary 3.5. *L_0 is an epimorphism on X.*

Proposition 3.5. *The following equalities are valid in a BCH-algebra $(X;*,0)$ for all x, y, $z \in X$.*

(e) $0*(x*y) = 0^2*(y*x)$.

(f) $(0^2*z)*(y*x) = 0^2*(x*(y*z))$.

Proof. (e)

$$\begin{aligned} 0*(x*y) &= 0^3*(x*y) \\ &= 0^2*(0*(x*y))\ [by\ H6] \\ &= 0^2*((0*x)*(0*y))\ [by\ H3] \\ &= 0^2*((0*(0*y))*x)\ [by\ (a)] \\ &= 0^2*((0^2*y)*x)\ [by\ remark] \\ &= 0^2*((0^2*y))*(0^2*x)\ [by\ Proposition\ 3.9] \\ &= (0^2*y)*(0^2*x) \\ &= 0^2*(y*x). \end{aligned}$$

(f)

$$\begin{aligned}(0^2 * z) * (y * x) &= (0 * (0 * z)) * (y * x) \\ &= (0 * (y * x)) * (0 * z) \\ &= 0 * ((y * x) * z) \\ &= 0 * ((y * z) * x) \\ &= 0^2 * (x * (y * z)).\end{aligned}$$

This ends the proof. □

Corollary 3.6.

$$0 * (x * (x * y)) = 0 * y \ \forall \ x \ , y \in X.$$

Corollary 3.7.

$$0 * ((x * y) * (x * z)) = 0 * (z * y) \ \forall \ x \ , y, \ z \in X.$$

Corollary 3.8.

$$(0^2 * y) * x = (0 * (x * y)) \ \forall \ x \ , y \in X.$$

It is known that a BCH-algebra $(X; *, 0)$ is not BCI-algebra,if

$$((x * y) * (x * z)) * (z * y) \neq 0$$

for at least one trio x, y, z of elements of BCH-algebra X.

Theorem 3.2. *The BCH-subalgebras $L_0(X) = \{0 * x : x \in X\}$ and $L_0^2(X) = \{0 * (0 * x) : x \in X\}$ of $(X; *, 0)$ are its BCI-subalgebras.*

Proof. $L_0(X)$ and $L_0^2(x)$ are images of X under L_0 and L_0^2 respectively. They form subalgebras of X since

$$L_0(x) * L_0(y) = L_0(x * y) \in L_0(X)$$

and

$$L_0^2(x) * L_0^2(y) = L_0^2(x * y) \in L_0^2(X)$$

for all $x, y \in X$. $L_0(X)$ is a BCI-algebra since, for all $L_0(x)$, $L_0(y)$, $L_0(z) \in L_0(X)$

$$\begin{aligned}L_0(((x * y) * (x * z)) * (z * y)) &= L_0((x * y) * (x * z)) * L_0(z * y) \\ &= L_0(z * y) * L_0(z * y) \\ &= 0.\end{aligned}$$

Similarly, $L_0^2(X)$ is a BCI-algebra. □

4. Order relations on a BCH-algebra

On a BCH-algebra $(X;*,0)$, a natural order $\sim$ is defined by $x \sim y$ if and only if $x * y = 0$. This *order* is reflexive, antisymmetric but not transitive in general. It is locally transitive at 0, since if $0 \sim x$ and $x \sim y$ then $0 \sim y$. We introduce another relation $\approx$ which is symmetric and transitive closure of $\sim$ and is defined by $x \approx y$ if and only if $x * y \in KerL_0$, then:

Lemma 4.1. *Let $(X;*,0)$ be a BCH-algebra and relation $\approx$ be defined on X. Then $x \approx y$ if and only if $0 * x = 0 * y$, x, $y \in X$.*

Proof. Suppose that $x \approx y$. Then $x * y \in KerL_0$ and hence,

$$\begin{aligned} 0 * (x * y) &= 0 \\ (0 * x) * (0 * y) &= 0 \ (I) \\ (0 * (0 * y)) * x &= 0 \\ 0 * ((0 * (0 * y)) * x) &= 0 \\ (0 * (0 * (0 * y))) * (0 * x) &= 0 \\ (0 * y) * (0 * x) &= 0 \ (II) \end{aligned}$$

From (I) and (II) it follows that $0 * x = 0 * y$.
Conversely, if $0 * x = 0 * y$ then

$$0 = (0 * x) * (0 * y) = 0 * (x * y).$$

□

Corollary 4.1. *The relation $\approx$ is symmetric on X.*

Corollary 4.2. *The relation $\approx$ is generally transitive on X.*

Definition 4.1. *Let there be a relation $\sim$ on X. A relation $\approx$ on X is called an equivalence closure on X if*

(i) $\sim \subseteq \approx$,
(ii) *$\approx$ is an equivalence relation.*

Theorem 4.1. *Let $(X;*,0)$ be a BCH-algebra and a relation $\sim$ be defined on X by, $x \sim y$ if and only if $x * y = 0$. Then there exists an equivalence closure $\approx$ of $\sim$ on X , as defined by $x \approx y$ if and only if $x * y \in KerL_0$.*

Proof. We see that the relation $\approx$ on X is reflexive since $x \approx x = 0 \in KerL_0$, for all $x \in X$. The symmetric and transitive properties of $\approx$ on X follow from the Lemma 4.1. Hence the assertion of the theorem is proved. □

Corollary 4.3. *If $KerL_0 = X$ then the equivalence class $C_0 = X$ and the BCH-quotient algebra $X/KerL_0$ is trivial.*

If $KerL_0 \subset X$ and there is one element in X invariant by L_0, then there exists an elementary abelian 2-group in X as a BCH-subalgebra. We characterize that:

Theorem 4.2. *In a BCH-algebra $(X; *, 0)$ if $0 * x = x$, for all $x \in X$ then $(X; *, 0)$ forms an elementary abelian 2-group .*

Definition 4.2. *A BCH-algebra is said to be $*$-commutative if $x * y = y * x$, for all $x,\ y \in X$.*

Theorem 4.3. *A BCH-algebra $(X; *, 0)$ forms an elementary abelian 2-group if and only if BCH-algebra $(X; *, 0)$ is $*$-commutative.*

Corollary 4.4. *There exists no $*$-commutative proper BCH-subalgebra.*

We have proved that L_0 is the only endomorphism of a BCH-algebra $(X; *, 0)$ with homomorphic image $L_0(X)$. L_0 is in fact an epimorphism on X. If $KerL_0 = \{x \in X : 0 * x = 0\}$ is proper ideal of X then the quotient algebra $X/KerL_0$ is a BCH-algebra and $n_0 : X \to X/KerL_0$ is a natural BCH-homomorphism defined by $n_0(x) = x * KerL_0$, $x \in X$ where

$$n_0(x * y) = (x * y) * KerL_0 = (x * KerL_0) * (y * KerL_0)$$

for all $x,\ y \in X$. Since $X/KerL_0$ and $L_0(X)$ are BCH-algebras and a map $\eta : X/KerL_0 \to L_0(X)$ is defined by $\eta(x * KerL_0) = 0 * x$, for all $x \in X$ is well-defined, since the map η is a BCH-isomorphism where

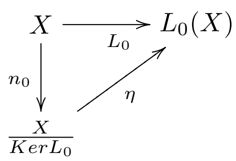
$$\begin{aligned} \eta((x * KerL_0) * (y * KerL_0)) &= \eta((x * y) * KerL_0) = 0 * (x * y) \\ &= (0 * x) * (0 * y) \\ &= \eta(x * KerL_0) * \eta(y * KerL_0) \end{aligned}$$

and $Ker\eta = KerL_0$. Thus we establish the fundamental Theorem:

Theorem 4.4. *Let $(X; *, 0)$ be a BCH-algebra and L_0 an epimorphism on X. Then*

$$X/KerL_0 \cong L_0(X).$$

Proof. The proof is exhibited by the following commutative diagram:



□

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