

On the Ghermănescu's Sequence

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ABSTRACT. The aim of this work is to establish some properties of the Ghermănescu's sequence.

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1. Introduction

Mihail Ghermănescu (1899–1962) was a notable Romanian mathematician recognized for his contributions across several mathematical disciplines, including Mathematical analysis, or Number theory. His research was both deep and diverse, as some of his key contributions include the introducing the concept of areolar derivatives, or studying and extending results related to Riccati's equation, Laplace's equation, and equations by Pierre Humbert, Weyl, Fredholm, and Volterra.

He introduced the following sequence

$$G_n = \frac{(n+2)^{n+1}}{(n+1)^n} - \frac{(n+1)^n}{n^{n-1}},$$

now known as the *Ghermănescu's sequence*.

The Ghermănescu's sequence is closely related to the Keller's sequence:

$$K_n = (n+1) \left(1 + \frac{1}{n}\right)^n - n \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

See, e.g. [1]. Keller's sequence was widely studied by the researchers in the recent past. See, e.g. [5]-[9].

The aim of this work is to establish some properties of the Ghermănescu's sequence.

2. Inequalities related to the Ghermănescu's sequence

Remark that the sequence G_n can be written in terms of the sequence

$$e_n = \left(1 + \frac{1}{n}\right)^n.$$

More precisely, we have the representation:

$$G_n = (n+1)e_{n+1} - ne_n.$$

A direct consequence is that

$$\lim_{n \rightarrow \infty} G_n = e.$$

Some estimates of G_n can be obtained by using estimates of e_n .

Mortici and Jang [9] provided the following inequalities, for all $0 < x \leq 1$:

$$\begin{aligned} & e \left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \frac{2447}{5760}x^4 - \frac{959}{2304}x^5 \right) \\ & < (1+x)^{1/x} \\ & < e \left(1 - \frac{1}{2}x + \frac{11}{24}x^2 - \frac{7}{16}x^3 + \frac{2447}{5760}x^4 \right). \end{aligned} \quad (1)$$

By replacing x by $1/n$, for all integers $n \geq 1$, we obtain:

$$u(n) < \frac{1}{e}e_n < v(n), \quad (2)$$

where

$$u(n) = 1 - \frac{1}{2n} + \frac{11}{24n^2} - \frac{7}{16n^3} + \frac{2447}{5760n^4} - \frac{959}{2304n^5}$$

and

$$v(n) = 1 - \frac{1}{2n} + \frac{11}{24n^2} - \frac{7}{16n^3} + \frac{2447}{5760n^4}.$$

We give the next

Theorem 2.1. *The following inequality holds true, for all integers $n \geq 1$:*

$$1 - \frac{11}{24n^2} < \frac{1}{e}G_n < 1 - \frac{11}{24n^2} + \frac{4}{3n^3}. \quad (3)$$

Proof. By using (2), we obtain

$$(n+1)u(n+1) - nv(n) < \frac{1}{e}G_n < (n+1)v(n+1) - nu(n)$$

and we have to prove that $\varphi < 0$ and $\psi > 0$, where

$$\varphi(x) = \left(1 - \frac{11}{24x^2} \right) - ((x+1)u(x+1) - xv(x))$$

and

$$\psi(x) = \left(1 - \frac{11}{24x^2} + \frac{4}{3x^3} \right) - ((x+1)v(x+1) - xu(x)).$$

This is true, since

$$\varphi(x) = -\frac{A(x-1)}{11520x^3(x+1)^4} < 0$$

and

$$\psi(x) = \frac{B(x-1)}{11520x^4(x+1)^3} > 0,$$

where

$$A(x) = 15360x^4 + 83003x^3 + 163485x^2 + 130145x + 29409$$

and

$$B(x) = 30287x^3 + 126918x^2 + 168844x + 67418.$$

□

From (3) it follows:

Theorem 2.2. *The following asymptotic formula holds true, as $n \rightarrow \infty$:*

$$G_n = e - \frac{11e}{24n^2} + O\left(\frac{1}{n^3}\right).$$

In consequence,

$$\lim_{n \rightarrow \infty} n^2(e - G_n) = \frac{11e}{24}. \tag{4}$$

3. The asymptotic series of Ghermănescu's sequence

As the Ghermănescu's sequence G_n is given in terms of e_n , we will search an asymptotic series for e_n , in order to provide an asymptotic series for G_n .

Brothers and Knox [2] (see also [3]) provided the following expansion, as $x \rightarrow \infty$:

$$\frac{1}{e} \left(1 + \frac{1}{x}\right)^x \sim 1 - \frac{1}{2x} + \frac{11}{24x^2} - \frac{7}{16x^3} + \frac{2447}{5760x^4} - \frac{959}{2304x^5} + \frac{238043}{580608x^6} - \dots$$

Note that the first terms of this expansion appear in (1). Later, Chen and Choi [3] established the general term of this expansion. More precisely, they proved the formula:

$$\frac{1}{e} \left(1 + \frac{1}{x}\right)^x \sim \sum_{j=0}^{\infty} \frac{(-1)^j b_j}{x^j} \quad (x \rightarrow \infty), \tag{5}$$

where the coefficients b_j are given by

$$b_j = \sum_{k_1+2k_2+\dots+jk_j=j} \frac{\left(\frac{1}{2}\right)^{k_1} \left(\frac{1}{3}\right)^{k_2} \dots \left(\frac{1}{j+1}\right)^{k_j}}{k_1!k_2!\dots k_j!}, \quad j \geq 1, \tag{6}$$

with $b_0 = 1$.

Now we are in a position to construct the asymptotic series of Ghermănescu's sequence. We have:

$$\begin{aligned} G_n &= (n+1)e_{n+1} - ne_n \\ &= (n+1)e \sum_{j=0}^{\infty} \frac{(-1)^j b_j}{(n+1)^j} - ne \sum_{j=0}^{\infty} \frac{(-1)^j b_j}{n^j} \\ &= e \sum_{j=0}^{\infty} \frac{(-1)^j b_j}{(n+1)^{j-1}} - e \sum_{j=0}^{\infty} \frac{(-1)^j b_j}{n^{j-1}}. \end{aligned}$$

Finally, the first series can be transformed in a series in n^{-j} powers by using the binomial expansion:

$$\frac{1}{(y-1)^{j-1}} = \frac{1}{y^{j-1}} \left(1 - \frac{1}{y}\right)^{-(j-1)} = \sum_{i=0}^{\infty} \binom{j-2+i}{j-2} \frac{1}{y^{j-1+i}}, \quad \frac{1}{y} < 1, \quad j = 1, 2, 3, \dots,$$

to obtain the entire asymptotic series of G_n . The first terms are the following:

$$\begin{aligned} \frac{1}{e} G_n \sim & 1 - \frac{11}{24n^2} + \frac{4}{3n^3} - \frac{1949}{640n^4} + \frac{289}{45n^5} \\ & - \frac{7609663}{580608n^6} + \frac{5905}{224n^7} - \frac{10486803319}{199065600n^8} + \dots \end{aligned} \tag{7}$$

4. Conclusions

Chen and Paris [4, Th. 2.1] provided an extension of the double-inequality (1). More precisely, they proved that inequalities of type (1) hold true by truncation the asymptotic series (5) at any term:

$$\sum_{j=0}^{2m+1} \frac{(-1)^j b_j}{x^j} < \frac{1}{e} \left(1 + \frac{1}{x}\right)^x < \sum_{j=0}^{2m} \frac{(-1)^j b_j}{x^j} \quad (m \geq 0). \tag{8}$$

For $m = 2$, the inequality (1) is obtained.

Recall that inequalities (1) were the main tool for establishing the under- and upper-approximations (3) for the sequence G_n . By using the more general inequalities (8), increasingly stronger inequalities for the sequence G_n can be obtained. These are truncations of the series (7). Note that inequalities (1) involve the first three terms of the series (7). As an example we refer to the more accurate under- and upper-approximations obtained by truncation the series (7) at the fifth term:

$$\begin{aligned} & 1 - \frac{11}{24n^2} + \frac{4}{3n^3} - \frac{1949}{640n^4} \\ & < \frac{1}{e} G_n \\ & < 1 - \frac{11}{24n^2} + \frac{4}{3n^3} - \frac{1949}{640n^4} + \frac{289}{45n^5}. \end{aligned} \tag{9}$$

These inequalities can be used to establish further limits of the form (4). For instance, we have:

$$\lim_{n \rightarrow \infty} \left(\left(n^2 (e - G_n) - \frac{11}{24} e \right) n + \frac{4}{3} e \right) n = \frac{1949}{640} e.$$

Inequalities (9) permit us to provide a computational proof of the monotonicity of Ghermănescu’s sequence.

Theorem 4.1. *The sequence G_n is strictly increasing.*

Proof. The inequalities $G_1 < G_2 < G_3 < G_4$ are true (they can be directly verified). By using (9), we deduce that

$$\begin{aligned} \frac{1}{e} G_{n+1} - \frac{1}{e} G_n &> \left(1 - \frac{11}{24(n+1)^2} + \frac{4}{3(n+1)^3} - \frac{1949}{640(n+1)^4} \right) \\ &\quad - \left(1 - \frac{11}{24n^2} + \frac{4}{3n^3} - \frac{1949}{640n^4} + \frac{289}{45n^5} \right) \\ &= \frac{P(n-4)}{5760n^5(n+1)^4} \\ &> 0, \end{aligned}$$

for all integers $n \geq 4$. Here,

$$\begin{aligned} P(n) &= 5280n^6 + 116\,880n^5 + 1068\,052n^4 + 5075\,630n^3 \\ &\quad + 12\,743\,100n^2 + 14\,439\,365n + 3308\,116. \end{aligned}$$

In conclusion the sequence G_n is strictly increasing. □

Next we discuss the following series involving the Ghermănescu’s sequence.

Theorem 4.2. a) *The following inequality holds true, for all integers $n \geq 1$:*

$$\frac{11e}{24(n+1)(n+2)} < e - G_n < \frac{11e}{24n(n+1)}.$$

The following series:

$$\sum_{n=1}^{\infty} (e - G_n)$$

is convergent. For all integers $n \geq 1$, it holds:

$$\frac{11(n+1)e}{24(n+2)} < \sum_{k=1}^n (e - G_k) < \frac{11ne}{24(n+1)}. \tag{10}$$

Proof. From (9), we get:

$$\alpha(n) < e - G_n < \beta(n), \tag{11}$$

where

$$\alpha(n) = \frac{11e}{24n^2} - \frac{4e}{3n^3} + \frac{1949e}{640n^4} - \frac{289e}{45n^5}$$

and

$$\beta(n) = \frac{11e}{24n^2} - \frac{4e}{3n^3} + \frac{1949e}{640n^4}.$$

Further, for all integers $n \geq 8$, we have:

$$\frac{11e}{24(n+1)(n+2)} < \alpha(n), \quad \beta(n) < \frac{11e}{24n(n+1)}, \tag{12}$$

since

$$\frac{11e}{24(n+1)(n+2)} - \alpha(n) = -\frac{Z(n-8)}{5760n^5(n+2)(n+1)}e < 0$$

and

$$\beta(n) - \frac{11e}{24n(n+1)} = -\frac{W(n-8)}{1920n^4(n+1)}e < 0.$$

Here,

$$Z(n) = 240n^4 + 7461n^3 + 87175n^2 + 377914n + 207120$$

and

$$W(n) = 1680n^2 + 23593n + 75377.$$

By using (11)-(12), we deduce that, for all integers $n \geq 8$,

$$\frac{11e}{24(n+1)(n+2)} < e - G_n < \frac{11e}{24n(n+1)}. \tag{13}$$

In fact, this inequality is true for all integers $n \geq 1$ (it can be directly verified for $n = 1, 2, \dots, 7$).

The inequalities (10) follows by summing the inequalities (13) from $n = 1$ to $n = n$. □

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