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On Subclasses of $K(G)$-algebras

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Abstract. In this paper, we introduce the class of $K(G)$-algebras as a super class of the classes of $BCH/BCI/BCK$-algebras and the class of B-algebras, when the group $(G, \cdot)$ is abelian and non-abelian respectively.

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1. Introduction

Imai and Iséki introduced two classes of logical algebras: BCK-algebras and BCI-algebras in [15, 16]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Hu and Li introduced a wide class of logical algebras: BCH-algebras in [9, 10]. They have demonstrated that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers and Kim [17] introduced the notion of B-algebras which is equivalent in some sense to the groups.
Dar and Akram introduced wider class of logical algebras on a group $(G, \cdot)$: K-algebras in [1, 2]. A K-algebra was built on the group G(briefly, $K(G)$-algebra) by using the induced binary operation $\odot$ on $(G, \cdot)$ as $x \odot y = x \cdot y^{-1} = xy^{-1}$, for all x, $y \in G$. This paper describes the classes of $BCH/BCI/BCK$-algebras as subclasses of the class of $K(G)$-algebras when $(G, \cdot)$ is an abelian group, and the class of B-algebras as a subclass of K(G)-algebras if $(G, \cdot)$ is a non-abelian group.

2. Preliminaries

In this section we review some elementary aspects that are necessary for this paper:

Definition 2.1. [1] *A $K(G)$-algebra $(G; \cdot, \odot, e)$ is an algebra defined on the group $(G, \cdot)$ in which each non-identity element is not of order 2 with the following axioms:*
(K1) $(x \odot y) \odot (x \odot z) = (x \odot (z^{-1} \odot y^{-1})) \odot x$,
(K2) $x \odot (x \odot y) = (x \odot y^{-1}) \odot x$,
(K3) $(x \odot x) = e$,
(K4) $(x \odot e) = x$,
(K5) $(e \odot x) = x^{-1}, \forall\ x, y, z \in G$.

If group $(G, \cdot)$ is an abelian , then the axioms (K1) and (K2) hold respectively as follows:
($\overline{K1}$) $(x \odot y) \odot (x \odot z) = z \odot y$.
($\overline{K2}$) $x \odot (x \odot y) = y$.

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Example 2.1. [1] *Consider the $K(G)$-algebra $(G, \cdot, \odot, e)$, where $G = \{e, a, a^2, a^3, a^4\}$ is the cyclic group of order 5 and $\odot$ is given by the following Cayley table:*

$\odot$	e	a	a^2	a^3	a^4
e	e	a^4	a^3	a^2	a
a	a	e	a^4	a^3	a^2
a^2	a^2	a	e	a^4	e^3
a^3	a^3	a^2	a	e	e^4
a^4	a^4	a^3	a^2	a	e

Example 2.2. [1] *Consider the $K(S_3)$-algebra $(S_3, \cdot, \odot, e)$ on the symmetric group $S_3 = \{e, a, b, x, y, z\}$ where $e = (1)$, $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, and $\odot$ is given by the following Cayley table:*

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

Proposition 2.1. [2] *If $K(G) = (G; \cdot, \odot, e)$ is a $K(G)$-algebra on an abelian group G, then*
(a) $(e \odot x) \odot (e \odot y) = y \odot x = e \odot (x \odot y)$,
(b) $(x \odot z) \odot (y \odot z) = x \odot y$,
(c) $e \odot (e \odot x) = x$,
(d) $x \odot (e \odot y) = y \odot (e \odot x)$,
for any $x, y, z \in G$.

Definition 2.2. [17] *A groupoid $(X; *, 0)$ with a special element 0 is called a B-algebra if , for all x, y, $z \in X$ the following axioms hold:*
(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B3) $(x * y) * z = x * (z * (0 * y))$.

In B-algebras, the following identities hold:
(B4) $0 * (0 * x) = x$. [17]
(B5) $0 * (x * y) = y * x$. [18]
(B6) $x * (x * (0 * x)) = 0 * x$. [17]
(B7) $x * (y * z) = (x * (0 * z)) * y$. [18]
(B8) $x * (z * (0 * x)) = (0 * x) * (z * x) = 0 * z$. [18]
(B9) $(x * y) * (0 * y) = x$. [17].

Definition 2.3. [17] *A B-algebras $(X; *, 0)$ is said to be commutative if, $x * (0 * y) = y * (0 * x)$, for all x, $y \in X$.*

We characterize commutative B-algebras in the following proposition:

Proposition 2.2. *In B-algebras $(X; *, 0)$, the following statements are equivalent:*
(a) *B-algebra $(X; *, 0)$ is commutative,*
(b) $x * (0 * y) = y * (0 * x)$,
(c) $x * (x * y) = y$,
(d) $(x * y) * z = (x * z) * y$,

(e) $(0 * x) * (0 * y) = 0 * (x * y)$,
(f) $(x * y) * (x * z) = z * y$,
for all $x,y,\ z \in X$.

Definition 2.4. [9] *A groupoid* $(X; *, 0)$ *with a special element 0 is called a BCH-algebra if, for all* $x,\ y,\ z \in X$, *the following axioms hold:*
(BCH1) $x * x = 0$,
(BCH2) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$,
(BCH3) $(x * y) * z = (x * z) * y$.

The following identities hold in BCH-algebras:
(BCH4) $x * 0 = x$.
(BCH5) $x * 0 = 0 \Rightarrow x = 0$.
(BCH6) $0 * (x * y) = (0 * x) * (0 * y)$.
(BCH7) $0 * (0 * (0 * x)) = 0 * x$.
(BCH8) $\{x * (x * y)\} * y = 0$.

Definition 2.5. *An algebra* $(X; *, 0)$ *of type (2, 0) is called a BCI-algebra if , for all* $x,\ y,\ z \in X$ *the following axioms hold:*
(BCI1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI2) $(x * (x * y)) * y = 0$,
(BCI3) $x * x = 0$,
(BCI4) $x * y = 0$ *and* $y * x = 0 \Rightarrow x = y$.

In BCI-algebras, the following hold:
(BCI5) $(x * y) * z = (x * z) * y$.
(BCI6) $(x * 0) = x$.
(BCI7) $(x * y) * z = x * (y * (0 * z))$.
(BCI8) $0 * (y * x) = (0 * y) * (0 * x) = x * y$.
(BCI9) $0 * (0 * x) = x$.
(BCI10) $x * (0 * y) = y * (0 * x)$.

Definition 2.6. *A* BCI*-algebra* $(X; *, 0)$ *is a BCK-algebra , if it satisfies*
(BCK) $0 * x = 0$, *for all* $x \in X$.

3. B-algebra as a subclass

It is realized in [17] that a B-algebra $(X; *, 0)$ admits the structure of a group $(X, \circ)$, where $x \circ y = x * (0 * y)$, for all $x,\ y \in X$. It is easy to remark that :
(B10) B-algebra is commutative if and only if the group $(X, \circ)$ is commutative.
(B11) B-algebra is not proper if each non-identity element of the group $(X, \circ)$ is its own inverse.
(B12) $x^{-1} = 0 * x$.
(B13) $x \circ y = x * (0 * y) = x * y^{-1}$.
(B14) 0 is the identity of the group.

Theorem 3.1. *Let* $(X; *, 0)$ *be a* B*-algebra and* $(X, \circ)$ *be the group admissible on* B*-algebra. Then* B*-algebra* $(X; *, 0)$ *is equivalent to a* $K(G)$*-algebra built on the group* $G = (X, \circ)$.

Proof. If the binary operation $'*'$ introduced on the group $(X, \circ)$ which is defined by $x * y = x \circ y^{-1}$, for all $x,\ y \in X$, then the properties $K3$, $K4$ and $K5$ of definition

2.1 are easily verified by the algebra $(X;\circ,*)$ built on the group $(X,\circ)$, $K1$ and $K2$ are verified by the B-algebra $(X;*,\circ)$ since

$$\begin{aligned}(x*y)*(x*z) &= x*((x*z)*(0*y))\\ &= x*(x*((0*y)*(0*z)) \quad [by\ B3]\\ &= x*(x*(y^{-1}\circ z)) \quad [(X,\circ)\ is\ a\ group]\\ &= (x*(0*(y^{-1}\circ z)))*x \quad [by\ B3]\\ &= (x*(y^{-1}\circ z)^{-1})*x \quad [0*x=x^{-1}]\\ &= (x*(z^{-1}\circ y))*x \quad [\circ\ is\ a\ group\ operation]\\ &= (x*(z^{-1}*y^{-1}))*x, \quad for\ all x,\ y,\ z\in X\end{aligned}$$

and

$$\begin{aligned}x*(x*y) &= (x*(0*y))*x \quad [by\ B3]\\ &= (x*y^{-1})*x, \quad for\ all\ x,\ y,\ z\ \in X.\end{aligned}$$

Thus B-algebras is the desired $K(G)$-algebra built on the group $G=(X,\circ)$. □

Corollary 3.1. *The binary operations* $'\circ'$ *and* $'*'$ *are right inverses of each other.*

4. BCH-algebra as a subclass

From the definition 2.6 and Proposition 2.2(d), it is seen that every BCH-algebras is a commutative B-subalgebra and hence admits an abelian group $(X,*)$, where $x*y=x\cdot y^{-1}=x*(0*y)=y*(0*x)$, for all $x,\ y\in X$. Thus

Theorem 4.1. *A* BCH*-algebra* $(X;*,\circ)$ *is* $K(G)$*-algebra built on an abelian group* $(X,\circ)$*, for all* $x,\ y\in X$.

Proof. Let $(X,\circ)$ be an abelian group. Then B-algebra as $K(G)$-subalgebra built on $(X,\circ)$ has to be abelian [by $B10$]. By proposition 2.2(d), it implies that $K(G)$-algebra $(X;\circ,\odot)$ is equivalent to a BCH-algebra $(X,\circ,*)$,
where $x\odot y=x\circ y^{-1}=x*(0*y)$. □

5. BCI-algebra as a subclass

It is known that a BCI-algebra admits the structure of an abelian group $(X,\cdot)$ by an induced binary operation $x\cdot y=x*(0*y)$ for all $x,\ y\in X$.
deduced by

$$x*y=x\cdot y^{-1}\quad\forall\ x,\ y\in X. \tag{1}$$

The (1)-equation admits the structure of $K(G)$-algebra [1] on the BCI-group $(X,\cdot)$. We prove that a class of BCI-algebras is equivalent to a class of $K(G)$-algebra built on an abelian BCI-group $(X,\cdot)$.

Theorem 5.1. *Let* $(X,\cdot)$ *be* BCI*-group having identity element* 0*. Then the* BCI*-algebra* $(X;*,0)$ *is equivalent to a* $K(G)$*-algebra* $(X;\odot,0)$ *built on the* BCI*-group* $(X,\cdot)$.

Proof. The properties (K3), (K4) and (K5) of $K(G)$-algebra can easily be verified from the properties (BCI3), (BCI6) and (1)-equation of BCI-algebra $(X;*,0)$. The properties $\overline{K1}$ and $\overline{K2}$ are verified in BCI-algebra since

$$\begin{aligned}
(x*y)*(x*z) &= x*(y*(0*(x*z))) \quad [by\ BCI7] \\
&= x*(y*(z*x)) \quad [by\ BCI8] \\
&= x*((y*z)*(0*x)) \quad [by\ BCI7] \\
&= (x*(y*z))*(0*(0*x)) \quad [by\ BCI7] \\
&= (x*(y*z))*x \quad [by\ BCI9] \\
&= (x*x)*(y*z) \quad [by\ BCI5] \\
&= 0*(y*z) \quad [by\ BCI3] \\
&= z*y. \quad [by\ BCI8]
\end{aligned}$$

The axiom $\overline{K2}$ can also be verified directly from $BCI2$ of definition 2.5. This ends the proof. □

Corollary 5.1. *All the classes of BCK-algebras form subclasses of $K(G)$-algebras built on abelian groups.*

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