Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 34, 2007, Pages 11–20
ISSN: 1223-6934

Fuzzy ideals of K-algebras

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Abstract. The fuzzy setting of an ideal in a K-algebra is presented and some properties are investigated. Properties of homomorphic image and inverse image of fuzzy ideals of K-algebras are discussed. A characterization theorem of fuzzy fully invariant is given. Fuzzy relations on K-algebras are discussed. A characterization theorem of fuzzy ideals in terms of the strongest fuzzy relations on K-algebras is also studied.

2000 Mathematics Subject Classification. 06F35, 03G25, 03B52.
Key words and phrases. K-algebras; Ideals; Fully invariant; Fuzzy relations; Strongest fuzzy relations.

1. Introduction

A K-algebra $(G, \cdot, \odot, e)$, introduced by Dar and Akram [4], is an algebra built on a group $(G, \cdot, e)$ with identity e and adjoined with an induced binary operation $\odot$ on G. It is non-commutative and non-associative with a right identity e. It is proved in [2, 4] that a K-algebra on an abelian group is equivalent to a p-semisimple BCI-algebra. For the convenience of study, authors renamed *a K-algebra built on a group G* as a $K(G)$-algebra [5]. The $K(G)$-algebra has been characterized by using its left and right mappings in [5]. Recently, Dar and Akram [7] have further proved that the class of $K(G)$-algebras is a generalized class of B-algebras [12] when $(G, \cdot, e)$ is a non-abelian group, and they also proved that the $K(G)$-algebra is a generalized class of the class of $BCH/BCI/BCK$-algebras [8, 9, 10] when $(G, \cdot, e)$ is an abelian group.
After the introduction of fuzzy sets by Zadeh [14], the fuzzy set theory developed by Zadeh himself and others in many directions and found applications in various areas of sciences. The study of fuzzy algebraic structures started with introduction of the concept of the fuzzy subgroup of a group in the pioneering paper of Rosenfeld [13]. Since then many researchers have been engaged in extending the concepts and results of abstract algebra to broader framework of the fuzzy setting. Akram *et al.* introduced the notions of subalgebras and fuzzy (maximal) ideals of K-algebras in [1] and further studied by Jun *et al.* in [11]. As a continuation of [1, 11], further some properties of the fuzzy ideals in a K-algebra are investigated. Properties of homomorphic image and inverse image of fuzzy ideals of K-algebras are discussed. A characterization theorem of fuzzy fully invariant is given. Fuzzy relations on a K-algebra are also discussed.

Received: February 13, 2007.
This research work of the first author is supported by PUCIT, Pakistan.
The second author is supported by HEC-Islamabad, Pakistan.

2. Preliminaries

In this section we review some elementary aspects that are necessary for this paper: Let $(G, \cdot, e)$ be a group with the identity e such that $x^2 \neq e$ for some $x(\neq e) \in G$. A *K-algebra built on G* (briefly, *K-algebra*) is a structure $\mathcal{K} = (G, \cdot, \odot, e)$, where "$\odot$" is a binary operation on G which is induced from the operation "$\cdot$", that satisfies the following:
(k1) $(\forall a, x, y \in G)$ $((a \odot x) \odot (a \odot y) = (a \odot (y^{-1} \odot x^{-1})) \odot a)$,
(k2) $(\forall a, x \in G)$ $(a \odot (a \odot x) = (a \odot x^{-1}) \odot a)$,
(k3) $(\forall a \in G)$ $(a \odot a = e)$,
(k4) $(\forall a \in G)$ $(a \odot e = a)$,
(k5) $(\forall a \in G)$ $(e \odot a = a^{-1})$.
If G is abelian, then conditions (k1) and (k2) are replaced by:
(k1′) $(\forall a, x, y \in G)$ $((a \odot x) \odot (a \odot y) = y \odot x)$,
(k2′) $(\forall a, x \in G)$ $(a \odot (a \odot x) = x)$,
respectively. A nonempty subset H of a K-algebra $\mathcal{K}$ is called a *subalgebra* of $\mathcal{K}$ if it satisfies:

- $(\forall a, b \in H)$ $(a \odot b \in H)$.

Note that every subalgebra of a K-algebra $\mathcal{K}$ contains the identity e of the group $(G, \cdot)$. A mapping $f : \mathcal{K}_1 \to \mathcal{K}_2$ of K-algebras is called a *homomorphism* if $f(x \odot y) = f(x) \odot f(y)$ for all $x, y \in \mathcal{K}_1$. Note that if f is a homomorphism, then $f(e) = e$. A nonempty subset I of a K-algebra $\mathcal{K}$ is called an *ideal* of $\mathcal{K}$ if it satisfies:
(i) $e \in I$,
(ii) $(\forall x, y \in G)$ $(x \odot y \in I,\ y \odot (y \odot x) \in I \Rightarrow x \in I)$.
Let μ be a *fuzzy set* on G, i.e., a map $\mu : G \to [0, 1]$. A fuzzy set μ in a K-algebra $\mathcal{K}$ is called a *fuzzy subalgebra* of $\mathcal{K}$ if it satisfies:

- $(\forall x, y \in G)$ $(\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\})$.

Note that every fuzzy subalgebra μ of a K-algebra $\mathcal{K}$ satisfies the following inequality:

$$(\forall x \in G)\,(\mu(e) \geq \mu(x)).$$

3. Fuzzy ideals of K-algebra

Definition 3.1. *A fuzzy set μ in a K-algebra $\mathcal{K}$ is called a fuzzy ideal of $\mathcal{K}$ if it satisfies:*
(i) $(\forall x \in G)$ $(\mu(e) \geq \mu(x))$,
(ii) $(\forall x, y \in G)$ $(\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\})$.

Example 3.1. *Consider the K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ on the cyclic group $G = \{0, a, b, c, d, f\}$, where $0 = e$, $a = a$, $b = a^2$, $c = a^3$, $d = a^4$, $f = a^5$ and $\odot$ is given by the following Cayley's table:*

$\odot$	0	a	b	c	d	f
0	0	f	d	c	b	a
a	a	0	f	d	c	b
b	b	a	0	f	d	c
c	c	b	a	0	f	d
d	d	c	b	a	0	f
f	f	d	c	b	a	0

Let μ be a fuzzy set in G defined by $\mu(e) = t_1$ and $\mu(x) = t_2$ for all $x \neq 0$ in G, where $t_1 > t_2$ in [0, 1]. Then, by routine verification, it is easy to see that μ is a fuzzy ideal of $\mathcal{K}$.

The following propositions are obvious.

Proposition 3.1. *[1] Let μ be a fuzzy set in a K-algebra $\mathcal{K}$. Then μ is a fuzzy ideal of $\mathcal{K}$ if and only if the set $U(\mu;t) := \{x \in G \mid \mu(x) \geq t\}$, $t \in [0,1]$, is an ideal of $\mathcal{K}$ when it is nonempty.*

Proposition 3.2. *Let μ be a fuzzy ideal of a K-algebra $\mathcal{K}$ and let $x \in \mathcal{K}$. Then $\mu(x) = t$ if and only if $x \in U(\mu;t)$ and $x \notin U(\mu;s)$ for all $s > t$.*

Proposition 3.3. *If μ and λ are fuzzy ideals of $\mathcal{K}$. Then $\mu \cap \lambda$ is also a fuzzy ideals of $\mathcal{K}$.*

Definition 3.2. *For a family of fuzzy sets $\{\mu_i | i \in I\}$ in a K-algebra $\mathcal{K}$, define the join $\bigvee_{i\in I} \mu_i$ and meet $\bigwedge_{i\in I} \mu_i$ of $\{\mu_i | i \in I\}$ as follows:*

$$(\bigvee_{i\in I} \mu_i)(x) = \sup\{\mu_i(x) | i \in I\},$$

$$(\bigwedge_{i\in I} \mu_i)(x) = \inf\{\mu_i(x) | i \in I\}$$

for each $x \in G$.

Theorem 3.1. *The family of fuzzy ideals of $\mathcal{K}$ is a completely distributive lattice with respect to the meet and the join.*

Proof. Let $\{\mu_i | i \in I\}$ be a family of fuzzy ideals of $\mathcal{K}$. Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\bigvee_{i\in I} \mu_i$ and $\bigwedge_{i\in I} \mu_i$ are fuzzy ideals of $\mathcal{K}$. For any $x \in \mathcal{K}$,

$$(\bigvee_{i\in I} \mu_i)(e) = \sup_{i\in I} \mu_i(e) \geq \sup_{i\in I} \mu_i(x) = (\bigvee_{i\in I} \mu_i)(x),$$

and

$$(\bigwedge_{i\in I} \mu_i)(e) = \inf_{i\in I} \mu_i(e) \geq \inf_{i\in I} \mu_i(x) = (\bigwedge_{i\in I} \mu_i)(x).$$

Let x, $y \in \mathcal{K}$. Then

$$\begin{aligned}
(\bigvee \mu_i)(x) &= \sup\{\mu_i(x) | i \in I\} \\
&\geq \sup\{\max(\mu_i(x \odot y), \mu_i(y \odot (y \odot x))) | i \in I\} \\
&= \max(\sup\{\mu_i(x \odot y) | i \in I\}, \sup\{\mu_i(y \odot (y \odot x)) | i \in I\}) \\
&= \max((\bigvee \mu_i)(x \odot y), (\bigvee \mu_i)(y \odot (y \odot x))),
\end{aligned}$$

$$\begin{aligned}
(\bigwedge \mu_i)(x) &= \inf\{\mu_i(x) | i \in I\} \\
&\geq \inf\{\min(\mu_i(x \odot y), \mu_i(y \odot (y \odot x))) | i \in I\} \\
&= \min(\inf\{\mu_i(x \odot y) | i \in I\}, \inf\{\mu_i(y \odot (y \odot x)) | i \in I\}) \\
&= \min((\bigwedge \mu_i)(x \odot y), (\bigwedge \mu_i)(y \odot (y \odot x))).
\end{aligned}$$

Hence $\bigvee_{\in I} \mu_i$ and $\bigwedge_{i\in I} \mu_i$ are fuzzy ideals of $\mathcal{K}$. □

Theorem 3.2. *If μ is a fuzzy ideal of a K-algebra $\mathcal{K}$, then for all $x \in G$*

$$\mu(x) = \sup\{t \in [0,1] \mid x \in U(\mu;t)\}.$$

Proof. Let $s := \sup\{t \in [0,1] \mid x \in U(\mu;t)\}$, and let $\epsilon > 0$. Then $s - \epsilon < t$ for some $t \in [0,1]$ such that $x \in U(\mu;t)$, and so $s - \epsilon < \mu(x)$. Since ϵ is an arbitrary, it follows that $s \le \mu(x)$. Now let $\mu(x) = v$, then $x \in U(\mu;v)$ and so $v \in \{t \in [0,1] \mid x \in U(\mu;t)\}$. Thus $\mu(x) = v \le \sup\{t \in [0,1] \mid x \in U(\mu;t)\} = s$. Hence $\mu(x) = s$. This completes the proof. □

We now consider the converse of Theorem 3.2.

Theorem 3.3. *Let Ω be a nonempty finite subset of $[0,1]$. Let $\{G_w \mid w \in \Omega\}$ be a collection of ideals of a K-algebra $\mathcal{K}$ such that*
(i) $G = \bigcup_{w\in\Omega} G_w$,
(ii) $\alpha > \beta$ *if and only if* $G_\alpha \subset G_\beta$ *for all* $\alpha,\ \beta \in \Omega$.
Then a fuzzy set μ in G defined by

$$\mu(x) = \sup\{w \in \Omega \mid x \in G_w\}$$

is a fuzzy ideal of $\mathcal{K}$.

Proof. In view of Proposition 3.1, it is sufficient to show that every nonempty level set $U(\mu;\alpha)$ is an ideal of $\mathcal{K}$. Assume $U(\mu;\alpha) \ne \alpha$ for some $\alpha \in [0;1]$. Then the following cases arise:

$$1.\ \alpha = \sup\{\beta \in \Omega \mid \beta < \alpha\} = \sup\{\beta \in \Omega \mid G_\alpha \subset G_\beta\},$$
$$2.\ \alpha \ne \sup\{\beta \in \Omega \mid \beta < \alpha\} = \sup\{\beta \in \Omega \mid G_\alpha \subset G_\beta\}.$$

Case(1) implies that

$$\begin{aligned} x \in U(\mu;\alpha) \quad &\Leftrightarrow \quad x \in G_w\ \forall\ w < \alpha \\ &\Leftrightarrow \quad x \in \bigcap_{w<\alpha} G_w. \end{aligned}$$

Hence $U(\mu;\alpha) = \bigcap_{w<\alpha} G_w$, which is an ideal of $\mathcal{K}$.
For case (2), there exists $\epsilon > 0$ such that $(\alpha-\epsilon,\alpha)\bigcap\Omega = \emptyset$. We claim that in this case $U(\mu;\alpha) = \bigcup_{\beta\ge\alpha} G_\beta$. Indeed, if $x \in \bigcup_{\beta\ge\alpha} G_\beta$, then $x \in G_\beta$ for some $\beta \ge \alpha$, which gives $\mu(x) \ge \beta \ge \alpha$. Thus $x \in U(\mu;\alpha)$, i.e., $\bigcup_{\beta\ge\alpha} G_\beta \subseteq U(\mu;\alpha)$. On the other hand, if $x \notin \bigcup_{\beta\ge\alpha} G_\beta$, then $x \notin G_\beta$ for all $\beta \ge \alpha$, which implies that $x \notin G_\beta$ for all $\beta > \alpha-\epsilon$, i.e., if $x \in G_\beta$ then $\beta \le \alpha - \epsilon$. Thus $\mu(x) \le \alpha - \epsilon$. So $x \notin U(\mu;\alpha)$. Thus $U(\mu;\alpha) \subseteq \bigcup_{\beta\ge\alpha} G_\beta$. Hence $U(\mu;\alpha) = \bigcup_{\beta\ge\alpha} G_\beta$, which is an ideal of $\mathcal{K}$. This completes the proof. □

Theorem 3.4. *Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If ν is a fuzzy ideal of $\mathcal{K}_2$ and μ is the pre-image of ν under f. Then μ is a fuzzy ideal of $\mathcal{K}_1$.*

Proof. It is easy to see that $\mu(e) \ge \mu(x)$ for all $x \in \mathcal{K}_1$. For any $x,\ y \in \mathcal{K}_1$,

$$\begin{aligned} \mu(x) &= \nu(f(x)) \ge \min(\nu(f(x\odot y)), \nu(f(y\odot(y\odot x)))) \\ &= \min(\mu(x\odot y), \mu(y\odot(y\odot x))). \end{aligned}$$

Hence μ is a fuzzy ideal of $\mathcal{K}_1$. □

Definition 3.3. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two K-algebras and let f be a function from $\mathcal{K}_1$ into $\mathcal{K}_2$. If ν is a fuzzy set in $\mathcal{K}_2$, then the preimage of ν under f is the fuzzy set in $\mathcal{K}_1$ defined by*

$$f^{-1}(\nu)(x) = \nu(f(x)) \quad \forall \ \ x \in G.$$

Theorem 3.5. *Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If ν is a fuzzy ideal in $\mathcal{K}_2$, then $f^{-1}(\nu)$ is a fuzzy ideal in $\mathcal{K}_1$.*

Proof. It is easy to see that $f^{-1}(\nu)(e) \geq f^{-1}(\nu)(x)$ for all $x \in \mathcal{K}$. Let x, $y \in \mathcal{K}$, then

$$\begin{aligned} f^{-1}(\nu)(x) &= \nu(f(x)) \\ &\geq \min(\nu(f(x \odot y), f(y \odot (y \odot x))) \\ &= \min(\nu(f(x \odot y)), \nu(f(y \odot (y \odot x)))) \\ &= \min(f^{-1}(\nu)(x \odot y), f^{-1}(\nu)(y \odot (y \odot x))). \end{aligned}$$

Hence $f^{-1}(\nu)$ is a fuzzy ideal in $\mathcal{K}_1$. □

Definition 3.4. *Let a mapping $f : \mathcal{K}_1 \to \mathcal{K}_2$ from $\mathcal{K}_1$ into $\mathcal{K}_2$ of K-algebras and let μ be a fuzzy set of $\mathcal{K}_2$. The map μ^f is called the pre-image of μ under f, if $\mu^f(x) = \mu(f(x))$, for all $x \in G$.*

Theorem 3.6. *Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If μ is a fuzzy ideal of $\mathcal{K}_2$, then μ^f is a fuzzy ideal of $\mathcal{K}_1$.*

Proof. For any $x \in \mathcal{K}$, we have $\mu^f(e_1) = \mu(f(e_1)) = \mu(e_2) \geq \mu(f(x)) = \mu^f(x)$. For any $x, y \in \mathcal{K}$, since μ is a fuzzy ideal of $\mathcal{K}_1$,

$$\begin{aligned} \mu^f(x) &= \mu(f(x)) \\ &\geq \min\{\mu(f(x \odot y)), \mu(f(y) \odot f(y \odot x))\} \\ &= \min\{\mu(f((x \odot y)), \mu(f(y \odot (y \odot x)))\} \\ &= \min\{\mu^f((x \odot y)), \mu^f(y \odot (y \odot x))\} \end{aligned}$$

proving that μ^f is a fuzzy ideal of $\mathcal{K}_1$. □

Theorem 3.7. *Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras. If μ^f is a fuzzy ideal of $\mathcal{K}_2$, then μ is a fuzzy ideal of $\mathcal{K}_1$.*

Proof. Since there exists $x \in \mathcal{K}_1$ such that $y = f(x)$ for any $y \in \mathcal{K}_2$, $\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(e_1) = \mu(f(e_1)) = \mu(e_2)$.
For any $x, y \in \mathcal{K}_2$, there exist $a, b, c \in \mathcal{K}_1$ such that $x = f(a)$ and $y = f(b)$. It follows that

$$\begin{aligned} \mu(x) &= \mu(f(a)) \\ &= \mu^f(a) \\ &\geq \min\{\mu^f((a \odot b)), \mu^f(b \odot (b \odot a))\} \\ &= \min\{\mu(f(a \odot b)), \mu(f(b \odot (b \odot a)))\} \\ &= \min\{\mu((f(a) \odot f(b))), \mu(f(b) \odot (f(b) \odot f(a)))\} \\ &= \min\{\mu((x \odot y)), \mu(y \odot (y \odot x))\} \end{aligned}$$

proving that μ is a fuzzy ideal of $\mathcal{K}_1$. □

Definition 3.5. *An ideal H of K-algebra is said to be fully invariant if $f(H) \subseteq H$, for all $f \in End(\mathcal{K})$, where $End(\mathcal{K})$ is the set of all endomorphisms of a K-algebra $\mathcal{K}$. A fuzzy ideal μ of a K-algebra $\mathcal{K}$ is called a fuzzy fully invariant if $\mu^f(x) \leq \mu(x)$ for all $x \in G$ and $f \in End(\mathcal{K})$.*

Theorem 3.8. *A fuzzy ideal is fully invariant if and only if each its level set is a fully invariant ideal.*

Proof. Suppose that μ is fuzzy fully invariant and let $t \in Im(\mu)$, $f \in End(\mathcal{K})$ and $x \in U(\mu, t)$. Then

$$\begin{array}{rcl} \mu^f(x) & \leq & \mu(x) \geq t \\ & \Rightarrow & \mu(f(x)) \geq t \\ & \Rightarrow & f(x) \in U(\mu; t). \end{array}$$

Thus $f(U(\mu; t)) \subseteq U(\mu; t)$, i.e., $U(\mu; t)$ is fully invariant.
Conversely, suppose that each level ideal of μ is fully invariant and let $x \in G$, $f \in End(\mathcal{K})$ and $\mu(x) = t$. Then, by virtue of Proposition 3.2 , $x \in U(\mu; t)$ and $x \notin U(\mu; s)$, for all $s > t$. It follows from the assumption that $f(x) \in f(U(\mu; t)) \leq U(\mu; t)$, so that $\mu^f(x) \leq \mu(x) \geq t$. Let $s = \mu^f(x)$ and assume that $s > t$. Then $f(x) \in U(\mu; s) = f(U(\mu; s))$, which implies from the injectivity of f that $x \in U(\mu; s)$, a contradiction. Hence $\mu^f(x) = \mu(f(x)) \leq \mu(x) = t$ showing that μ is fuzzy fully invariant. □

Definition 3.6. *Let μ be a fuzzy ideal of K-algebra $\mathcal{K}$ and $x \in \mathcal{K}$. The fuzzy subset μ^*_x of $\mathcal{K}$ defined by*

$$\mu^*_x(a) = \mu(a \odot x) \quad \forall a \in \mathcal{K}$$

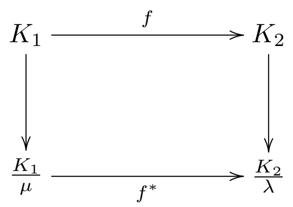
is termed as the fuzzy coset determined by x and μ.

Proposition 3.4. *Let μ be a fuzzy ideal of $\mathcal{K}$. Then $\mathcal{K}/\mu$, the set of all fuzzy cosets of μ in $\mathcal{K}$, is a K-algebra under the following operation:*

$$\mu^*_x \odot \mu^*_y = \mu^*_{x \odot y} \quad \forall x, y \in \mathcal{K}.$$

Proof. Straightforward. □

Theorem 3.9. *Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be homomorphism of K-algebras and let μ be fuzzy ideal of $\mathcal{K}_1$ and λ of $\mathcal{K}_2$ such that $f(\mu) \subseteq \lambda$, there is a homomorphism of K-algebras $f^* : \mathcal{K}_1/\mu \to \mathcal{K}_2/\lambda$ where $f^*(\mu^*_x) = \lambda^*_{f(x)}$ such that the following diagram commutes.*



.

Proof. Let $\mu^*_x = \mu^*_y$ then $\mu(x \odot y) = \mu(e)$. Thus

$$\begin{array}{rcl} \lambda(f(x) \odot f(y)) & = & \lambda(f(x \odot y)) = f^{-1}(\lambda)(x \odot y) \\ & \geq & \mu(x \odot y) = \mu(e). \end{array}$$

That is $\lambda(f(x)) = \lambda(f(y))$. Hence f^* is well-defined. Finally, f^* is homomorphism because

$$\begin{array}{rcl} f^*(\mu^*_x \odot \mu^*_y) & = & f^*(\mu^*_{x \odot y}) = \lambda^*_{f(x) \odot f(y)} \\ & = & \lambda^*_{f(x)} \odot \lambda^*_{f(y)} = f^*(\mu^*_x) \odot f^*(\lambda^*_{f(y)}). \end{array}$$

This completes the proof. □

4. Cartesian product of fuzzy ideals

Definition 4.1. [3] *A fuzzy relation on any set G is a fuzzy set $\mu : G \times G \to [0,1]$.*

Definition 4.2. [3] *If μ is a fuzzy relation on a set G and λ is a fuzzy set in G, then μ is a fuzzy relation on λ if*

$$\mu(x,y) \leq \min\{\lambda(x), \lambda(y)\} \quad \forall\ x,\ y \quad G.$$

Definition 4.3. [3] *Let μ and λ be the fuzzy sets in a set G. The cartesian product of μ and λ is defined by $(\mu \times \lambda)(x,y) = \min\{\mu(x), \lambda(x)\}$, $\forall\ x,\ y \in G$.*

Lemma 4.1. [3] *Let μ and λ be fuzzy sets in a set G. Then*
(i) *$\mu \times \lambda$ is a fuzzy relation on G,*
(ii) *$U(\mu \times \lambda; t) = U(\mu; t) \times U(\lambda; t)$ for all $t \in [0,1]$.*

Definition 4.4. [3] *Let λ be a fuzzy set in a set G, the strongest fuzzy relation on G that is fuzzy relation on λ is μ_λ, given by $\mu_\lambda(x,y) = \min\{\lambda(x), \lambda(y)\}$ for all x, y $\in G$.*

Lemma 4.2. [3] *For a given fuzzy set λ in a set G, let μ_λ be the strongest fuzzy relation on G. Then $U(\mu_\lambda; t) = U(\lambda; t) \times U(\lambda; t)$ for $t \in [0,1]$.*

Proposition 4.1. *For a given fuzzy set λ in a set G, let μ_λ be the strongest fuzzy relation on G. If μ_λ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$, then $\lambda(x) \leq \lambda(e)$ for all $x \in \mathcal{K}$.*

Proof. If μ_λ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$, then $\mu_\lambda(x,x) \leq \mu_\lambda(e,e)$ for all $x \in \mathcal{K}$. This means that
$\min\{\lambda(x), \lambda(x)\} \leq \min\{\lambda(e), \lambda(e)\}$, which implies that $\lambda(x) \leq \lambda(e)$. □

Proposition 4.2. *If μ_λ is a fuzzy ideal of a K-algebra $\mathcal{K}$, then the level ideals of μ_λ are given by $U(\mu_\lambda; t) = U(\lambda; t) \times U(\lambda; t)$ for all $t \in [0,1]$.*

Proof. Follows immediately from Lemma 4.2. □

Theorem 4.1. *If μ and λ are fuzzy ideals of a K-algebra $\mathcal{K}$. Then $\mu \times \lambda$ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$.*

Proof. For any $(x,y) \in \mathcal{K} \times \mathcal{K}$, we have

$$(\mu \times \lambda)(e,e) = \min\{\mu(e), \lambda(e)\} \geq \min\{\mu(x), \lambda(y)\} = (\mu \times \lambda)(x,y).$$

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathcal{K} \times \mathcal{K}$. Then

$$\begin{aligned}
(\mu \times \lambda)(x) &= (\mu \times \lambda)((x_1, x_2)) = \min\{\mu(x_1), \lambda(x_2)\} \\
&\geq \min\{\min\{\mu(x_1 \odot y_1), \mu(y_1 \odot (y_1 \odot x_1))\} \\
&, \ \min\{\lambda(x_2 \odot y_2), \lambda(y_2 \odot (y_2 \odot x_2))\}\} \\
&= \min\{\min\{\mu(x_1 \odot y_1), \lambda(x_2 \odot y_2)\} \\
&, \ \min\{\mu(y_1 \odot (y_1 \odot x_1)), \lambda(y_2 \odot (y_2 \odot x_2))\}\} \\
&= \min\{(\mu \times \lambda)((x_1 \odot y_1, x_2 \odot y_2)) \\
&, \ (\mu \times \lambda)((y_1 \odot (y_1 \odot x_1)), y_2 \odot (y_2 \odot x_2))\} \\
&= \min\{(\mu \times \lambda)((x_1, x_2) \odot (y_1, y_2)) \\
&, \ (\mu \times \lambda)((y_1, y_2) \odot ((y_1, y_2) \odot (x_1, x_2)))\} \\
&= \min\{(\mu \times \lambda)(x \odot y), (\mu \times \lambda)(y \odot (y \odot x))\}.
\end{aligned}$$

Hence $\mu \times \lambda$ is a fuzzy K-ideal of $\mathcal{K} \times \mathcal{K}$. □

The converse of Theorem 4.1 may not be true as seen in the following example.

Example 4.1. *Let $\mathcal{K}$ be a K-algebra and let $s, t \in [0,1)$ such that $s \leq t$. Define fuzzy sets μ_1 and μ_2 in $\mathcal{K}$ by $\mu_1(x) = s$ and*

$$\mu_2(x) = \begin{cases} t & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases}$$

for all $x \in \mathcal{K}$, respectively.
If $x \neq 0$, then $\mu_2(x) = 1$, and thus

$$(\mu_1 \times \mu_2)(x,x) = \min(\mu_1(x), \mu_2(x)) = \min(s,1) = s.$$

If $x = 0$, then $\mu_2(x) = t < 1$, and thus

$$(\mu_1 \times \mu_2)(x,x) = \min(\mu_1(x), \mu_2(x)) = \min(s,t) = s.$$

That is, $\mu_1 \times \mu_2$ is a constant function and so $\mu_1 \times \mu_2$ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$. Now μ_1 is a fuzzy ideal of $\mathcal{K}$, but μ_2 is not a fuzzy ideal of $\mathcal{K}$ since for $x \neq 0$, we have $\mu_2(0) = t < 1 = \mu_2(x)$.

Theorem 4.2. *Let μ and λ be fuzzy sets in a K-algebra $\mathcal{K}$ such that $\mu \times \lambda$ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$, then*
(i) *Either $\mu(e) \geq \mu(x)$ or $\lambda(e) \geq \lambda(x)$, $\forall x \in \mathcal{K}$.*
(ii) *If $\mu(e) \geq \mu(x)$, $\forall x \in \mathcal{K}$, then either $\lambda(e) \geq \mu(x)$ or $\lambda(e) \geq \lambda(x)$.*
(iii) *If $\lambda(e) \geq \lambda(x)$, $\forall x \in \mathcal{K}$, then either $\mu(e) \geq \mu(x)$ or $\mu(e) \geq \lambda(x)$.*
(iv) *If $\mu(x) \leq \mu(e)$ for all $x \in \mathcal{K}$ and $\lambda(y) > \mu(e)$ for some $y \in \mathcal{K}$, then μ is a fuzzy ideal ideal of $\mathcal{K}$.*
(v) *If $\lambda(x) \leq \mu(e)$ for any $x \in \mathcal{K}$, then λ is a fuzzy ideal of $\mathcal{K}$.*

Proof. (i) We prove it using reductio ad absurdum.
Assume $\mu(x) > \mu(e)$ and $\lambda(y) > \lambda(e)$, for some x, y $\in \mathcal{K}$. Then

$$\begin{aligned} (\mu \times \lambda)(x,y) &= \min\{\mu(x), \lambda(y)\} > \min\{\mu(e), \lambda(e)\} = (\mu \times \lambda)(e,e) \\ &\Rightarrow (\mu \times \lambda)(x,y) > (\mu \times \lambda)(e,e), \forall\, x,\ y \in \mathcal{K}, \end{aligned}$$

which is a contradiction. Hence (i) is proved.

(ii) Again, we use reduction to absurdity.
Assume $\lambda(e) < \mu(x)$ and $\lambda(e) < \lambda(y)$, $\forall\, x,\ y \in \mathcal{K}$. Then,

$$(\mu \times \lambda)(e,e) = \min\{\mu(e), \lambda(e)\} = \lambda(e),$$

$$\begin{aligned} &(\mu \times \lambda)(x,y) = \min\{\mu(x), \lambda(y)\} > \lambda(e) = (\mu \times \lambda)(e,e) \\ &\Rightarrow (\mu \times \lambda)(x,y) > (\mu \times \lambda)(e,e), \end{aligned}$$

which is a contradiction. Hence (ii) is proved.

(iii) The proof is similar to (ii).

(iv) Assume that $\mu(x) \leq \mu(e)$ for all $x \in \mathcal{K}$ and $\lambda(y) > \mu(e)$ for some $y \in \mathcal{K}$. Then $\lambda(e) \geq \lambda(y) > \mu(e)$. Since $\mu(e) \geq \mu(x)$ for all $x \in \mathcal{K}$, it follows that $\lambda(e) > \mu(x)$ for all $x \in \mathcal{K}$. So

$$(\mu \times \lambda)(x,e) = \min\{\mu(x), \lambda(x)\} = \mu(x) \quad \forall\, x \in \mathcal{K}.$$

Thus

$$\begin{aligned} \mu(x) &= (\mu \times \lambda)(x,e) \geq \min\{(\mu \times \lambda)(x \odot y, e), (\mu \times \lambda)(y \odot (y \odot x), e)\} \\ &= \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\}. \end{aligned}$$

Hence μ is a fuzzy ideal of $\mathcal{K}$.

(v) If $\lambda(x) \leq \mu(x)$ for any $x \in \mathcal{K}$, then

$$\begin{aligned} \lambda(x) &= \min\{\mu(e), \lambda(x)\} = (\mu \times \lambda)(e, x) \\ &\geq \min\{(\mu \times \lambda)(e, x \odot y), (\mu \times \lambda)(e, y \odot (y \odot x))\} \\ &= \min\{\min\{\mu(e), \lambda(x \odot y)\}, \min\{\mu(e), \lambda(y \odot (y \odot x))\}\} \\ &= \min\{\lambda(x \odot y), \lambda(y \odot (y \odot x))\}. \end{aligned}$$

Hence λ is a fuzzy ideal of $\mathcal{K}$.

□

The counter example 4.1 shows: if $\mu \times \lambda$ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$, then μ and λ may not be both fuzzy ideals of $\mathcal{K}$. Now we give the partial converse of the Theorem 4.1 is the following Theorem.

Theorem 4.3. *If $\mu \times \lambda$ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$, then either μ or λ is a fuzzy ideal of $\mathcal{K}$.*

Proof. By Theorem 4.2(i), without loss of generality we assume that $\lambda(e) \geq \lambda(x)$, $\forall$ $x \in \mathcal{K}$.
It follows from Theorem 4.2(iii) that either $\mu(e) \geq \mu(x)$ or $\mu(e) \geq \lambda(x)$. If $\mu(e) \geq \lambda(x)$, $\forall$ $x \in \mathcal{K}$. Then

$$(\mu \times \lambda)(e, x) = \min\{\mu(e), \lambda(x)\} = \lambda(x) \cdots \qquad (I)$$

Since $\mu \times \lambda$ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$,
$(\mu\times\lambda)(x_1, x_2) \geq \min\{(\mu\times\lambda)((x_1, x_2)\odot(y_1, y_2)), (\mu\times\lambda)((y_1, y_2)\odot((y_1, y_2)\odot(x_1, x_2)))\}$.
It implies that
$(\mu\times\lambda)(x_1, x_2) \geq \min\{(\mu\times\lambda)((x_1\odot y_1, x_2\odot y_2)), (\mu\times\lambda)(y_1\odot(y_1\odot x_1), y_2\odot(y_2\odot x_2))\}$.
Putting $x_1 = y_1 = e$ gives

$$(\mu \times \lambda)(e, x_2) \geq \min\{(\mu \times \lambda)((e, x_2 \odot y_2)), (\mu \times \lambda)(e, y_2 \odot (y_2 \odot x_2))\}.$$

Using equation(I), we have

$$\lambda(x_2) \geq \min\{\lambda(x_2 \odot y_2), \lambda(y_2 \odot (y_2 \odot x_2))\}.$$

This proves that λ is a fuzzy ideal of $\mathcal{K}$. The second part is similar. This completes the proof. □

Theorem 4.4. *Let λ be a fuzzy set in a K-algebra $\mathcal{K}$ and μ_λ be the strongest fuzzy relation on $\mathcal{K}$. Then λ is a fuzzy ideal of $\mathcal{K}$ if and only if μ_λ is a fuzzy ideal of $\mathcal{K}\times\mathcal{K}$.*

Proof. Suppose that λ is a fuzzy ideal of $\mathcal{K}$. Then

$$\mu_\lambda(e, e) = \min\{\lambda(e), \lambda(e)\} \geq \min\{\lambda(x), \lambda(y)\} = \mu_\lambda(x, y) \quad \forall\ (x, y) \in \mathcal{K} \times \mathcal{K}.$$

For any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathcal{K} \times \mathcal{K}$,

$$\begin{aligned} \mu_\lambda(x) &= \mu_\lambda(x_1, x_2) = \min\{\lambda(x_1), \lambda(x_2)\} \\ &\geq \min\{\min\{\lambda(x_1 \odot y_1), \lambda(y_1 \odot (y_1 \odot x_1))\} \\ &, \min\{\lambda(x_2 \odot y_2), \lambda(y_2 \odot (y_2 \odot x_2))\}\} \\ &= \min\{\min\{\lambda(x_1 \odot y_1), \lambda(x_2 \odot y_2)\} \\ &, \min\{\lambda(y_1 \odot (y_1 \odot x_1)), \lambda(y_2 \odot (y_2 \odot x_2))\}\} \\ &= \min\{\mu_\lambda(x_1 \odot y_1, x_2 \odot y_2), \mu_\lambda(y_1 \odot (y_1 \odot x_1), y_2 \odot (y_2 \odot x_2))\} \\ &= \min\{\mu_\lambda((x_1, x_2) \odot (y_1, y_2)), \mu_\lambda((y_1, y_2) \odot ((y_1, y_2) \odot (x_1, x_2)))\} \\ &= \min\{\mu_\lambda(x \odot y), \mu_\lambda(y \odot (y \odot x))\}. \end{aligned}$$

Hence μ_λ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$.
Conversely, suppose that μ_λ is a fuzzy ideal of $\mathcal{K} \times \mathcal{K}$. Then

$$\min\{\lambda(e), \lambda(e)\} = \mu_\lambda(e,e) \geq \mu_\lambda(x,y) = \min\{\lambda(x), \lambda(y)\} \quad \forall (x,y) \in \mathcal{K} \times \mathcal{K}.$$

It follows that $\lambda(x) \leq \lambda(e)$, $\forall x \in \mathcal{K}$. For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathcal{K} \times \mathcal{K}$,

$$\begin{aligned}
\min\{\lambda(x_1), \lambda(x_2)\} &= \mu_\lambda(x_1, x_2) \\
&\geq \min\{\mu_\lambda((x_1,x_2) \odot (y_1,y_2)), \mu_\lambda((y_1,y_2) \odot ((y_1,y_2) \odot (x_1,x_2))\} \\
&= \min\{\mu_\lambda(x_1 \odot y_1, x_2 \odot y_2), \mu_\lambda(y_1 \odot (y_1 \odot x_1), y_2 \odot (y_2 \odot x_2))\} \\
&= \min\{\min\{\lambda(x_1 \odot y_1), \lambda(x_2 \odot y_2)\} \\
&, \quad \min\{\lambda(y_1 \odot (y_1 \odot x_1)), \lambda(y_2 \odot (y_2 \odot x_2))\}\} \\
&= \min\{\min\{\lambda(x_1 \odot y_1), \lambda(y_1 \odot (y_1 \odot x_1))\} \\
&, \quad \min\{\lambda(x_2 \odot y_2), \lambda(y_2 \odot (y_2 \odot x_2))\}\}.
\end{aligned}$$

Putting $x_2 = y_2 = e$ gives
$\lambda(x_1) \geq \min\{\lambda(x_1 \odot y_1), \lambda(y_1 \odot (y_1 \odot x_1))\}$. Hence λ is a fuzzy ideal of $\mathcal{K}$. □

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