Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 34, 2007, Pages 21–28
ISSN: 1223-6934

Valuations on residuated lattices

Cătălin Buşneag

Abstract. The aim of this paper is to introduce the notions of pseudo-valuation (valuation) on residuated lattices and to prove some theorems of extension for these (using the model of Hilbert algebra (see [5])).

2000 Mathematics Subject Classification. 03G10, 06B20, 08A72, 03G25.
Key words and phrases. Residuated lattice, pseudo-valuation, valuation.

1. Preliminaries

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([13]), Dilworth ([7]), Ward and Dilworth ([18]), Ward ([17]), Balbes and Dwinger ([1]) and Pavelka ([15]).

In [9], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: *BCK- latices* in [8], *full BCK- algebras* in [13], FL_{ew}*- algebras* in [14], and *integral, residuated, commutative l-monoids* in [3].

Definition 1.1. *A* residuated lattice *(*[2]*,* [16]*) is an algebra* $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *of type (2,2,2,2,0,0) equipped with an order* $\leq$ *satisfying the following:*

(LR_1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice,*

(LR_2) $(A, \odot, 1)$ *is a commutative ordered monoid,*

(LR_3) $\odot$ *and* $\rightarrow$ *form an adjoint pair, i.e.* $c \leq a \rightarrow b$ *iff* $a \odot c \leq b$ *for all* $a, b, c \in A$.

The relations between the pair of operations $\odot$ and $\rightarrow$ expressed by Definition 1.1 (LR_3), is a particular case of the *law of residuation* ([2]). Namely, let A and B two posets, and $f : A \rightarrow B$ a map. Then f is called *residuated* if there is a map $g : B \rightarrow A$, such that for any $a \in A$ and $b \in B$, we have $f(a) \leq b$ iff $b \leq g(a)$ (this is also expressed by saying that the pair (f, g) is a *residuated pair*).

Now setting A a residuated lattice, $B = A$, and defining, for any $a \in A$, two maps $f_a, g_a : A \rightarrow A, f_a(x) = x \odot a$ and $g_a(x) = a \rightarrow x$, for any $x \in A$, we see that $x \odot a = f_a(x) \leq y$ iff $x \leq g_a(y) = a \rightarrow y$ for every $x, y \in A$, that is, for every $a \in A$, (f_a, g_a) is a pair of residuation.

The symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and logical equivalence.

Proposition 1.1. *(*[9]*) The class* $\mathcal{RL}$ *of residuated lattices is equational.*

Example 1.1. *Let* p *be a fixed natural number and* $A = [0, 1]$ *the real unit interval. If for* $x, y \in A$*, we define* $x \odot y = 1 - \min\{1, [(1-x)^p + (1-y)^p]^{1/p}\}$ *and* $x \rightarrow y = \sup\{z \in [0, 1] : x \odot z \leq y\}$*, then* $(A, \max, \min, \odot, \rightarrow, 0, 1)$ *is a residuated lattice.*

Received: February 13, 2007.

Example 1.2. *If we preserve the notation from Example 1, and we define for* $x, y \in A$, $x \odot y = (\max\{0, x^p + y^p - 1\})^{1/p}$ *and* $x \to y = \min\{1, (1 - x^p + y^p)^{1/p}\}$, *then* $(A, \max, \min, \odot, \to, 0, 1)$ *become a residuated lattice called generalized Łukasiewicz structure. For* $p = 1$ *we obtain the notion of Łukasiewicz structure* ($x \odot y = \max\{0, x + y - 1\}, x \to y = \min\{1, 1 - x + y\}$).

Example 1.3. *If on* $A = [0, 1]$, *for* $x, y \in A$ *we define* $x \odot y = \min\{x, y\}$ *and* $x \to y = 1$ *if* $x \leq y$ *and* y *otherwise, then* $(A, \max, \min, \odot, \to, 0, 1)$ *is a residuated lattice (called Gődel structure).*

Example 1.4. *If consider on* $A = [0, 1]$, $\odot$ *to be the usual multiplication of real numbers and for* $x, y \in A, x \to y = 1$ *if* $x \leq y$ *and* y/x *otherwise, then* $(A, \max, \min, \odot, \to, 0, 1)$ *is a residuated lattice (called Products structure or Gaines structure).*

Example 1.5. *If* $(A, \vee, \wedge, ', 0, 1)$ *is a Boolean algebra, then if we define for* $x, y \in A, x \odot y = x \wedge y$ *and* $x \to y = x' \vee y$, *then* $(A, \vee, \wedge, \odot, \to, 0, 1)$ *become a residuated lattice.*

Definition 1.2. *(*[16]*) A residuated lattice* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *is called BL-algebra, if the following two identities hold in* A:

(BL_1) $x \odot (x \to y) = x \wedge y$;

(BL_2) $(x \to y) \vee (y \to x) = 1$.

Remark 1.1. *Łukasiewicz structure, Gődel structure and Product structure are* $BL-$ *algebras. Not every residuated lattice, however, is a BL-algebra (see* [16], *p.16).*

Remark 1.2. *If in a* $BL-$ *algebra* A, $x^{**} = x$ *for all* $x \in A$, *and for* $x, y \in A$ *we denote* $x \oplus y = (x^* \odot y^*)^*$ *then we obtain an algebra* $(A, \oplus, ^*, 0)$ *of type* $(2, 1, 0)$ *called* $MV-$ *algebras (see* [16]*).*

Remark 1.3. *(*[16]*) A residuated lattice* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *is an* MV*-algebra iff it satisfies an additional condition:* $(x \to y) \to y = (y \to x) \to x$, *for any* $x, y \in A$.

In what follows by A *we denote a residuated lattice*; for $x \in A$ and a natural number n, we define $x^* = x \to 0, (x^*)^* = x^{**}$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

Theorem 1.1. *(*[12], [16]*) Let* $x, x_1, x_2, y, y_1, y_2, z \in A$. *Then we have the following rules of calculus:*

(c_1) $1 \to x = x, x \to x = 1, y \leq x \to y, x \to 1 = 1, 0 \to x = 1$;

(c_2) $x \odot y \leq x, y$, *hence* $x \odot y \leq x \wedge y$ *and* $x \odot 0 = 0$;

(c_3) $x \odot y \leq x \to y$;

(c_4) $x \leq y$ *iff* $x \to y = 1$;

(c_5) $x \to y = y \to x = 1 \Leftrightarrow x = y$;

(c_6) $x \odot (x \to y) \leq y, x \leq (x \to y) \to y, ((x \to y) \to y) \to y = x \to y$;

(c_7) $x \odot (y \to z) \leq y \to (x \odot z) \leq (x \odot y) \to (x \odot z)$;

(c_8) $x \to y \leq (x \odot z) \to (y \odot z); x \to y \leq (x \wedge z) \to (y \wedge z); x \to y \leq (x \vee z) \to (y \vee z)$;

(c_9) $x \leq y$ *implies* $x \odot z \leq y \odot z$;

(c_{10}) $x \to y \leq (z \to x) \to (z \to y)$;

(c_{11}) $x \to y \leq (y \to z) \to (x \to z)$;

(c_{12}) $x \leq y$ *implies* $z \to x \leq z \to y, y \to z \leq x \to z$ *and* $y^* \leq x^*$,

(c_{13}) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(c_{14}) $x_1 \to y_1 \leq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$.

(c_{15}) $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \leq y^*$;

(c_{16}) $x \leq x^{**}, x^{**} \leq x^* \to x$;

(c_{17}) $1^* = 0$, $0^* = 1$;
(c_{18}) $x \to y \leq y^* \to x^*$;
(c_{19}) $x^{***} = x^*, (x \odot y)^* = x \to y^* = y \to x^* = x^{**} \to y^*$.

Theorem 1.2. *([12], [16]) If A is a complete residuated lattice, $x \in A$ and $(y_i)_{i\in I}$ a family of elements of A, then :*
(c_{20}) $x \odot (\bigvee\limits_{i\in I} y_i) = \bigvee\limits_{i\in I} (x \odot y_i)$;
(c_{21}) $x \odot (\bigwedge\limits_{i\in I} y_i) \leq \bigwedge\limits_{i\in I} (x \odot y_i)$;
(c_{22}) $x \to (\bigwedge\limits_{i\in I} y_i) = \bigwedge\limits_{i\in I} (x \to y_i)$;
(c_{23}) $(\bigvee\limits_{i\in I} y_i) \to x = \bigwedge\limits_{i\in I} (y_i \to x)$;
(c_{24}) $\bigvee\limits_{i\in I} (y_i \to x) \leq (\bigwedge\limits_{i\in I} y_i) \to x$;
(c_{25}) $\bigvee\limits_{i\in I}(x \to y_i) \leq x \to (\bigvee\limits_{i\in I} y_i)$;
(c_{26}) $(\bigvee\limits_{i\in I} y_i)^* = \bigwedge\limits_{i\in I} y_i^*$;
(c_{27}) $(\bigwedge\limits_{i\in I} y_i)^* \geq \bigvee\limits_{i\in I} y_i^*$.

Corollary 1.1. *([6]) If $x, x', y, y', z \in A$ then:*
(c_{28}) $x \vee y = 1$ *implies* $x \odot y = x \wedge y$;
(c_{29}) $x \to (y \to z) \geq (x \to y) \to (x \to z)$;
(c_{30}) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$, *hence* $x^m \vee y^n \geq (x \vee y)^{mn}$, *for any* m, n *natural numbers;*
(c_{31}) $(x \to y) \odot (x' \to y') \leq (x \vee x') \to (y \vee y')$;
(c_{32}) $(x \to y) \odot (x' \to y') \leq (x \wedge x') \to (y \wedge y')$.

2. Boolean center and deductive systems of a residuated lattice

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Recall that an element $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such element b exists it is called a *complement* of a. We will denote $b = a'$ and the set of all complemented elements in L by $B(L)$. Complements are generally not unique, unless the lattice is distributive.

In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.

Lemma 2.1. *([12]) Suppose that $a \in A$ have a complement $b \in A$. Then, the following hold:*
(i) *If c is another complement of a in A, then $c = b$;*
(ii) $a' = b$ *and* $b' = a$;
(iii) $a^2 = a$.

Let $B(A)$ the set of all complemented elements of A.

Lemma 2.2. *([6]) If $e \in B(A)$, then $e' = e^*$ and $e^{**} = e$.*

Remark 2.1. *([12]) If $e, f \in B(A)$, then $e\wedge f, e\vee f \in B(A)$. Moreover, $(e\vee f)' = e'\wedge f'$ and $(e \wedge f)' = e' \vee f'$. So, $e \to f = e' \vee f \in B(A)$ and*
(c_{33}) $e \odot x = e \wedge x$, *for every* $x \in A$.

Corollary 2.1. *([12]) The set $B(A)$ is the universe of a Boolean subalgebra of A, called the Boolean center of A.*

Proposition 2.1. *([6]) For $e \in A$ the following are equivalent:*
(i) $e \in B(A)$,
(ii) $e \vee e^* = 1$.

Proposition 2.2. *([6]) For $e \in A$ we consider the following assertions:*
(1) $e \in B(A)$,
(2) $e^2 = e$ *and* $e = e^{**}$,
(3) $e^2 = e$ *and* $e^* \to e = e$,
(4) $(e \to x) \to e = e$, *for every* $x \in A$,
(5) $e \wedge e^* = 0$. *Then:*
(i) $(1) \Rightarrow (2), (3), (4)$ *and* (5),
(ii) $(2) \nRightarrow (1), (3) \nRightarrow (1), (4) \nRightarrow (1), (5) \nRightarrow (1)$.

Lemma 2.3. *([6]) If $e, f \in B(A)$ and $x, y \in A$, then:*
(c_{34}) $x \odot (x \to e) = e \wedge x, e \odot (e \to x) = e \wedge x$;
(c_{35}) $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$;
(c_{36}) $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$;
(c_{37}) $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)]$;
(c_{38}) $x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)]$;
(c_{39}) $e \to (x \to y) = (e \to x) \to (e \to y)$.

Definition 2.1. *([4])Let A and B be residuated lattices. A function $f : A \to B$ is a* morphism of residuated lattices *if f is morphism of bounded lattices and for every $x, y \in A : f(x \odot y) = f(x) \odot f(y)$ and $f(x \to y) = f(x) \to f(y)$.*

Definition 2.2. *([12], [16]) A non empty subset $D \subseteq A$ is called a* deductive system *of A,* ***ds*** *for short, if the following conditions are satisfied:*
(D_1) $1 \in D$;
(D_2) *If $x, x \to y \in D$, then $y \in D$.*

Clearly $\{1\}$ and A are **ds** ; a **ds** D of A is called *proper* if $D \neq A$.

Remark 2.2. *([12], [16]) A nonempty subset $D \subseteq A$ is a ds of A iff for all $x, y \in A$:*
(D_1') *If x, $y \in D$, then $x \odot y \in D$;*
(D_2') *If $x \in D, y \in A, x \leq y$, then $y \in D$.*

Remark 2.3. *Deductive systems are called also congruence filters in literature. To avoid confusion we reserve, however in this paper, the name filter to lattice filters and deductive system for implicative congruence filters. From $(lr - c_2)$ and Remark 2.2 we deduce that every* ***ds*** *of A is a filter for $L(A)$, but filters of $L(A)$ are not, in general, deductive systems for A (see [16]).*

We denote by $Ds(A)$ the set of all deductive systems of A.

3. Valuations on residuated lattices

Throughout this paper, by A we denote a residuated lattice.

Definition 3.1. *A real function $v : A \to \mathbf{R}$ is called* a pseudo-valuation *on A if*
(v_1)**:** $v(1) = 0$;
(v_2)**:** $v(x \to y) \geq v(y) - v(x)$, *for every $x, y \in A$.*
The pseudo-valuation v is said to be a valuation *if*
(v_3)**:** $v(x) = 0 \Rightarrow x = 1$ $(x \in A)$.

If we interpret A as an implicational calculus, $x \to y$ as the proposition "$x \Rightarrow y$" and 1 as truth, the pseudo-valuation on A can be interpret as "falsity-valuation".

Example 3.1. *$v : A \to \mathbf{R}, v(x) = 0$, for every $x \in A$ is a pseudo-valuation on A (called trivial).*

Example 3.2. *If $D \in Ds(A)$ and $r \in \mathbf{R}_+$, then $v_D : A \to \mathbf{R}, v(x) = \{ \begin{array}{l} 0,\ if\ x \in D, \\ r,\ if\ x \notin D \end{array}$ is a pseudo-valuation on A and a valuation iff $D = \{1\}$ and $r > 0$.*

Example 3.3. *If M is a finite set with n elements and $A = P(M)$ is the Boolean lattice of the power set of M, then $v : P(M) \to \mathbf{R}, v(X) = n - n(X)$ is a valuation on A (where $n(X)$ is the number of elements of X).*

Lemma 3.1. *A pseudo-valuation v on A is a non-negative decreasing function satisfying*
(c_{40}): *$v(x \to z) \leq v(x \to y) + v(y \to z)$, for every $x, y, z \in A$.*

Proof. If in (v_2) we put $y = 1$, we obtain that for every $x \in A$, $v(x \to 1) \geq v(1) - v(x) \overset{(c_1)}{\Rightarrow} v(1) \geq v(1) - v(x) \Rightarrow v(x) \geq 0$.

If $x \leq y$, then $x \to y = 1$ and by (v_2) we deduce that for every $x, y \in A$, $v(x \to y) \geq v(y) - v(x) \Leftrightarrow 0 = v(1) \geq v(y) - v(x) \Leftrightarrow v(x) \geq v(y)$.

Let now $x, y, z \in A$; by (c_{11}) $[x \to y \leq (y \to z) \to (x \to z)] \Rightarrow v(x \to y) \leq v[(y \to z) \to (x \to z)] \geq v(x \to z) - v(y \to z) \Rightarrow$
$v(x \to z) \leq v(x \to y) + v(y \to z)$.■

Lemma 3.2. *If $x_1, ..., x_n \in A$ and $v : A \to \mathbf{R}$ is a pseudo-valuation on A, then*
(c_{41}): *$v(\overset{n}{\underset{i=1}{\odot}} x_i) \leq \sum\limits_{i=1}^{n} v(x_i)$, so (since $\overset{n}{\underset{i=1}{\odot}} x_i \leq \overset{n}{\underset{i=1}{\wedge}} x_i$) then $v(\overset{n}{\underset{i=1}{\wedge}} x_i) \leq \sum\limits_{i=1}^{n} v(x_i)$.*

Proof. Mathematical induction relative to n; for $n = 2$ we have $x_1 \to (x_2 \to (x_1 \odot x_2)) \overset{(c_{13})}{=} (x_1 \odot x_2) \to (x_1 \odot x_2) = 1$, hence $0 = v(1) \geq v(x_2 \to (x_1 \odot x_2)) - v(x_1) \geq v(x_1 \odot x_2) - v(x_2) - v(x_1) \Rightarrow v(x_1 \odot x_2) \leq v(x_1) + v(x_2)$.■

Lemma 3.3. *Let $v : A \to \mathbf{R}$ a pseudo-valuation (valuation) on A. If we define $d_v : A \times A \to \mathbf{R}, d_v(x, y) = v(x \to y) + v(y \to x)$, for $(x, y) \in A \times A$, then (A, d_v) is a pseudo-metric (metric) space satisfying for any $x, y, z \in A$:*
(c_{42}): *$max\{d_v(x \to z, y \to z), d_v(z \to x, z \to y)\} \leq d_v(x, y)$;*
(c_{43}): *$d_v(x \wedge z, y \wedge z) \leq d_v(x, y)$;*
(c_{44}): *$d_v(x \vee z, y \vee z) \leq d_v(x, y)$;*
(c_{45}): *$d_v(x \odot z, y \odot z) \leq d_v(x, y)$;*

Proof. Let $x, y, z \in A$; clearly, $d_v(x, y) = d_v(y, x) \geq 0$, while $d_v(x, x) = v(x \to x) + v(x \to x) = v(1) + v(1) = 0 + 0 = 0$. Also, $d_v(x, y) + d_v(y, z) = [v(x \to y) + v(y \to x)] + [v(y \to z) + v(z \to y)] = [v(x \to y) + v(y \to z)] + [v(z \to y) + v(y \to x)] \overset{(c_{40})}{\geq} v(x \to z) + v(z \to x) = d_v(x, z)$, hence d_v is a pseudo-metric on A. Suppose v is a valuation on A and let $x, y \in A$ such that $d_v(x, y) = 0$. Then $v(x \to y) = v(y \to x) = 0$, hence $x \to y = y \to x = 1 \Rightarrow x = y$, that is, d_v is a metric on A.

Suppose d_v is a metric on A and let $x \in A$ such that $v(x) = 0$.

Since $d_v(x, 1) = v(x \to 1) + v(1 \to x) = v(1) + v(x) = 0 + 0 = 0$, then $x = 1$, that is, v is a valuation on A.

(c_{42}). We have $d_v(x \to z, y \to z) = v((x \to z) \to (y \to z)) + v((y \to z) \to (x \to z))$. Since by (c_{11}), $x \to y \leq (y \to z) \to (x \to z)$ and $y \to x \leq (x \to z) \to (y \to z)$ we deduce that $v(x \to y) \geq v[(y \to z) \to (x \to z)]$ and $v(y \to x) \geq v[(x \to z) \to (y \to z)]$, hence $d_v(x, y) = v(x \to y) + v(y \to x) \geq v[(y \to z) \to (x \to z)] + v[(x \to z) \to (y \to z)] = d_v(x \to z, y \to z)$.

Since by (c_{10}), $x \to y \leq (z \to x) \to (z \to y)$ and $y \to x \leq (z \to y) \to (z \to x)$, analogously as above we deduce that $d_v(x, y) \geq d_v(z \to x, z \to y)$.

So, $max\{d_v(x \to z, y \to z), d_v(z \to x, z \to y)\} \leq d_v(x, y)$.

(c_{43}). We have $d_v(x \wedge z, y \wedge z) = v[(x \wedge z) \to (y \wedge z)] + v[(y \wedge z) \to (x \wedge z)]$; since by (c_8), $x \to y \leq (x \wedge z) \to (y \wedge z)$ and $y \to x \leq (y \wedge z) \to (x \wedge z)$ we deduce that $v(x \to y) \geq v[(x \wedge z) \to (y \wedge z)]$ and $v(y \to x) \geq v[(y \wedge z) \to (x \wedge z)]$, hence $d_v(x, y) = v(x \to y) + v(y \to x) \geq v[(x \wedge z) \to (y \wedge z)] + v[(y \wedge z) \to (x \wedge z)] = d_v(x \wedge z, y \wedge z)$.

$(c_{44}) - (c_{45})$. Analogously (see (c_{43})). ■

Corollary 3.1. *Let $v : A \to \mathbf{R}$ a valuation. In the metric space (A, d_v), the functions $\wedge, \vee, \to, \odot : A \times A \to \mathbf{R}$ are uniformly continuous.*

Proof. For $\wedge$: let $x, x', y, y' \in A$ and $0 < \varepsilon \in \mathbf{R}$. If $\overline{d}_v : A \times A \to \mathbf{R}$ is the natural metric on $A \times A$ (defined by $\overline{d}_v((x, y), (x', y')) = \max\{d_v(x, x'), d_v(y, y')\}$, then if $\overline{d}_v((x, y), (x', y')) < \varepsilon$ and $d_v(x, x'), d_v(y, y') < \varepsilon/2$, we have $d_v(x \wedge y, x' \wedge y') \leq d_v(x \wedge y, x' \wedge y) + d_v(x' \wedge y, x' \wedge y') \leq d_v(x, x') + d_v(y, y') \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$, that is, $\wedge$ is uniformly continuous. Analogously for functions $\vee, \to, \odot$ (see (c_{44}) and (c_{45})). ■

4. Theorem of extension for pseudo-valuations

Let A, B two residuated lattices such that A is a residuated sublattice of B. We have the following theorem of extension:

Theorem 4.1. *For every pseudo-valuation (valuation) $v : A \to \mathbf{R}$ there exists a pseudo-valuation $v' : B \to \mathbf{R}$ such that $v'_{|A} = v$.*

Proof. Let $v : A \to \mathbf{R}$ a pseudo-valuation. For $x \in B$ we define $v' : B \to R, v'(x) = \inf\left\{\sum_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A \text{ and } x_1 \odot ... \odot x_n \leq x\right\}$.

Since $1 \in A$ and $1 \leq 1 \Rightarrow v'(1) \leq v(1) = 0$, hence $v'(1) = 0$. Let now $x, y \in B$ and $x_1, ..., x_n, y_1, ..., y_m \in A$ such that $x_1 \odot ... \odot x_n \leq x$ and $y_1 \odot ... \odot y_m \leq x \to y$. Then $x_1 \odot ... \odot x_n \odot y_1 \odot ... \odot y_m \leq x \odot (x \to y) \leq y$, hence $v'(y) \leq \sum_{i=1}^{n} v(x_i) + \sum_{i=1}^{m} v(y_i) \Rightarrow v'(y) \leq \inf\{\sum_{i=1}^{n} v(x_i)\} + \inf\{\sum_{i=1}^{m} v(y_i)\} = v'(x) + v'(x \to y)$, hence $v'(x \to y) \geq v'(y) - v'(x)$. If $x \in A$, since $x \leq x \Rightarrow v'(x) \leq v(x)$.

Let now $x_1, ..., x_n \in A$ such that $x_1 \odot ... \odot x_n \leq x \Rightarrow v(x) \leq v(x_1 \odot ... \odot x_n) \overset{(c_{41})}{\leq} \sum_{i=1}^{n} v(x_i) \Rightarrow v(x) \leq \inf\{\sum_{i=1}^{n} v(x_i)\} = v'(x) \Rightarrow v'(x) = v(x) \Rightarrow v'_{|A} = v$.■

We recall (see [6]) that a subset $S \subseteq A$ is called a $\wedge-$*closed system* if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

We denote by $S(A)$ the set of all $\wedge-$closed systems of A (clearly $\{1\}, A \in S(A)$).

For $S \in S(A)$, on A we consider the relation θ_S defined by $(x, y) \in \theta_S$ iff there is $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$.

Lemma 4.1. *The relation θ_S is a congruence on A.*

For $x \in A$ we denote by x/S the equivalence class of x relative to θ_S and by $A[S] = A/\theta_S$. By $p_S : A \to A[S]$ we denote the canonical mapping defined by $p_S(x) = x/S$, for every $x \in A$. Clearly $A[S]$ become a residuated lattice, where $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$, $(x/S) \wedge (y/S) = (x \wedge y)/S$, $(x/S) \vee (y/S) = (x \vee y)/S$, $(x/S) \odot (y/S) = (x \odot y)/S$ and $(x/S) \to (y/S) = (x \to y)/S$. So, p_S is an onto morphism of residuated lattices.

Theorem 4.2. *If $S \in S(A)$ and $v : A \to \mathbf{R}$ is a pseudo-valuation on A, then the following are equivalent:*

(i)**:** *There exists a pseudo-valuation $v' : A[S] \to \mathbf{R}$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "p_S"] \arrow[dr, "v"'] & A[S] \arrow[d, "v'"] \\ & \mathbf{R} \end{tikzcd}$$

is commutative (i.e. $v' \circ p_S = v$);

(ii)**:** $v(s) = 0$ *for every* $s \in S \cap B(A)$.

Proof. $(i) \Rightarrow (ii)$. Let $v' : A[S] \to \mathbf{R}$ a valuation such that $v' \circ p_S = v$ and $s \in S \cap B(A)$. Since $s \wedge s = s \wedge 1$ we deduce that $(s, 1) \in \theta_S$, so $p_S(s) = p_S(1)$, hence $v(s) = (v' \circ p_S)(s) = v'(p_S(s)) = v'(p_S(1)) = (v' \circ p_S)(1) = v(1) = 0$.

$(ii) \Rightarrow (i)$. For $x \in A$ we define $v'(x/S) = v(x)$. If $x, y \in S$ and $x/S = y/S$ then there exists $s \in S \cap B(A)$ such that $s \wedge x = s \wedge y$. Since $s \wedge x \leq x$, we deduce $v(x) \leq v(s \wedge x) = v(s \wedge y) \overset{(c_{14})}{\leq} v(y) + v(s) = v(y) + 0 = v(y)$ and analogously $v(y) \leq v(x)$, hence $v(x) = v(y)$, that is, v' is correctly defined.

We have $v'(1/S) = v(1) = 0$ and for $x, y \in A$, $v'(x/S \to y/S) = v'((x \to y)/S) = v(x \to y) \geq v(y) - v(x) = v'(y/S) - v'(x/S)$, hence v' is a pseudo-valuation on A. If v is a valuation, then v' is a valuation because $v'(x/S) = 0$, for $x \in A$, then $v(x) = 0$, hence $x = 1$ and $x/S = \mathbf{1}$. ■

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(Cătălin Buşneag) Faculty of Mathematics and Computer Science,
University of Craiova,
13, Al.I. Cuza st., 200585, Craiova, Romania
E-mail address: catalinbusneag@yahoo.com