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On perfect pseudo-BCK algebras with pseudo-product

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Abstract. Pseudo-BCK algebras were introduced by G. Georgescu and A. Iorgulescu as a generalization of BCK algebras. Their properties and connections with other fuzzy structures were established by A. Iorgulescu and J. Kühr. In this paper we study the class of perfect pseudo-BCK algebras with pseudo-product and we prove that any perfect pseudo-BCK algebra with pseudo-product is strongly bipartite. Another main result of the paper states that any perfect pseudo-BCK algebra with pseudo-product admits a unique Bosbach state on it.

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1. Introduction

Pseudo-BCK algebras were introduced in [7] by G. Georgescu and A. Iorgulescu as a generalization of BCK algebras in order to give a corresponding structure to pseudo-MV algebras, since the bounded commutative BCK algebras correspond to MV algebras. Properties of pseudo-BCK algebras and their connections with others fuzzy structures were established by A.Iorgulescu in [9], [10], [11], [12]. One of the most important class of pseudo-BCK algebras is that having the pseudo-product property (pP for short). This property proved to be very important to establish the connections of pseudo-BCK algebras with other fuzzy structures. It was proved in [11] that the pseudo-BCK(pP) algebras are categorically equivalent with the partial ordered residuated integral monoids (porims) and it was proved in [9] that the pseudo-BCK(pP) lattices are termwise equivalent with the residuated lattices which generalize other structures such as pseudo-MTL algebras, bounded divisible non-commutative algebras (Rℓ-monoids), pseudo-BL algebras and pseudo-MV algebras. J. Kühr proved in [14] that every pseudo-BCK algebra is a subreduct of a residuated lattice. It was proved in [5] that every pseudo-hoop is a pseudo-BCK(pP) algebra. The class of perfect multiple-valued logic algebras proved to be very important for the study of the existence of states on these structure. For the commutative case, the perfect structures play an important role for the study of different kinds of convergences on these algebras.

Perfect MV-algebras were studied in [1], perfect BL-algebras were studied in [18], while perfect bounded commutative Rℓ-monoids were investigated in [17]. For the case of non-commutative structures, perfect pseudo-MV algebras were presented in [15], perfect pseudo-BL algebras in [8], perfect pseudo-MTL algebras in [3] and perfect residuated lattices in [2]. Recently, the properties of perfect bounded non-commutative Rℓ-monoids were investigated in [16]. The perfect pseudo-BCK(pP) algebras were

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introduced and studied in [6]. In this paper we obtain new results regarding perfect pseudo-BCK(pP) algebras, more precisely we define the notion of bipartite and strong bipartite pseudo-BCK(pP) algebras and we prove that any perfect pseudo-BCK(pP) algebra is strongly bipartite. We prove that the class of local pseudo-BCK(pP) algebras can be classified in perfect, locally finite and peculiar subclasses. One of the main results of the paper consists of proving that any perfect pseudo-BCK(pP) algebra admits at least a Bosbach state on it. We define the notion of bipartite and strong bipartite pseudo-BCK(pP) algebras and we prove that any perfect pseudo-BCK(pP) algebra is strongly bipartite.

2. Pseudo-BCK algebras and their basic properties

Definition 2.1. ([9]) *A pseudo-BCK algebra (more precisely, reversed left-pseudo-BCK algebra) is a structure* $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ *where* $\leq$ *is a binary relation on* A, $\rightarrow$ *and* $\rightsquigarrow$ *are binary operations on* A *and* 1 *is an element of* A *satisfying, for all* $x, y, z \in A$, *the axioms:*
(A_1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $\quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$;
(A_2) $x \leq (x \rightarrow y) \rightsquigarrow y$, $\quad x \leq (x \rightsquigarrow y) \rightarrow y$;
(A_3) $x \leq x$;
(A_4) $x \leq 1$;
(A_5) *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$;
(A_6) $x \leq y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$.

A pseudo-BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ is *commutative* iff $\rightarrow = \rightsquigarrow$. Any commutative pseudo-BCK algebra is a BCK algebra.
We will also refer to a pseudo-BCK algebra by its universe A.

Example 2.1. ([4]) *Consider* $A = \{o_1, a_1, b_1, c_1, o_2, a_2, b_2, c_2, 1\}$ *with* $o_1 < a_1, b_1 < c_1 < 1$ *and* a_1, b_1 *incomparable,* $o_2 < a_2, b_2 < c_2 < 1$ *and* a_2, b_2 *incomparable. We also assume that any element of the set* $\{o_1, a_1, b_1, c_1\}$ *is incomparable with any element of the set* $\{o_2, a_2, b_2, c_2\}$. *Consider the operations* $\rightarrow, \rightsquigarrow$ *given by the following tables:*

$\rightarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	o_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	a_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	o_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	c_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	o_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

$\rightsquigarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	b_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	o_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	b_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	b_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	b_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

.

Then, $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ *is a proper pseudo-BCK algebra.*

Proposition 2.1. ([11], [12]) *In any pseudo-BCK algebra the following properties hold:*
(c_1) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;
(c_2) $x \leq y, y \leq z$ *implies* $x \leq z$;
(c_3) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$ *and* $x \rightsquigarrow (y \rightarrow z) = y \rightarrow (x \rightsquigarrow z)$;

(c_4) *$z \leq y \to x$ iff $y \leq z \rightsquigarrow x$;*
(c_5) $z \to x \leq (y \to z) \to (y \to x) \quad z \rightsquigarrow x \leq (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)$;
(c_6) $x \leq y \to x, \quad x \leq y \rightsquigarrow x$;
(c_7) $1 \to x = x = 1 \rightsquigarrow x$;
(c_8) *$x \leq y$ implies $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$;*
(c_9) $[(y \to x) \rightsquigarrow x] \to x = y \to x, \quad [(y \rightsquigarrow x) \to x] \rightsquigarrow x = y \rightsquigarrow x$.

Definition 2.2. *([9]) If there is an element 0 of a pseudo-BCK algebra $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$, such that $0 \leq x$ (i.e. $0 \to x = 0 \rightsquigarrow x = 1$), for all $x \in A$, then 0 is called the* zero *of $\mathcal{A}$. A pseudo-BCK algebra with zero is called* bounded pseudo-BCK algebra *and it is denoted by $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$.*

Example 2.2. *([4]) Consider $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$ and a, b incomparable. Consider the operations $\to, \rightsquigarrow$ given by the following tables:*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

.

Then, $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra.

Definition 2.3. *([9]) A pseudo-BCK algebra with* (pP) condition *(i.e. with* pseudo-product *condition) or a* pseudo-BCK(pP) algebra *for short, is a pseudo-BCK algebra $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ satisfying (pP) condition:*
(pP) there exists, for all $x, y \in A$, $x \odot y = min\{z \mid x \leq y \to z\} = min\{z \mid y \leq x \rightsquigarrow z\}$.

Definition 2.4. *([9]) (1) Let $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. If the poset $(A, \leq)$ is a lattice, then we say that $\mathcal{A}$ is a* pseudo-BCK lattice.
(2) Let $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ be a pseudo-BCK(pP) algebra. If the poset $(A, \leq)$ is a lattice, then we say that $\mathcal{A}$ is a pseudo-BCK(pP) lattice. *A pseudo-BCK(pP) lattice $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ will be denoted by $\mathcal{A} = (A, \vee, \wedge, \to, \rightsquigarrow, 1)$.*

Remark 2.1. *Pseudo-BCK algebras are connected with other structures as follows:*
(1) *Pseudo-BCK(pP) algebras are caregorically isomorphic with* left-porims *(partially ordered, residuated, integral left-monoids) ([11]).*
(2) *(Bounded) pseudo-BCK(pP) lattices are categorically isomorphic with (bounded) residuated lattices ([9]) .*
(3) *Every pseudo-BCK algebra is a subreduct of a residuated lattice ([14]).*
(4) *Every pseudo-hoop is a pseudo-BCK(pP) algebra ([5]).*

Example 2.3. (1) *If $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ is the bounded pseudo-BCK lattice from Example 2.2, then $min\{z \mid b \leq a \to z\} = min\{a, b, c, 1\}$ and $min\{z \mid a \leq b \rightsquigarrow z\} = min\{a, b, c, 1\}$ do not exist. Thus, $b \odot a$ does not exist, so $\mathcal{A}$ is not a pseudo-BCK(pP) algebra. Moreover, since $(A, \leq)$ is a lattice, it follows that $\mathcal{A}$ is a pseudo-BCK lattice.*
(2) *If $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ is a reduct of a residuated lattice, then it is obvious that $\mathcal{A}$ is a bounded pseudo-BCK(pP) algebra.*

Example 2.4. *([10]) Take $A = \{0, a_1, a_2, s, a, b, n, c, d, m, 1\}$ with $0 < a_1 < a_2 < s < a$, $b < n < c$, $d < m < 1$ (a is incomparable with b and c is incomparable with d).*

Consider the operations $\rightarrow, \rightsquigarrow$ *given by the following tables:*

$\rightarrow$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1	1	1
a_1	a_1	1	1	1	1	1	1	1	1	1	1
a_2	a_1	a_1	1	1	1	1	1	1	1	1	1
s	0	a_1	a_2	1	1	1	1	1	1	1	1
a	0	a_1	a_2	m	1	m	1	1	1	1	1
b	0	a_1	a_2	m	m	1	1	1	1	1	1
n	0	a_1	a_2	m	m	m	1	1	1	1	1
c	0	a_1	a_2	m	m	m	m	1	m	1	1
d	0	a_1	a_2	m	m	m	m	m	1	1	1
m	0	a_1	a_2	m	m	m	m	m	m	1	1
1	0	a_1	a_2	s	a	b	n	c	d	m	1

$\rightsquigarrow$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1	1	1
a_1	a_2	1	1	1	1	1	1	1	1	1	1
a_2	0	a_1	1	1	1	1	1	1	1	1	1
s	0	a_1	a_2	1	1	1	1	1	1	1	1
a	0	a_1	a_2	m	1	m	1	1	1	1	1
b	0	a_1	a_2	m	m	1	1	1	1	1	1
n	0	a_1	a_2	m	m	m	1	1	1	1	1
c	0	a_1	a_2	m	m	m	m	1	m	1	1
d	0	a_1	a_2	m	m	m	m	m	1	1	1
m	0	a_1	a_2	m	m	m	m	m	m	1	1
1	0	a_1	a_2	s	a	b	n	c	d	m	1

Then, $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ *is a bounded pseudo-BCK(pP) algebra. The operation* $\odot$ *is given by the following table:*

$\odot$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0	0	0	0
a_1	0	0	0	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	0	a_1	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2
s	0	a_1	a_2	s	s	s	s	s	s	s	s
a	0	a_1	a_2	s	s	s	s	s	s	s	a
b	0	a_1	a_2	s	s	s	s	s	s	s	b
n	0	a_1	a_2	s	s	s	s	s	s	s	n
c	0	a_1	a_2	s	s	s	s	s	s	s	c
d	0	a_1	a_2	s	s	s	s	s	s	s	d
m	0	a_1	a_2	s	s	s	s	s	s	s	m
1	0	a_1	a_2	s	a	b	n	c	d	m	1

Remark 2.2. *Any bounded linearly ordered pseudo-BCK algebra is with (pP) condition (see* [9]*). If the pseudo-BCK algebra is not bounded this result is not always valid, as we can see in the following example communicated by J.* Kühr.
Let $(Q, +, 0, \leq)$ *be the additive group of rationals with the usual linear order and take* $A = \{x \in Q : -\sqrt{2} < x \leq 0\}$. *Then* $(A, \rightarrow, 0)$ *is a linear BCK algebra with* $x \rightarrow y = min\{0, y - x\}$. *We have* $\{z \in A : (-1) \leq (-1) \rightarrow z = min\{0, z + 1\}\} = A$, *thus* $(-1) \odot (-1) = min A$ *doesn't exist in* $(A, \rightarrow, 0)$.

Proposition 2.2. *([12]) In any pseudo-BCK algebra(pP) the following properties hold:*
(c_{10}) $x \odot y \leq x, y$;
(c_{11}) $(x \to y) \odot x \leq x, y, \quad x \odot (x \rightsquigarrow y) \leq x, y$;
(c_{12}) $y \leq x \to (y \odot x), \quad y \leq x \rightsquigarrow (x \odot y)$;
(c_{13}) $x \to y \leq (x \odot z) \to (y \odot z), \quad x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y)$;
(c_{14}) $x \odot (y \to z) \leq y \to (x \odot z), \quad (y \rightsquigarrow z) \odot x \leq y \rightsquigarrow (z \odot x)$;
(c_{15}) $(y \to z) \odot (x \to y) \leq x \to z, \quad (x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leq x \rightsquigarrow z$;
(c_{16}) $x \to (y \to z) = (x \odot y) \to z, \quad x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$;
(c_{17}) $(x \odot z) \to (y \odot z) \leq x \to (z \to y), \quad (z \odot x) \rightsquigarrow (z \odot y) \leq x \rightsquigarrow (z \rightsquigarrow y)$;
(c_{18}) $x \to y \leq (x \odot z) \to (y \odot z) \leq x \to (z \to y)$,
$x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y) \leq x \rightsquigarrow (z \rightsquigarrow y)$
(c_{19}) $x \leq y$ *implies* $x \odot z \leq y \odot z$ *and* $z \odot x \leq z \odot y$.

Let $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo-BCK algebra. We define two negations $^-$ and $^\sim$ ([12]): for all $x \in A$,

$$x^- = x \to 0, \quad x^\sim = x \rightsquigarrow 0.$$

Proposition 2.3. *([12]) In a bounded pseudo-BCK algebra the following hold:*
(c_{20}) $1^- = 0 = 1^\sim, \quad 0^- = 1 = 0^\sim$;
(c_{21}) $x \leq (x^-)^\sim, \quad x \leq (x^\sim)^-$;
(c_{22}) $x \to y \leq y^- \rightsquigarrow x^-, \quad x \rightsquigarrow y \leq y^\sim \to x^\sim$;
(c_{23}) $x \leq y$ *implies* $y^- \leq x^-$ *and* $y^\sim \leq x^\sim$;
(c_{24}) $x \to y^\sim = y \rightsquigarrow x^-$ *and* $x \rightsquigarrow y^- = y \to x^\sim$;
(c_{25}) $((x^-)^\sim)^- = x^-, \quad ((x^\sim)^-)^\sim = x^\sim$.

Proposition 2.4. *([4]) In a bounded pseudo-BCK algebra the following hold:*
(c_{26}) $x \to y^{-\sim} = y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim}$ *and*
$x \rightsquigarrow y^{\sim-} = y^\sim \to x^\sim = x^{\sim-} \rightsquigarrow y^{\sim-}$;
(c_{27}) $x \to y^\sim = y^{\sim-} \rightsquigarrow x^- = x^{-\sim} \to y^\sim$ *and* $x \rightsquigarrow y^- = y^{-\sim} \to x^\sim = x^{\sim-} \rightsquigarrow y^-$;
(c_{28}) $(x \to y^{\sim-})^{\sim-} = x \to y^{\sim-}$ *and* $(x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}$.

Proposition 2.5. *([4]) In a bounded pseudo-BCK(pP) algebra the following hold:*
(c_{29}) $(x_{n-1} \to x_n) \odot (x_{n-2} \to x_{n-1}) \odot ... \odot (x_1 \to x_2) \leq x_1 \to x_n$ *and*
$(x_1 \rightsquigarrow x_2) \odot (x_2 \rightsquigarrow x_3) \odot ... \odot (x_{n-1} \rightsquigarrow x_n) \leq x_1 \rightsquigarrow x_n$;
(c_{30}) $x \odot 0 = 0 \odot x = 0$;
(c_{31}) $x \odot 1 = 1 \odot x = x$;
(c_{32}) $x^- \odot x = 0$ *and* $x \odot x^\sim = 0$;
(c_{33}) $x \leq y^-$ *iff* $x \odot y = 0$ *and* $x \leq y^\sim$ *iff* $y \odot x = 0$;
(c_{34}) $x \to y^- = (x \odot y)^-$ *and* $x \rightsquigarrow y^\sim = (y \odot x)^\sim$;
(c_{35}) $x \leq y^-$ *iff* $y \leq x^\sim$;
(c_{36}) $x \leq x^\sim \to y$ *and* $x \leq x^- \rightsquigarrow y$.

Definition 2.5. *A bounded pseudo-BCK algebra $\mathcal{A}$ is called* good *if* $(x^-)^\sim = (x^\sim)^-$ *for all* $x \in A$.

Remark 2.3. *It is easy to show that any bounded pseudo-BCK algebra can be embedded into a good one. Indeed, consider the bounded pseudo-BCK algebra $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ and an element $0_1 \notin A$. Consider a new pseudo-BCK algebra $\mathcal{A}_1 = (A_1, \leq, \to_1, \rightsquigarrow_1, 0_1, 1)$, where $A_1 = A \cup \{0_1\}$ and the operations $\to_1$ and $\rightsquigarrow_1$*

are defined as follows:

$$x \to_1 y = \begin{cases} x \to y, & \text{if } x, y \in A, \\ 1, & \text{if } x = 0_1,\ y \in A_1, \\ 0_1, & \text{if } x \in A,\ y = 0_1, \end{cases}$$

$$x \rightsquigarrow_1 y = \begin{cases} x \rightsquigarrow y, & \text{if } x, y \in A, \\ 1, & \text{if } x = 0_1,\ y \in A_1, \\ 0_1, & \text{if } x \in A,\ y = 0_1. \end{cases}$$

One can easily check that $\mathcal{A}$ as a subalgebra of $\mathcal{A}_1$ and $\mathcal{A}_1$ is a good pseudo-BCK algebra.

Example 2.5. *([10]) Consider the pseudo-BCK lattice $\mathcal{A}$ from Example 2.4. Since $(a_1^-)^\sim = a_2$ and $(a_1^\sim)^- = a_1$, it follows that $\mathcal{A}$ is not good. $\mathcal{A}$ is embedded into the good pseudo-BCK algebra (see [10]) $\mathcal{A}_1 = (A_1, \leq, \to, \rightsquigarrow, 0, 1)$, where $A_1 = \{0, a_1, a_2, b_2, s, a, b, n, c, d, m, 1\}$ with $0 < a_1 < a_2 < b_2 < s < a, b < n < c, d < m < 1$ (a is incomparable with b and c is incomparable with d). The operations $\to$ and $\rightsquigarrow$ are defined as follows:*

$\to$	0	a_1	a_2	b_2	s	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1	1	1	1
a_1	0	1	1	1	1	1	1	1	1	1	1	1
a_2	0	a_2	1	1	1	1	1	1	1	1	1	1
b_2	0	a_2	a_2	1	1	1	1	1	1	1	1	1
s	0	a_1	a_2	b_2	1	1	1	1	1	1	1	1
a	0	a_1	a_2	b_2	m	1	m	1	1	1	1	1
b	0	a_1	a_2	b_2	m	m	1	1	1	1	1	1
n	0	a_1	a_2	b_2	m	m	m	1	1	1	1	1
c	0	a_1	a_2	b_2	m	m	m	m	1	m	1	1
d	0	a_1	a_2	b_2	m	m	m	m	m	1	1	1
m	0	a_1	a_2	b_2	m	m	m	m	m	m	1	1
1	0	a_1	a_2	b_2	s	a	b	n	c	d	m	1

$\rightsquigarrow$	0	a_1	a_2	b_2	s	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1	1	1	1
a_1	0	1	1	1	1	1	1	1	1	1	1	1
a_2	0	b_2	1	1	1	1	1	1	1	1	1	1
b_2	0	a_1	a_2	1	1	1	1	1	1	1	1	1
s	0	a_1	a_2	b_2	1	1	1	1	1	1	1	1
a	0	a_1	a_2	b_2	m	1	m	1	1	1	1	1
b	0	a_1	a_2	b_2	m	m	1	1	1	1	1	1
n	0	a_1	a_2	b_2	m	m	m	1	1	1	1	1
c	0	a_1	a_2	b_2	m	m	m	m	1	m	1	1
d	0	a_1	a_2	b_2	m	m	m	m	m	1	1	1
m	0	a_1	a_2	b_2	m	m	m	m	m	m	1	1
1	0	a_1	a_2	b_2	s	a	b	n	c	d	m	1

One can easily check that $\mathcal{A}_1 = (A_1, \leq, \to, \rightsquigarrow, 0, 1)$ is a good pseudo-BCK(pP) algebra. The operation $\odot$ is given by the following table:

$\odot$	0	a_1	a_2	b_2	s	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a_1	0	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	0	a_1	a_1	a_1	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2
b_2	0	a_1	a_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2	b_2
s	0	a_1	a_2	b_2	s	s	s	s	s	s	s	s
a	0	a_1	a_2	b_2	s	s	s	s	s	s	s	a
b	0	a_1	a_2	b_2	s	s	s	s	s	s	s	b
n	0	a_1	a_2	b_2	s	s	s	s	s	s	s	n
c	0	a_1	a_2	b_2	s	s	s	s	s	s	s	c
d	0	a_1	a_2	b_2	s	s	s	s	s	s	s	d
m	0	a_1	a_2	b_2	s	s	s	s	s	s	s	m
1	0	a_1	a_2	b_2	s	a	b	n	c	d	m	1.

Example 2.6. *([4]) Consider the pseudo-BCK lattice $\mathcal{A}$ from Example 2.2. Since $(a^-)^\sim = 1$ and $(a^\sim)^- = a$, it follows that $\mathcal{A}$ is not good. $\mathcal{A}$ is embedded into the good pseudo-BCK lattice $\mathcal{A}_1 = (A_1, \leq, \to, \rightsquigarrow, 0, 1)$, where $A_1 = \{0, a, b, c, d, 1\}$ (in the construction given in Remark 2.3 we replaced c by d, b by c, a by b, 0 by a and 0_1 by 0, so $0 < a < b, c < d < 1$ and b, c are incomparable). The operations $\to$ and $\rightsquigarrow$ are defined as follows:*

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	a	1	c	1	1
c	0	b	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	c	1	c	1	1
c	0	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

.

One can easily check that $\mathcal{A}_1$ is a good pseudo-BCK algebra. Moreover, we can see that:

$$min\{z \mid c \leq b \to z\} = min\{b, c, d, 1\} \text{ and}$$
$$min\{z \mid b \leq c \rightsquigarrow z\} = min\{b, c, d, 1\}$$

do not exist. Thus, $c \odot b$ do not exists, so $\mathcal{A}_1$ is without (pP) condition.
Since $(A_1, \leq)$ is a lattice, it follows that $\mathcal{A}_1$ is a good pseudo-BCK lattice without (pP) condition.

Definition 2.6. *Let $\mathcal{A}$ be a pseudo-BCK algebra. The subset $D \subseteq A$ is called* deductive system *of A if it satisfies the following conditions:*
(DS_1) $1 \in D$;
(DS_2) for all $x, y \in A$, if $x, x \to y \in D$, then $y \in D$.

The condition (DS_2) is equivalent with the following condition:
(DS_2') for all $x, y \in A$, if $x, x \rightsquigarrow y \in D$, then $y \in D$.

Proposition 2.6. *If A is a bounded pseudo-BCK(pP) algebra, then the sets*

$$A_0^- = \{x \in A \mid x^- = 0\} \text{ and } A_0^\sim = \{x \in A \mid x^\sim = 0\}$$

are proper deductive systems of A.

Proof. If $x, y \in A_0^-$, then $(x \odot y)^- = x \to y^- = x \to 0 = x^- = 0$, so $x \odot y \in A_0^-$.
If $x \in A_0^-, y \in A$ such that $x \leq y$, then $y^- \leq x^- = 0$, so $y^- = 0$, that is $y \in A_0^-$.
Because $0 \notin A_0^-$, we conclude that A_0^- is a proper deductive system of A.
Similarly for the case of $A_0^\sim$. ∎

We will denote by $\mathcal{DS}(A)$ the set of all deductive systems of A.
Obviously, $\{1\}, A \in \mathcal{DS}(A)$.
A deductive system D of a pseudo-BCK algebra $\mathcal{A}$ is called *proper* if $D \neq A$.

Definition 2.7. *A deductive system D of a pseudo-BCK algebra $\mathcal{A}$ is called* normal *if it satisfies the condition:*
(DS_3) *for all $x, y \in A$, $x \to y \in D$ iff $x \rightsquigarrow y \in D$.*

The normal deductive system is called *compatible deductive* system in [13], but for an easier connection with the previous results in this paper we will use the notion of normal deductive system.
We will denote by $\mathcal{DS}_n(A)$ the set of all normal deductive systems of A.
It is obvious that $\{1\}, A \in \mathcal{DS}_n(A)$ and $\mathcal{DS}_n(A) \subseteq \mathcal{DS}(A)$.

Proposition 2.7. *Let A be a bounded pseudo-BCK algebra and $H \in \mathcal{DS}_n(A)$. Then:*
(1) *$x^- \in H$ iff $x^\sim \in H$;*
(2) *$x \in H$ implies $(x^-)^- \in H$ and $(x^\sim)^\sim \in H$.*

Proof. (1): It follows by taking $y = 0$ in the definition of a normal deductive system.
(2): From $x \in H$ and $x \leq (x^-)^\sim$ we get $(x^-)^\sim \in H$, that is $x^- \rightsquigarrow 0 \in H$. Hence, $x^- \to 0 \in H$, so $(x^-)^- \in H$. Similarly, $(x^\sim)^\sim \in H$. $\square$

Definition 2.8. *A deductive system is called* maximal *if it is proper and not strictly contained in any other deductive system. Denote:*
$Max(A) = \{F \mid F \text{ is maximal deductive system of } A\}$ *and*
$Max_n(A) = \{F \mid F \text{ is maximal normal deductive system of } A\}$.
Clearly, $Max_n(A) \subseteq Max(A)$.

Proposition 2.8. *Any proper deductive system of a pseudo-BCK algebra A can be extended to a maximal deductive system of A.*

Proof. It is an immediate consequence of Zorn's lemma. $\square$

Example 2.7. (1) *Let A be the pseudo-BCK(pP) algebra A from Example 2.4 and $D_1 = \{s, a, b, n, c, d, m, 1\}$, $D_2 = \{a_2, s, a, b, n, c, d, m, 1\}$. Then:*
$\mathcal{DS}(A) = \{\{1\}, D_1, D_2, A\}$, $\mathcal{DS}_n(A) = \{\{1\}, D_1, A\}$, $Max(A) = \{D_2\}$, $Max_n(A) = \emptyset$.
(2) *In the case of the pseudo-BCK(pP) algebra A_1 from Example 2.5, denoting by $D_1 = \{a_1, a_2, b_2, s, a, b, n, c, d, m, 1\}$, $D_2 = \{b_2, s, a, b, n, c, d, m, 1\}$ and $D_3 = \{s, a, b, n, c, d, m, 1\}$, we have: $\mathcal{DS}(A_1) = \{\{1\}, D_1, D_2, D_3, A\}$, $\mathcal{DS}_n(A_1) = \{\{1\}, D_1, D_3, A_1\}$, $Max(A_1) = \{D_1\}$, $Max_n(A_1) = \{D_1\}$.*

Definition 2.9. *Let $\mathcal{A}$ be pseudo-BCK(pP) algebra. The subset $\emptyset \neq F \subseteq A$ is called* filter *of A if it satisfies the following conditions:*
(F_1) *$x, y \in F$ implies $x \odot y \in F$;*
(F_2) *$x \in F$, $y \in A$, $x \leq y$ implies $y \in F$.*

One can easily check that in the case of a pseudo-BCK(pP) algebra the definition of the filter is equivalent with the definition of the deductive system.

If A is a pseudo-BCK(pP) algebra, then for any $n \in \mathbb{N}$, $x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \odot x = x \odot x^n$. If A is bounded, the *order* of $x \in A$, denoted $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $ord(x) = \infty$.

Definition 2.10. *For every subset $X \subseteq A$, the smallest deductive system of A containing X (i.e. the intersection of all deductive systems $D \in \mathcal{DS}(A)$ such that $X \subseteq D$) is called the deductive system generated by X and will be denoted by $< X >$. If $X = \{x\}$ we write $< x >$ instead of $< \{x\} >$.*

Lemma 2.1. *([8]) Let A be a bounded pseudo-BCK(pP) algebra and $x, y \in A$. Then:*
(1) $< x >$ is proper iff $ord(x) = \infty$;
(2) if $x \leq y$ and $ord(y) < \infty$, then $ord(x) < \infty$;
(3) if $x \leq y$ and $ord(x) = \infty$, then $ord(y) = \infty$.

Definition 2.11. *A bounded pseudo-BCK(pP) algebra A is locally finite if for any $x \in A$, $x \neq 1$ implies $ord(x) < \infty$.*

Proposition 2.9. *Let A be a bounded pseudo-BCK(pP) algebra. The following are equivalent:*
(a) A is locally finite;
(b) $\{1\}$ is the unique proper deductive system of A.

Proof. According to Lemma 2.1(1), A is locally finite iff for every $x \in A \setminus \{1\}$, $< x >= A$ iff $\{1\}$ is the unique proper deductive system of A. □

3. Perfect pseudo-BCK algebras with pseudo-product

First, we recall some definitions and results regarding the classes of local and perfect pseudo-BCK(pP) algebras. For more details, we refer the reader to [6].

Definition 3.1. *A pseudo-BCK(pP) algebra is called* local *if it has a unique maximal deductive system.*

In this section by a pseudo-BCK(pP) algebra we mean a bounded pseudo-BCK(pP) algebra, even though some notions and properties are valid for an arbitrary pseudo-BCK(pP) algebra.
We will denote:

$$D(A) = \{x \in A \mid ord(x) = \infty\} \quad \text{and} \quad D(A)^* = \{x \in A \mid ord(x) < \infty\}.$$

Obviously, $D(A) \cap D(A)^* = \emptyset$ and $D(A) \cup D(A)^* = A$.
We also can remark that $1 \in D(A)$ and $0 \in D(A)^*$.

Let A be a pseudo-BCK(pP) algebra and $D \in \mathcal{DS}(A)$. We will use the following notations:
$D^*_{-} = \{x \in A \mid x \leq y^{-}$ for some $y \in D\}$, $D^*_{\sim} = \{x \in A \mid x \leq y^{\sim}$ for some $y \in D\}$.

Proposition 3.1. *Let A be a local pseudo-BCK(pP) algebra. Then:*
(1) any proper deductive system of A is included in the unique maximal deductive system of A;
(2) A_0^{-} and $A_0^{\sim}$ are included in the unique maximal deductive system of A.

Proof. (1) It follows applying proposition 2.8 and taking into consideration that A has a unique maximal deductive system;
(2) Apply proposition 2.6 and (1). □

Theorem 3.1. *([6]) Let A be a pseudo-BCK(pP) algebra. Then the following are equivalent:*
(a) *$D(A)$ is a deductive system of A;*
(b) *$D(A)$ is a proper deductive system of A;*
(c) *A is local;*
(d) *$D(A)$ is the unique maximal deductive system of A;*
(e) *for all $x, y \in A$, $ord(x \odot y) < \infty$ implies $ord(x) < \infty$ or $ord(y) < \infty$.*

Corollary 3.1. *If A is a local pseudo-BCK(pP) algebra, then:*
(1) *for any $x \in A$, $ord(x) < \infty$ or $(ord(x^-) < \infty$ and $ord(x^\sim) < \infty)$;*
(2) *$D(A)^*_- \subseteq D(A)^*$ and $D(A)^*_\sim \subseteq D(A)^*$;*
(3) *$D(A) \cap D(A)^*_- = D(A) \cap D(A)^*_\sim = \emptyset$.*

Example 3.1. *Consider the pseudo-BCK(pP) algebra A from Example 2.4. One can easily check that $D(A) = \{a_2, s, a, b, n, c, d, m, 1\}$ which is a deductive system of A, so A is a local pseudo-BCK(pP) algebra.*

Proposition 3.2. *Any linearly ordered pseudo-BCK(pP) algebra is local.*

Proof. Assume that A is a linearly ordered pseudo-BCK(pP) algebra and consider $x, y \in A$ such that $ord(x \odot y) < \infty$. Since A is a linearly ordered, we have $x \le y$ or $y \le x$. Assume that $x \le y$. It follows that $x \odot x \le x \odot y$, so $ord(x \odot x) < \infty$. Hence, $ord(x) < \infty$. Similarly, from $y \le x$ we get $ord(y) < \infty$. Thus, according to theorem 3.1, A is a local pseudo-BCK(pP) algebra. □

Proposition 3.3. *Any locally finite pseudo-BCK(pP) algebra is local.*

Proof. Since $D(A) = \{1\}$, that is $D(A)$ is a deductive system of A, and applying theorem 3.1 it follows that A is local. □

Definition 3.2. *A pseudo-BCK(pP) algebra A is called perfect if it satisfies the following conditions:*
(1) *A is a local good pseudo-BCK(pP) algebra;*
(2) *for any $x \in A$, $ord(x) < \infty$ iff $ord(x^-) = \infty$* and $\mathrm{ord(x^\sim)} = \infty$.

Proposition 3.4. *([6]) Let A be a local good pseudo-BCK(pP) algebra. Then the following are equivalent:*
(a) *A is perfect;*
(b) *$D(A)^*_- = D(A)^*_\sim = D(A)^*$.*

Corollary 3.2. *If A is a perfect pseudo-BCK(pP) algebra, then*

$$D(A)^* = \{x^- \mid x \in D(A)\} = \{x^\sim \mid x \in D(A)\}.$$

Corollary 3.3. *Let A be a local good pseudo-BCK(pP) algebra. Then the following are equivalent:*
(a) *A is perfect;*
(b) *$D(A) \cup D(A)^*_- = D(A) \cup D(A)^*_\sim = A$.*

Example 3.2. (1) *Consider the pseudo-BCK(pP) algebra A from Example 2.4. Since A is not good, it follows that it is not a perfect pseudo-BCK(pP) algebra.*
(2) *If A_1 is the good pseudo-BCK(pP) algebra from Example 2.5, we have $D(A_1) = \{a_1, a_2, b_2, s, a, b, n, c, d, m, 1\}$ and $D(A_1)^* = \{0\}$. Since $ord(0^-) = ord(0^\sim) = \infty$, it follows that A_1 is a perfect pseudo-BCK(pP) algebra.*

Definition 3.3. *Let A be a pseudo-BCK(pP) algebra. The intersection of all maximal deductive systems of A is called the radical of A and it is denoted by $Rad(A)$.*

Proposition 3.5. *If A is a perfect pseudo-BCK(pP) algebra, then $Rad(A) = D(A)$.*

Proof. By theorem 3.1 it follows that $D(A)$ is the unique maximal deductive system of A, so $Rad(A) = D(A)$. □

Example 3.3. *Consider the perfect pseudo-BCK(pP) A_1 from Example 2.5. One can easily check that $Rad(A_1) = Rad_n(A_1) = D(A_1) = \{a_1, a_2, b_2, s, a, b, n, c, d, m, 1\}$.*

Remark 3.1. *If A is a local pseudo-BCK(pP) algebra and $x \in Rad(A)^*$, $y \in A$ such that $y \leq x$, then $y \in Rad(A)^*$.*

Theorem 3.2. *([6]) If A is a perfect pseudo-BCK(pP) algebra, then $Rad(A)$ is a normal deductive system of A.*

Remark 3.2. *If the pseudo-BCK(pP) algebra A is not perfect, then the above result is not always valid. Indeed, consider the pseudo-BCK(pP) algebra A from Example 2.4. Since A is not good, it is not a perfect pseudo-BCK(pP) algebra. Moreover, $D = \{a_2, s, a, b, n, c, d, 1\}$ is the unique maximal deductive system of A, so $Rad(A) = D$. But D is not a normal deductive system.*

Definition 3.4. *A pseudo-BCK(pP) algebra A is called peculiar if it satisfies the following conditions:*
(1) *A is a local good pseudo-BCK(pP) algebra;*
(2) *there is $x \in A \setminus \{1\}$ such that $ord(x) = \infty$;*
(3) *there is $x \in A$ such that $ord(x) < \infty$ and $(ord(x^-) < \infty$ or $ord(x^\sim) < \infty)$.*

Example 3.4. *a) The pseudo-BCK(pP) algebra A_1 from Example 2.5 is not peculiar. Indeed, the only element $x \in A_1$ such that $ord(x) < \infty$ is $x = 0$, but $ord(0^-) = ord(0^\sim) = ord(1) = \infty$. Hence, A_1 does not satisfy the condition (3) in the definition of a peculiar pseudo-BCK(pP) algebra.*
b) The pseudo-BCK(pP) algebra A from Example 2.4 is peculiar. Indeed, conditions $(1)-(3)$ in the definition of a peculiar pseudo-BCK(pP) algebra are satisfied:
(1) *A is a local pseudo-BCK(pP) algebra (see Example 3.1);*
(2) *$ord(a_2) = \infty$;*
(3) *$ord(a_1) < \infty$ and $ord(a_1^-) = ord(a_1) < \infty$.*

We denote by:
$\mathcal{PF}-$ the class of perfect pseudo-BCK(pP) algebras;
$\mathcal{LF}-$ the class of locally finite pseudo-BCK(pP) algebras;
$\mathcal{PC}-$ the class of peculiar pseudo-BCK(pP) algebras.

Theorem 3.3. *Let A be a local pseudo-BCK(pP) algebra, $A \neq \mathbf{L}_2 = \{0, 1\}$. Then exactly one of the following holds:*
(1) *$A \in \mathcal{PF}$;*
(2) *$A \in \mathcal{LF}$;*
(3) *$A \in \mathcal{PC}$.*

Proof. Assume that $A \notin \mathcal{PF}$ and $A \notin \mathcal{LF}$. Since $A \notin \mathcal{LF}$, it follows that there is $x \in A \setminus \{1\}$ such that $ord(x) = \infty$. From $A \notin \mathcal{PF}$ we get that there is $x \in A$ such that $ord(x) < \infty$ and $(ord(x^-) < \infty$ or $ord(x^\sim) < \infty)$. Thus, $A \in \mathcal{PC}$.
From the definitions of the classes $\mathcal{PF}$, $\mathcal{LF}$ and $\mathcal{PC}$ it follows that

$$\mathcal{PC} \cap \mathcal{LF} = \mathcal{PC} \cap \mathcal{PF} = \emptyset.$$

We prove that $A \in \mathcal{PF} \cap \mathcal{LF} = \mathbf{L}_2$.
Obviously, $\mathbf{L}_2$ is perfect and locally finite.
Let $A \neq \mathbf{L}_2 = \{0, 1\}$ be a locally finite pseudo-BCK(pP) algebra. Hence, there is

$x \in A$ such that $x \neq 0$ and $x \neq 1$. Since $a \neq 0$, we have $a^- \neq 1$. Indeed, if $a^- = 1$, we get $1 = a \to 0$, so $a = 1 \odot a \leq 0$, so $a = 0$, a contradiction.
Since A is locally finite, we get that $ord(x) < \infty$ and $ord(a^-) < \infty$.
It follows that A is not perfect. Hence, $A \in \mathcal{PF} \cap \mathcal{LF} = \mathbf{L}_2$.
Thus, exactly one of (1), (2), (3) holds. $\square$

Definition 3.5. *A pseudo-BCK(pP) algebra is called* bipartite *if there exists a maximal filter F of A such that $F \cup F^*_- = F \cup F^*_\sim = A$.*
A pseudo-BCK(pP) algebra A is called strongly bipartite *if $F \cup F^*_- = F \cup F^*_\sim = A$ for any maximal filter F of A.*

Theorem 3.4. *Any perfect pseudo-BCK(pP) algebra A is strongly bipartite.*

Proof. Since A is local, it follows that $D(A)$ is the unique maximal filter of A.
Applying Corollary 3.3 we have $D(A) \cup D(A)^*_- = D(A) \cup D(A)^*_\sim = A$. Thus, A is a strongly bipartite pseudo-BCK(pP) algebra. $\square$

4. Existence of Bosbach states on perfect pseudo-BCK algebras with pseudo-product

In this section we prove that any perfect pseudo-BCK algebra with pseudo-product admits at least a Bosbach state.
First, we recall some notions and results regarding the Bosbach state on pseudo-BCK algebras. For more details about this subject we refer the reader to [4].

Definition 4.1. *A Bosbach state on a bounded pseudo-BCK algebra A is a function $s : A \longrightarrow [0,1]$ such that the following conditions hold for any $x, y \in A$:*
(B_1) $s(x) + s(x \to y) = s(y) + s(y \to x)$;
(B_2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$;
(B_3) $s(0) = 0$ *and* $s(1) = 1$.

Example 4.1. *Consider the bounded pseudo-BCK lattice A_1 from Example 2.6. The function $s : A_1 \longrightarrow [0,1]$ defined by: $s(0) = 0, s(a) = 1, s(b) = 1, s(c) = 1, s(d) = 1, s(1) = 1$ is the unique Bosbach state on A_1.*

Not every bounded pseudo-BCK algebra however has a Bosbach state on it.

Example 4.2. *Consider the bounded pseudo-BCK lattice A from Example 2.2.*
One can prove that A has no Bosbach states on it.
Indeed, assume that A admits a Bosbach state s such that $s(0) = 0$, $s(a) = \alpha$, $s(b) = \beta$, $s(c) = \gamma$, $s(1) = 1$. From $s(x) + s(x \to y) = s(y) + s(y \to x)$, taking $x = a, y = 0$, $x = b, y = 0$ and respectively $x = c, y = 0$ we get $\alpha = 1$, $\beta = 0$, $\gamma = 1$.
On the other hand, taking $x = b, y = 0$ in $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$ we get $\beta + 0 = 0 + 1$, so $0 = 1$ which is a contradiction. Hence, A does not admit a Bosbach state.

Proposition 4.1. ([4]) *Let A be a bounded pseudo-BCK algebra and s a Bosbach state on A. Then, for all $x, y \in A$ the following properties hold:*
(1) $y \leq x$ *implies* $s(y) \leq s(x)$ *and* $s(x \to y) = s(x \rightsquigarrow y) = 1 - s(x) + s(y)$;
(2) $s(x \to y) = 1 - s(x \vee y) + s(y)$ *and* $s(x \rightsquigarrow y) = 1 - s(x \cup y) + s(y)$;

(3) $s(x \vee y) = s(y \vee x)$ *and* $s(x \cup y) = s(y \cup x)$;
(4) $s(x^-) = s(x^\sim) = 1 - s(x)$;
(5) $s(x^{-\sim}) = s(x^{\sim-}) = s(x^{--}) = s(x^{\sim\sim}) = s(x)$;
(6) $s(x^- \rightsquigarrow y^-) = s(y^{-\sim} \to x^{-\sim}) = s(y \to x^{-\sim})$ *and*
$s(x^\sim \to y^\sim) = s(y^{\sim-} \rightsquigarrow x^{\sim-}) = s(y \rightsquigarrow x^{\sim-})$;
(7) $s(x \to y^\sim) = s(y^{\sim-} \rightsquigarrow x^-) = s(x^{-\sim} \to y^\sim)$ *and*
$s(x \rightsquigarrow y^-) = s(y^{-\sim} \to x^\sim) = s(x^{\sim-} \rightsquigarrow y^-)$.

Theorem 4.1. *Any perfect pseudo-BCK(pP) algebra admits a Bosbach state.*

Proof. Let A be a perfect pseudo-BCK(pP) algebra, so $A = Rad(A) \cup Rad(A)^*$. Consider the map $s : A \to [0,1]$ defined by

$$s(x) = \begin{cases} 1, & \text{if } x \in Rad(A) \\ 0, & \text{if } x \in Rad(A)^*. \end{cases}$$

We will show hat s is a Bosbach state on A. Obviously, $s(1) = 1$ and $s(0) = 0$.
In order to prove conditions (B_1) and (B_2) we consider the following cases :
(1) $x, y \in Rad(A)$.
Obviously, $s(x) = s(y) = 1$. Since $Rad(A)$ is a filter of A and $x \leq y \to x$, $y \leq x \to y$, it follows that $x \to y, y \to x \in Rad(A)$. Hence, $s(x \to y) = s(y \to x) = 1$.
Similarly $s(x \rightsquigarrow y) = s(y \rightsquigarrow x) = 1$, so conditions (B_1) and (B_2) are verified.
(2) $x, y \in Rad(A)^*$.
In this case $s(x) = s(y) = 0$ and we will prove that $x \to y, y \to x \in Rad(A)$. Indeed, assume that $x \to y \in Rad(A)^*$. Since $x \leq x^{-\sim}$, it follows that $x^{-\sim} \to y \leq x \to y$, so $x^{-\sim} \to y \in Rad(A)^*$. But, $x^- \leq x^{-\sim} \to y$, hence $x^- \in Rad(A)^*$, that is, $x \in Rad(A)$ which is a contradiction. It follows that $x \to y \in Rad(A)$ and similarly, $y \to x \in Rad(A)$. Hence, $s(x \to y) = s(y \to x) = 1$. In the same way we can prove that $s(x \rightsquigarrow y) = s(y \rightsquigarrow x) = 1$, so (B_1) and (B_2) are verified.
(3) $x \in Rad(A), y \in Rad(A)^*$.
Obviously, $s(x) = 1$ and $s(y) = 0$. Because $x \leq y \to x$ we get $y \to x \in Rad(A)$.
We show that $x \to y \in Rad(A)^*$. Indeed, assume that $x \to y \in Rad(A)$.
Because $y \leq y^{-\sim}$ we have $x \to y \leq x \to y^{-\sim}$, so $x \to y^{-\sim} \in Rad(A)$. It means that $(x \odot y^\sim)^- \in Rad(A)$, that is, $x \odot y^\sim \in Rad(A)^*$. On the other hand, since $Rad(A)$ is a filter of A and $x, y^\sim \in Rad(A)$ we have $x \odot y^\sim \in Rad(A)$. We conclude that $x \to y \in Rad(A)^*$, so $s(x \to y) = 0$ and $s(y \to x) = 1$. Similarly, $s(x \rightsquigarrow y) = 0$ and $s(y \rightsquigarrow x) = 1$. Thus, conditions (B_1) and (B_2) are verified.
(4) $x \in Rad(A)^*, y \in Rad(A)$.
This case can be treated in the same manner as the case (3).
We conclude that s is a Bosbach state on A. □

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