Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 34(1), 2007, Pages 43–58
ISSN: 1223-6934

Ostrowski Type Inequalities on Time Scales

Cristian Dinu

Abstract. We present an improved variant of the Ostrowski's inequality for continuous functions on time scales.

2000 Mathematics Subject Classification. 26D15 .
Key words and phrases. Time scale, convex function, dynamic derivatives, Ostrowski's inequality.

1. Introduction

Recently, new developments of the theory and applications of dynamic derivatives on time scales were made. The study provides a unification and extension of traditional differential and difference equations and, in the same time, it is an unification of the discrete theory with the continuous theory, from the theoretical point of view. Moreover, it is a crucial tool in many computational and numerical applications. Based on the well-known Δ (delta) and ∇ (nabla) dynamic derivatives, a combined dynamic derivative, so called $\diamond_\alpha$ (diamond-α) dynamic derivative, was introduced as a linear combination of Δ and ∇ dynamic derivatives on time scales. The diamond-α dynamic derivative reduces to the Δ derivative for $\alpha = 1$ and to the ∇ derivative for $\alpha = 0$. On the other hand, it represents a "weighted dynamic derivative" on any uniformly discrete time scale when $\alpha = 1/2$. See [1], [2] and [4] for the basic rules of the calculus associated with the diamond-α dynamic derivatives.

In [7], A. M. Ostrowski proved an interesting and useful inequality

Theorem 1.1. *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b), whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b). Then, for all $x \in [a, b]$ we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty. \tag{1}$$

Here $\|f'\|_\infty = \sup_{x \in (a,b)} |f'(x)|$

In [6], B. G. Pachpatte improved Ostrowski's inequality and gave some applications.

The aim of this paper is to prove a variant of Ostrowski's inequality for the time scales.

In section 2 we review the necessary background on time scales. In section 3 we prove our main results and detail several applications.

Received: 10 February 2008.

2. Preliminaries

A *time scale* (or measure chain) is any nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$ (endowed with the topology of subspace of $\mathbb{R}$).

Throughout this paper $\mathbb{T}$ will denote a time scale.

For all $t, r \in \mathbb{T}$, we define the *forward jump operator* σ and the *backward jump operator* ρ by the formulas:

$$\sigma(t) = \inf\{\tau \in \mathbb{T} | \tau > t\} \in \mathbb{T}, \quad \rho(r) = \sup\{\tau \in \mathbb{T} | \tau < r\} \in \mathbb{T}.$$

In this definition

$$\inf \emptyset := \sup \mathbb{T}, \quad \sup \emptyset := \inf \mathbb{T}.$$

If $\sigma(t) > t$, then t is said to be *right-scattered*, and if $\rho(r) < r$, then r is said to be *left-scattered*. Points that are simultaneously right-scattered and left-scattered are called *isolated*. If $\sigma(t) = t$, then t is said to be *right dense*, and if $\rho(r) = r$, then r is said to be *left dense*. Points that are simultaneously right-dense and left-dense are called *dense*.

The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by

$$\mu(t) := \sigma(t) - t$$

and

$$\nu(t) := t - \rho(t)$$

are called, respectively, the *forward* and *backward graininess functions*.

If $\mathbb{T}$ has a right-scattered minimum m, then define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, then define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Finally, put $\mathbb{T}^\kappa_\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

Definition 2.1. *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define the delta derivative of f in t, to be the number denoted $f^\Delta(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, we define the nabla derivative of f in t, to be the number denoted $f^\nabla(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| < \varepsilon|\rho(t) - s|$$

for all $s \in V$.

We say that f is *delta differentiable* on $\mathbb{T}^\kappa$, provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and that f is *nabla differentiable* on $\mathbb{T}_\kappa$, provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

If $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f^\nabla(t) = f'(t).$$

If $\mathbb{T} = \mathbb{Z}$, then

$$f^\Delta(t) = f(t+1) - f(t)$$

is the forward difference operator, while

$$f^\nabla(t) = f(t) - f(t-1)$$

is the backward difference operator.

For a function $f : \mathbb{T} \to \mathbb{R}$ we define $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, (that is $f^\sigma = f \circ \sigma$). We also define $f^\rho : \mathbb{T} \to \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, (that is $f^\rho = f \circ \rho$).

For all $t \in \mathbb{T}^\kappa$ we have the following properties:

(i) If f is delta differentiable at t, then f is continuous at t.
(ii) If f is left continuous at t and t is right-scattered, then f is delta differentiable at t with $f^\Delta(t) = \frac{f^\sigma(t)-f(t)}{\mu(t)}$.
(iii) If t is right-dense, then f is delta differentiable at t if and only if the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^\Delta(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.
(iv) If f is delta differentiable at t, then $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$.

In the same manner, for all $t \in \mathbb{T}_\kappa$ we have the following properties:

(i) If f is nabla differentiable at t, then f is continuous at t.
(ii) If f is right continuous at t and t is left-scattered, then f is nabla differentiable at t with $f^\nabla(t) = \frac{f(t)-f^\rho(t)}{\nu(t)}$.
(iii) If t is left-dense, then f is nabla differentiable at t if and only if the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^\nabla(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.
(iv) If f is nabla differentiable at t, then $f^\rho(t) = f(t) + \nu(t)f^\nabla(t)$.

Definition 2.2. *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by C_{rd} the set of all rd-continuous functions.*

A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous, if it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits are finite at all right-dense points in $\mathbb{T}$. We denote by C_{ld} the set of all ld-continuous functions.

It is easy to remark that the set of continuous functions on $\mathbb{T}$ contains both C_{rd} and C_{ld}.

Definition 2.3. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$, for all $t \in \mathbb{T}^\kappa$. Then, we define the delta integral by $\int_a^t f(s)\Delta s = F(t) - F(a)$.*

A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $G^\nabla(t) = f(t)$, for all $t \in \mathbb{T}_\kappa$. Then, we define the nabla integral by $\int_a^t f(s)\Delta s = G(t) - G(a)$.

According to Theorem 1.74 in [2], every rd-continuous function has a delta antiderivative and every ld-continuous function has a nabla antiderivative.

Theorem 2.1. (Theorem 1.75, in [2])

(i) If $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$, then

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t).$$

(ii) If $f \in C_{ld}$ and $t \in \mathbb{T}^\kappa$, then

$$\int_{\rho(t)}^t f(s)\nabla s = \nu(t)f(t).$$

Proof. We will prove only the assertion (ii), the other one can be treated in a similar manner. Let F be the nabla antiderivative of f, and

$$\int_{\rho(t)}^t f(s)\nabla s = F(t) - F(\rho(t)) = \nu(t)F^\nabla(t) = \nu(t)f(t).$$

□

Theorem 2.2. (Theorem 1.77, in [2]) *If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$ and $f, g \in C_{rd}$, then*

(i) $\int_a^b (f(t)+g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;
(ii) $\int_a^b \beta f(t)\Delta t = \beta \int_a^b f(t)\Delta t$;
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;
(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;
(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$;
(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$;
(vii) $\int_a^a f(t)\Delta t = 0$;
(viii) *if $f(t) \geq 0$ for all t, then $\int_a^b f(t)\Delta t \geq 0$;*
(ix) *if $f(t) \leq g(t)$ for all t, then $\int_a^b f(t)\Delta t \leq \int_a^b g(t)\Delta t$;*
(x) *if $f(t) \geq 0$ for all t, then $f \equiv 0$ if and only if $\int_a^b f(t)\Delta t = 0$;*
(xi) *if $|f(t)| \leq g(t)$ on $[a, b)$, then*
$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t.$$

In Theorem 2.2, (xi), if we choose $g(t) = |f(t)|$ on $[a, b]$ we obtain
$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b |f(t)|\Delta t. \tag{2}$$

A similar theorem works for the nabla antiderivative, for $f, g \in C_{ld}$. Now, we give a brief introduction of the diamond-α dynamic derivative and the diamond-α integral.

Definition 2.4. *Let $\mathbb{T}$ be a time scale and for $s, t \in \mathbb{T}_\kappa^\kappa$ we define $\mu_{ts} = \sigma(t) - s$, $\nu_{ts} = \rho(t) - s$. For $f : \mathbb{T} \to \mathbb{R}$ we define the diamond-α dynamic derivative of f in t to be number denoted $f^{\diamond_\alpha}(t)$ (when it exists), with the property that, for any $\epsilon > 0$, there is a neighborhood U of t such that for all $s \in U$*
$$|\alpha[f(\sigma(t)) - f(s)]\nu_{ts} + (1-\alpha)[f(\rho(t)) - f(s)]\mu_{ts} - f^{\diamond_\alpha}(t)\mu_{ts}\nu_{ts}| < \epsilon|\mu_{ts}\nu_{ts}|.$$

A function is called *diamond-α differentiable* on $\mathbb{T}_\kappa^\kappa$ if $f^{\diamond_\alpha}(t)$ exists for all $t \in \mathbb{T}_\kappa^\kappa$. If $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}$ in the sense of Δ and ∇, then f is diamond-α differentiable at $t \in \mathbb{T}_\kappa^\kappa$, and the diamond-$\alpha$ derivative $f^{\diamond_\alpha}(t)$ is given by
$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha)f^{\nabla}(t), \quad 0 \leq \alpha \leq 1.$$

As it can be seen, f is diamond-α differentiable, for $0 \leq \alpha \leq 1$ if and only if f is Δ and ∇ differentiable. It is obvious that for $\alpha = 1$ the diamond-α derivative reduces to the standard Δ derivative and for $\alpha = 0$ the diamond-α derivative reduces to the standard ∇ derivative. For $\alpha \in (0, 1)$ it represents a "weighted dynamic derivative".

We present here some operations with the diamond-α derivative. For that, let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then,

- $f + g : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ and
$$(f+g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) + g^{\diamond_\alpha}(t)$$
- if $c \in \mathbb{R}$ and $cf : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ and
$$(cf)^{\diamond_\alpha}(t) = cf^{\diamond_\alpha}(t).$$
- $fg : \mathbb{T} \to \mathbb{R}$ is diamond-α differentiable at $t \in \mathbb{T}$ and
$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t).$$

Definition 2.5. *Let $f : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$, then the diamond-α integral of f from a to b is defined by*

$$\int_a^b f(t)\diamond_\alpha t = \alpha \int_a^b f(t)\Delta t + (1-\alpha) \int_a^b f(t)\nabla t, \quad 0 \le \alpha \le 1.$$

The combined derivative $\diamond_\alpha$ is not a dynamic derivative, since we do not have a $\diamond_\alpha$ anti-derivative. In general,

$$\left(\int_a^t f(s)\diamond_\alpha s \right)^{\diamond_\alpha} \ne f(t), \quad t \in \mathbb{R},$$

but we still have some of the "classical" properties, as one can easily be deduced from Theorem 2.2 and its analogue for the nabla integral.

Theorem 2.3. *If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$ and f, g continuous functions, then*

(i) $\int_a^b (f(t)+g(t))\diamond_\alpha t = \int_a^b f(t)\diamond_\alpha t + \int_a^b g(t)\diamond_\alpha t$;
(ii) $\int_a^b \beta f(t)\diamond_\alpha t = \beta \int_a^b f(t)\diamond_\alpha t$;
(iii) $\int_a^b f(t)\diamond_\alpha t = -\int_b^a f(t)\diamond_\alpha t$;
(iv) $\int_a^b f(t)\diamond_\alpha t = \int_a^c f(t)\diamond_\alpha t + \int_c^b f(t)\diamond_\alpha t$;
(v) $\int_a^a f(t)\diamond_\alpha t = 0$;
(vi) *if $f(t) \ge 0$ for all t, then $\int_a^b f(t)\diamond_\alpha t \ge 0$;*
(vii) *if $f(t) \le g(t)$ for all t, then $\int_a^b f(t)\diamond_\alpha t \le \int_a^b g(t)\diamond_\alpha t$;*
(viii) *if $f(t) \ge 0$ for all t, then $f \equiv 0$ if and only if $\int_a^b f(t)\diamond_\alpha t = 0$;*
(ix) *if $|f(t)| \le g(t)$ on $[a, b)$, then*

$$\left| \int_a^b f(t)\diamond_\alpha t \right| \le \int_a^b g(t)\diamond_\alpha t.$$

In Theorem 2.3, (ix), if we choose $g(t) = |f(t)|$ on $[a, b]$ we have

$$\left| \int_a^b f(t)\diamond_\alpha t \right| \le \int_a^b |f(t)|\diamond_\alpha t. \tag{3}$$

3. Main results

In this section we present an improved variant of Ostrowski's inequality, for time scales. For a function $f : \mathbb{T} \to \mathbb{R}$, which is delta and nabla differentiable, we define $\|f^\Delta\|_\infty = \sup_{t\in\mathbb{T}^\kappa} |f^\Delta(t)|$ and $\|f^\nabla\|_\infty = \sup_{t\in\mathbb{T}_\kappa} |f^\nabla(t)|$. We also define $\|f^{\diamond_\alpha}\|_\infty = \sup_{t\in\mathbb{T}^\kappa_\kappa} |f^{\diamond_\alpha}(t)|$. Obviously,

$$\begin{aligned} \|f^{\diamond_\alpha}\|_\infty &= \sup_{t\in\mathbb{T}^\kappa_\kappa} |\alpha f^\Delta(t) + (1-\alpha) f^\nabla(t)| \\ &\le \alpha \sup_{t\in\mathbb{T}^\kappa_\kappa} |f^\Delta(t)| + (1-\alpha) \sup_{t\in\mathbb{T}^\kappa_\kappa} |f^\nabla(t)| \\ &= \alpha \|f^\Delta\|_\infty + (1-\alpha)\|f^\nabla\|_\infty. \end{aligned} \tag{4}$$

We have equality in (4) if both f^Δ and f^∇ attain their maximum value at the same point.

We need the following technical lemmas, which work for all time scales.

Lemma 3.1. *Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function and $a, b \in \mathbb{T}$.*

(i) *If f is nondecreasing on $\mathbb{T}$ then*
$$(b-a)f(a) \le \int_a^b f(t)\Delta t \le \int_a^b \tilde{f}(t)dt \le \int_a^b f(t)\nabla t \le (b-a)f(b),$$
where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function such that $f(t) = \tilde{f}(t)$, for all $t \in \mathbb{T}$.

(ii) *If f is nonincreasing on $\mathbb{T}$ then*
$$(b-a)f(a) \ge \int_a^b f(t)\Delta t \ge \int_a^b \tilde{f}(t)dt \ge \int_a^b f(t)\nabla t \ge (b-a)f(b),$$
where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a continuous nonincreasing function such that $f(t) = \tilde{f}(t)$, for all $t \in \mathbb{T}$.

In both cases, there exists an $\alpha_T \in [0,1]$ such that
$$\int_a^b f(t)\diamond_{\alpha_T} t = \int_a^b \tilde{f}(t)dt.$$

Proof. (i) We start by noticing that if $\mathbb{T} = \{a, b\}$ then by Theorem 2.1, we have
$$\int_a^b f(t)\Delta t = \int_a^{\sigma(a)} f(t)\Delta t = f(a)(b-a),$$
while, if $\mathbb{T} = [a, b]$ then
$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

It suffices to prove that, for monotone functions, the value of $\int_a^b f(t)\Delta t$, for a general time scale $\mathbb{T}$, remains between the values of $\int_a^b f(t)\Delta t$ for $\mathbb{T} = \{a, b\}$ and for $\mathbb{T} = [a, b]$.

Now, let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function such that $f(t) = \tilde{f}(t)$, for all $t \in \mathbb{T}$. First, we will show that by adding a point or an interval, the corresponding integral increases.

Let us suppose that we add a point c to $\mathbb{T}$, where $a < c < b$. If $\mathbb{T}_1 = \mathbb{T} \cup \{c\}$ and $c \notin \mathbb{T}$ is an isolated point of $\mathbb{T}_1$ (with $\int_a^b f(t)\Delta_1 t$ the corresponding integral), then
$$\begin{aligned}
\int_a^b f(t)\Delta_1 t &= \int_a^c f(t)\Delta_1 t + \int_c^b f(t)\Delta_1 t \\
&= \int_a^{\rho(c)} f(t)\Delta_1 t + \int_{\rho(c)}^c f(t)\Delta_1 t + \int_c^{\sigma(c)} f(t)\Delta_1 t + \int_{\sigma(c)}^b f(t)\Delta_1 t \\
&= \int_a^{\rho(c)} f(t)\Delta t + \int_{\rho(c)}^c f(t)\Delta_1 t + \int_c^{\sigma(c)} f(t)\Delta_1 t + \int_{\sigma(c)}^b f(t)\Delta t \\
&= \int_a^b f(t)\Delta t - \int_{\rho(c)}^{\sigma(c)} f(t)\Delta t + \int_{\rho(c)}^c f(t)\Delta_1 t + \int_c^{\sigma(c)} f(t)\Delta_1 t \\
&= \int_a^b f(t)\Delta t - f(\rho(c))(\sigma(c)-\rho(c)) + f(\rho(c))(c-\rho(c)) + f(c)(\sigma(c)-c) \\
&= \int_a^b t\Delta t + (f(c) - f(\rho(c)))(\sigma(c)-c) \\
&\ge \int_a^b f(t)\Delta t.
\end{aligned}$$

In the same manner, we prove that if we add an interval, the corresponding integral remains in the same interval. So, let us denote $\mathbb{T}_1 = \mathbb{T} \cup [c, d]$, with $a < c < d < b$ and $\mathbb{T} \cap [c, d] = \emptyset$ then

$$\begin{aligned}
&\int_a^b f(t)\Delta_1 t \\
&= \int_a^{\rho(c)} f(t)\Delta_1 t + \int_{\rho(c)}^{c} f(t)\Delta_1 t + \int_c^d f(t)\Delta_1 t + \int_d^{\sigma(d)} f(t)\Delta_1 t + \int_{\sigma(d)}^b f(t)\Delta_1 t \\
&= \int_a^{\rho(c)} f(t)\Delta t + \int_{\rho(c)}^{c} f(t)\Delta_1 t + \int_c^d f(t)\Delta_1 t + \int_d^{\sigma(d)} f(t)\Delta_1 t + \int_{\sigma(d)}^b f(t)\Delta t \\
&= \int_a^b f(t)\Delta t - \int_{\rho(c)}^{\sigma(d)} f(t)\Delta t + \int_{\rho(c)}^{c} f(t)\Delta_1 t + \int_c^d f(t)\Delta_1 t + \int_d^{\sigma(d)} f(t)\Delta_1 t \\
&= \int_a^b f(t)\Delta t - f(\rho(c))(\sigma(d) - \rho(c)) + f(\rho(c))(c - \rho(c)) + \int_c^d \tilde{f}(t)dt + f(d)(\sigma(d) - d) \\
&\geq \int_a^b f(t)\Delta t - f(\rho(c))(d - c) + (d - c)\tilde{f}(s) \\
&\geq \int_a^b f(t)\Delta t,
\end{aligned}$$

where $s \in (c, d)$ is the point from Mean Value Theorem.

Using the same methods, we show that if we "extract" an isolated point or an interval from an initial times scale, the corresponding integral decreases. And so, the value of $\int_a^b f(t)\Delta t$ is between its minimum value (corresponding to $\mathbb{T} = \{a, b\}$) and its maximum value (corresponding to $\mathbb{T} = [a, b]$), that is

$$(b - a)f(a) \leq \int_a^b f(t)\Delta t \leq \int_a^b \tilde{f}(t)dt.$$

The proof is similar in the case of nonincreasing functions and also, for the nabla integral. The final conclusion of the Lemma 3.1 is now clear if we take

$$\alpha_T = \frac{\int_a^b \tilde{f}(t)dt - \int_a^b f(t)\nabla t}{\int_a^b f(t)\Delta t - \int_a^b f(t)\nabla t}.$$

Then

$$\int_a^b \tilde{f}(t)dt = \alpha_T \int_a^b f(t)\Delta t + (1 - \alpha_T) \int_a^b f(t)\nabla t,$$

that is

$$\int_a^b f(t)\diamond_{\alpha_T} t = \int_a^b \tilde{f}(t)dt.$$

□

Remark 3.1. (i) *If f is nondecreasing on $\mathbb{T}$ then for $\alpha \leq \alpha_T$, we have*

$$\int_a^b f(t)\diamond_{\alpha_T} t \geq \int_a^b \tilde{f}(t)dt$$

while, if $\alpha \geq \alpha_T$, we have

$$\int_a^b f(t)\diamond_{\alpha_T} t \leq \int_a^b \tilde{f}(t)dt.$$

(ii) *If f is nonincreasing on $\mathbb{T}$ then for $\alpha \leq \alpha_T$, we have*

$$\int_a^b f(t)\diamondsuit_{\alpha_T} t \leq \int_a^b \tilde{f}(t)dt$$

while, if $\alpha \geq \alpha_T$, we have

$$\int_a^b f(t)\diamondsuit_{\alpha_T} t \geq \int_a^b \tilde{f}(t)dt.$$

If $\mathbb{T} = [a, b]$ or if f is constant, then α_T can be any real number from $[0, 1]$. Otherwise, $\alpha_T \in (0, 1)$

Now we will prove that if $f : \mathbb{T} \to \mathbb{R}$ is a linear function, (that is $f(t) = ut+v$) then $\int_a^b f(t)\Delta t$ and $\int_a^b f(t)\nabla t$ are symmetric with respect to $\int_a^b \tilde{f}(t)dt$, where $\tilde{f} : [a, b] \to \mathbb{R}$ is the corresponding linear function, defined on the interval $[a, b]$.

Lemma 3.2. *Let $f : \mathbb{T} \to \mathbb{R}$ be a linear function and let $\tilde{f} : [a, b] \to \mathbb{R}$ be the corresponding linear function. If $\int_a^b f(t)\Delta t = \int_a^b \tilde{f}(t)dt - C$, with $C \in \mathbb{R}$, then $\int_a^b f(t)\nabla t = \int_a^b \tilde{f}(t)dt + C$.*

Proof. We will start by considering the case of $f : \mathbb{T} \to \mathbb{R}$, $f(t) = t$. If $\mathbb{T} = [a, b]$, then $c = 0$ and the conclusion is clear. If $\mathbb{T} = [a, b] \smallsetminus (c, d)$, then

$$\begin{aligned}
\int_a^b t\Delta t &= \int_a^c t\Delta t + \int_c^d t\Delta t + \int_d^b t\Delta t \\
&= \int_a^c tdt + \int_c^{\sigma(c)} t\Delta t + \int_d^b tdt \\
&= \int_a^b tdt - \int_c^d tdt + c(d-c) \\
&= \int_a^b tdt - (d-c)\frac{d+c}{2} + c(d-c) \\
&= \int_a^b tdt - \frac{(d-c)^2}{2}
\end{aligned}$$

while

$$\begin{aligned}
\int_a^b t\nabla t &= \int_a^c t\nabla t + \int_c^d t\nabla t + \int_d^b t\nabla t \\
&= \int_a^c tdt + \int_\rho (d)^d t\nabla t + \int_d^b tdt \\
&= \int_a^b tdt - \int_c^d tdt + d(d-c) \\
&= \int_a^b tdt - (d-c)\frac{d+c}{2} + d(d-c) \\
&= \int_a^b tdt + \frac{(d-c)^2}{2}
\end{aligned}$$

and, obvious, if we choose $C = \frac{(d-c)^2}{2}$ the conclusion is clear.

By repeating the same arguments several times, we can "extract" any number of intervals from $[a, b]$ and get the same conclusion.

If we "extract" an interval, but we "add" an isolated point (that is $\mathbb{T} = [a,b] \setminus ((c,e) \cup (e,d)) = [a,c] \cup \{e\} \cup [d,b]$), then

$$\begin{aligned}
\int_a^b t\Delta t &= \int_a^c t\Delta t + \int_c^e t\Delta t + \int_e^d t\Delta t + \int_d^b t\Delta t \\
&= \int_a^c tdt + \int_c^{\sigma(c)} t\Delta t + \int_e^{\sigma(e)} t\Delta t + \int_d^b tdt \\
&= \int_a^b tdt - \int_c^d tdt + c(e-c) + e(d-e) \\
&= \int_a^b tdt - (d-c)\frac{d+c}{2} + e(c+d) - c^2 - e^2 \\
&= \int_a^b tdt - \frac{d^2}{2} - \frac{c^2}{2} + e(c+d) - e^2
\end{aligned}$$

while

$$\begin{aligned}
\int_a^b t\nabla t &= \int_a^c t\nabla t + \int_c^e t\nabla t + \int_e^d t\nabla t + \int_d^b t\nabla t \\
&= \int_a^c tdt + \int_{\rho(e)}^e t\nabla t + \int_{\rho(d)}^d t\nabla t + \int_d^b tdt \\
&= \int_a^b tdt - \int_c^d tdt + e(e-c) + d(d-e) \\
&= \int_a^b tdt - (d-c)\frac{d+c}{2} - e(c+d) + d^2 + e^2 \\
&= \int_a^b tdt + \frac{d^2}{2} + \frac{c^2}{2} - e(c+d) + e^2
\end{aligned}$$

and thus, for $C = \frac{(e-c)^2}{2} + \frac{(d-e)^2}{2}$ we get the conclusion.

For a general linear function, $f(t) = ut + v$, we have

$$\int_a^b f(t)\Delta t = \int_a^b (ut+v)\Delta t = u(\int_a^b tdt - C) + v(b-a) = u\int_a^b tdt - uC + v(b-a)$$

and

$$\int_a^b f(t)\nabla t = \int_a^b (ut+v)\nabla t = u(\int_a^b tdt + C) + v(b-a) = u\int_a^b tdt + uC + v(b-a)$$

so that $\int_a^b f(t)\Delta t = \int_a^b \tilde{f}(t)dt - uC$ and $\int_a^b f(t)\nabla t = \int_a^b \tilde{f}(t)dt + uC$. $\square$

Definition 3.1. *Let $\mathbb{T}$ be a time scale. We define the* measure of graininess *between a and b to be the function $G : \mathbb{T} \times \mathbb{T} \to \mathbb{R}_+$ by*

$$G(a,b) = \sum_{a \le t \le b} \frac{\mu(t)^2}{2} = \sum_{a \le t \le b} \frac{\nu(t)^2}{2}.$$

In other words, the function G measures the square of distances between all scattered points between a and b and it depends on the "geometry" of the time scale $\mathbb{T}$.

Remark 3.2. *The difference between* $\int_a^b t\Delta t$ *and* $\int_a^b tdt$ *depends on the measure of graininess function. In fact, we have*

$$\int_a^b t\Delta t = \int_a^b tdt - G(a,b).$$

The proof uses the same methods as the prof of Lemma 3.2, so we will omit the details.

Notice that

$$\int_a^b t\nabla t = \int_a^b tdt + G(a,b).$$

Based on the previous remarks, we can compute $\int_a^b |t-s|\diamond_\alpha s$.

Corollary 3.1. *Let* $\mathbb{T}$ *be a time scale. Then*

$$\int_a^b |t-s|\diamond_\alpha s = \frac{(x-a)^2+(b-x)^2}{2} + (1-2\alpha)(G(x,b)-G(a,x))$$

where G *is the function introduced in Definition 3.1.*

Proof. Using Remark 3.2 we have

$$\begin{aligned}\int_a^b |t-s|\diamond_\alpha s &= \int_a^t (t-s)\diamond_\alpha s + \int_t^b (s-t)\diamond_\alpha s \\ &= t(t-a) - \int_a^t s\diamond_\alpha s + t(b-t) + \int_t^b s\diamond_\alpha s \\ &= \frac{(x-a)^2+(b-x)^2}{2} + (1-2\alpha)(G(x,b)-G(a,x)).\end{aligned}$$

□

Now we are able to state our main result.

Theorem 3.1. *Let* $f, g : \mathbb{T} \to \mathbb{R}$ *be continuous functions on* $\mathbb{T}$*, whose delta and nabla derivative are bounded (i.e.* $\|f^\Delta\|_\infty, \|g^\Delta\|_\infty, \|f^\nabla\|_\infty, \|g^\nabla\|_\infty < \infty$*). Then*

$$\begin{aligned}&\left| f(t)g(t) - \frac{1}{2(b-a)}\left[g(t)\int_a^b f(s)\diamond_\alpha s + f(t)\int_a^b g(s)\diamond_\alpha s\right]\right| \\ &\quad\le \frac{1}{2}\{\alpha[|g(t)|\|f^\Delta\|_\infty + |f(t)|\|g^\Delta\|_\infty] + (1-\alpha)[|g(t)|\|f^\nabla\|_\infty + |f(t)|\|g^\nabla\|_\infty]\} \\ &\qquad\cdot\left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2 + (1-2\alpha)\frac{G(t,b)-G(a,t)}{(b-a)^2}\right](b-a),\end{aligned}$$

for any $t \in \mathbb{T}$*, where* G *is the measure of graininess between* a *and* b*.*

Proof. For any $t, s \in \mathbb{T}$ with $a \le t, s \le b$ we have the following

$$f(t) - f(s) = \int_s^t f^\Delta(\tau)\Delta\tau \tag{5}$$

and

$$g(t) - g(s) = \int_s^t g^\Delta(\tau)\Delta\tau \tag{6}$$

If we multiply both sides of (5) and (6) by $f(t)$ and $g(t)$, respectively and summing up , we get

$$2f(t)g(t) - [g(t)f(s) + f(t)g(s)] = g(t)\int_s^t f^{\Delta}(\tau)\Delta\tau + f(t)\int_s^t g^{\Delta}(\tau)\Delta\tau. \tag{7}$$

And we do the same for nabla derivatives:

$$f(t) - f(s) = \int_s^t f^{\nabla}(\tau)\nabla\tau \tag{8}$$

and

$$g(t) - g(s) = \int_s^t g^{\nabla}(\tau)\nabla\tau. \tag{9}$$

If we multiply both sides of (8) and (9) by $f(t)$ and $g(t)$, respectively and summing, we get

$$2f(t)g(t) - [g(t)f(s) + f(t)g(s)] = g(t)\int_s^t f^{\nabla}(\tau)\nabla\tau + f(t)\int_s^t g^{\nabla}(\tau)\nabla\tau. \tag{10}$$

Multiplying both sides of (7) and (10) by α and $1-\alpha$, respectively and summing, we get

$$\begin{aligned}2f(t)g(t) - [g(t)f(s) + f(t)g(s)] =& g(t)\left(\alpha\int_s^t f^{\Delta}(\tau)\Delta\tau + (1-\alpha)\int_s^t f^{\nabla}(\tau)\nabla\tau\right)\\ &+ f(t)\left(\alpha\int_s^t g^{\Delta}(\tau)\Delta\tau + (1-\alpha)\int_s^t g^{\nabla}(\tau)\nabla\tau\right).\end{aligned} \tag{11}$$

If we take the diamond-α integral on both sides of (11) with respect to s, from a to b, after dividing all by $2(b-a)$, we have:

$$\begin{aligned}f(t)g(t) - &\frac{1}{2(b-a)}\left[g(t)\int_a^b f(s)\diamond_\alpha s + f(t)\int_a^b g(s)\diamond_\alpha s\right]\\ &= \frac{1}{2(b-a)}\int_a^b \left\{g(t)\left(\alpha\int_s^t f^{\Delta}(\tau)\Delta\tau + (1-\alpha)\int_s^t f^{\nabla}(\tau)\nabla\tau\right)\right.\\ &\quad\left. + f(t)\left(\alpha\int_s^t g^{\Delta}(\tau)\Delta\tau + (1-\alpha)\int_s^t g^{\nabla}(\tau)\nabla\tau\right)\right\}\diamond_\alpha s.\end{aligned} \tag{12}$$

Using the properties of modulus and (3) in (12), we have

$$\begin{aligned}
&\left|f(t)g(t) - \frac{1}{2(b-a)}\left[g(t)\int_a^b f(s)\diamond_\alpha s + f(t)\int_a^b g(s)\diamond_\alpha s\right]\right| \\
&\quad\le \frac{1}{2(b-a)}\int_a^b \{|g(t)|[\alpha\|f^\Delta\|_\infty + (1-\alpha)\|f^\nabla\|_\infty]|t-s| \\
&\qquad + |f(t)|[\alpha\|g^\Delta\|_\infty + (1-\alpha)\|^\nabla\|_\infty]|t-s|\}\diamond_\alpha s \\
&\quad= \frac{1}{2(b-a)}\{|g(t)|[\alpha\|f^\Delta\|_\infty + (1-\alpha)\|f^\nabla\|_\infty] \\
&\qquad + |f(t)|[\alpha\|g^\Delta\|_\infty + (1-\alpha)\|g^\nabla\|_\infty]\}\int_a^b |t-s|\diamond_\alpha s \\
&\quad= \frac{1}{2(b-a)}\{|g(t)|[\alpha\|f^\Delta\|_\infty + (1-\alpha)\|f^\nabla\|_\infty] \\
&\qquad + |f(t)|[\alpha\|g^\Delta\|_\infty + (1-\alpha)\|g^\nabla\|_\infty]\} \\
&\qquad \cdot\left[\frac{(t-a)^2+(b-t)^2}{2} + (1-2\alpha)(G(t,b)-G(a,t))\right] \\
&\quad= \frac{1}{2}\{|g(t)|[\alpha\|f^\Delta\|_\infty + (1-\alpha)\|f^\nabla\|_\infty] + |f(t)|[\alpha\|g^\Delta\|_\infty + (1-\alpha)\|g^\nabla\|_\infty]\} \\
&\qquad \cdot\left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2 + (1-2\alpha)\frac{G(t,b)-G(a,t)}{(b-a)^2}\right](b-a)
\end{aligned}$$

and the conclusion is now clear. $\square$

Remark 3.3. (i) *If $\mathbb{T} = \mathbb{R}$ and the functions are also differentiable, $f^\Delta = f'$, $g^\nabla = g'$ and $G(a,t) = G(t,b) = 0$, then we retrieve Theorem 2.1 from* [6].

(ii) *If $\mathbb{T}$ is a reunion of intervals from $\mathbb{R}$, then the functions are not differentiable in the end points of the intervales, but $f^\Delta = f'_-$, $f^\nabla = f'_+$, $g^\Delta = g'_-$, $g^\nabla = g'_+$, then we get an extended variant of Theorem 2.1 from* [6].

(iii) *If $\alpha = \frac{1}{2}$ or if $G(a,t) = G(t,b)$ for a time scale $\mathbb{T}$ (we call such time scale G-symmetric to t), then we have*

$$\begin{aligned}
&\left|f(t)g(t) - \frac{1}{2(b-a)}\left[g(t)\int_a^b f(s)\diamond_\alpha s + f(t)\int_a^b g(s)\diamond_\alpha s\right]\right| \\
&\quad\le \frac{1}{2}\{\alpha[|g(t)|\|f^\Delta\|_\infty + |f(t)|\|g^\Delta\|_\infty] + (1-\alpha)[|g(t)|\|f^\nabla\|_\infty + |f(t)|\|g^\nabla\|_\infty]\} \\
&\qquad \cdot\left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right](b-a).
\end{aligned}$$

(iv) *If $\alpha = 1$ and $\alpha = 0$, then one can find the delta variant and the nabla variant, respectively, of Theorem 3.1 for time scales.*

(v) *If both f^Δ and f^∇ attain their maximum in the same point then we have*

$$\left|f(t)g(t)-\frac{1}{2(b-a)}\left[g(t)\int_a^b f(s)\diamond_\alpha s+f(t)\int_a^b g(s)\diamond_\alpha s\right]\right|$$
$$\leq \frac{1}{2}\{|g(t)|\|f^{\diamond_\alpha}\|_\infty+|f(t)|\|g^{\diamond_\alpha}\|_\infty\}$$
$$\cdot\left[\frac{1}{4}+\frac{(t-\frac{a+b}{2})^2}{(b-a)^2}+(1-2\alpha)\frac{G(t,b)-G(a,t)}{(b-a)^2}\right](b-a).$$

Remark 3.4. *If we take $g(t)=1$ for all $t\in\mathbb{T}$, then $g^\Delta(t)=g^\nabla(t)=0$ and Theorem 3.1 gives us a time scale version of Ostrowski's inequality (5) (see also* [7]*):*

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\diamond_\alpha s\right|$$
$$\leq[\alpha f^\Delta\|_\infty+(1-\alpha)\|f^\nabla\|_\infty]\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2+(1-2\alpha)\frac{G(t,b)-G(a,t)}{(b-a)^2}\right](b-a).$$

On the other hand, if we take the diamond-α integral in both sides of (11) we obtain, after rewriting and using the properties of modulus, the following Grüs type inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)\diamond_\alpha t-\left(\int_a^b f(t)\diamond_\alpha t\right)\left(\frac{1}{b-a}\int_a^b g(t)\diamond_\alpha t\right)\right|$$
$$\leq\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left\{\alpha[|g(t)|\|f^\Delta\|_\infty+|f(t)|\|g^\Delta\|_\infty]\right.$$
$$\left.+(1-\alpha)[|g(t)|\|f^\nabla\|_\infty+|f(t)|\|g^\nabla\|_\infty]\right\}|t-s|\diamond_\alpha s\diamond_\alpha t.$$

The following theorem is a stronger variant of the Theorem 3.1.

Theorem 3.2. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be continuous functions on $\mathbb{T}$ whose delta and nabla derivative are bounded. Then*

$$\left|f(t)g(t)-\frac{1}{b-a}\left[g(t)\int_a^b f(s)\diamond_\alpha s+\right.\right.$$
$$\left.\left.f(t)\int_a^b g(s)\diamond_\alpha s\right]+\frac{1}{b-a}\int_a^b f(s)g(s)\diamond_\alpha s\right|$$
$$\leq\frac{1}{b-a}[\alpha\|f^\Delta\|_\infty+(1-\alpha)\|f^\nabla\|_\infty][\alpha\|g^\Delta\|_\infty+(1-\alpha)\|g^\nabla\|_\infty]\int_a^b|t-s|^2\diamond_\alpha s,$$

for all $t\in\mathbb{T}$ with $a\leq t\leq b$.

Proof. Since we have the same hypotheses as in Theorem 3.1, it is obvious that identities (5), (6), (8) and (9) remain true. Multiplying side by side (5) and (6), (8) and (9) , (5) and (9) and (6) and (8) respectively, we get

$$f(t)g(t)-[g(t)f(s)+f(t)g(s)]+f(s)g(s)=\begin{cases}\left\{\int_s^t f^{\Delta}(\tau)\Delta\tau\right\}\left\{\int_s^t g^{\Delta}(\tau)\Delta\tau\right\}\\ \left\{\int_s^t f^{\nabla}(\tau)\nabla\tau\right\}\left\{\int_s^t g^{\nabla}(\tau)\nabla\tau\right\}\\ \left\{\int_s^t f^{\Delta}(\tau)\Delta\tau\right\}\left\{\int_s^t g^{\nabla}(\tau)\nabla\tau\right\}\\ \left\{\int_s^t f^{\nabla}(\tau)\nabla\tau\right\}\left\{\int_s^t g^{\Delta}(\tau)\Delta\tau\right\}.\end{cases} \tag{13}$$

Multiplying the first and the third identity with α and $1-\alpha$ respectively, and adding them, then doing the same operation with the second and the forth identities in (13), we get

$$\begin{aligned} f(t)g(t)-[g(t)f(s)+f(t)g(s)]+f(s)g(s)&=\left\{\int_s^t f^{\Delta}(\tau)\Delta\tau\right\}\\ &\cdot\left\{\alpha\int_s^t g^{\Delta}(\tau)\Delta\tau+(1-\alpha)\int_s^t g^{\nabla}(\tau)\nabla\tau\right\}\end{aligned} \tag{14}$$

and

$$\begin{aligned} f(t)g(t)-[g(t)f(s)+f(t)g(s)]+f(s)g(s)&=\left\{\int_s^t f^{\nabla}(\tau)\nabla\tau\right\}\\ &\cdot\left\{\alpha\int_s^t g^{\Delta}(\tau)\Delta\tau+(1-\alpha)\int_s^t g^{\nabla}(\tau)\nabla\tau\right\}.\end{aligned} \tag{15}$$

Multiplying the identity from (14) with α and the identity from (15) with $1-\alpha$, after summing them, we get

$$\begin{aligned} &f(t)g(t)-[g(t)f(s)+f(t)g(s)]+f(s)g(s)\\ &=\left\{\alpha\int_s^t f^{\Delta}(\tau)\Delta\tau+(1-\alpha)\int_s^t f^{\nabla}(\tau)\nabla\tau\right\}\cdot\left\{\alpha\int_s^t g^{\Delta}(\tau)\Delta\tau+(1-\alpha)\int_s^t g^{\nabla}(\tau)\nabla\tau\right\}.\end{aligned} \tag{16}$$

Taking the diamond-α integral on both sides of (16) with respect to s, from a to b, we have after simplification by $(b-a)$:

$$\begin{aligned} f(t)g(t)-\frac{1}{b-a}&\left[g(t)\int_a^b f(s)\diamond_\alpha s+\right.\\ &\left.f(t)\int_a^b g(s)\diamond_\alpha s\right]+\frac{1}{b-a}\int_a^b f(s)g(s)\diamond_\alpha s\\ &\le\frac{1}{b-a}\int_a^b\left\{\alpha\int_s^t f^{\Delta}(\tau)\Delta\tau+(1-\alpha)\int_s^t f^{\nabla}(\tau)\nabla\tau\right\}\\ &\qquad\cdot\left\{\alpha\int_s^t g^{\Delta}(\tau)\Delta\tau+(1-\alpha)\int_s^t g^{\nabla}(\tau)\nabla\tau\right\}\diamond_\alpha s.\end{aligned} \tag{17}$$

Taking the modulus in (17) and using its properties and (3), we obtain the conclusion. □

Remark 3.5. (i) *If $\mathbb{T}=\mathbb{R}$ and the functions f,g are also differentiable, $f^{\Delta}=f^{\nabla}=f'$, $g^{\Delta}=g^{\nabla}=g'$, then we get Theorem 2.3 in* [6].

(ii) *If $\mathbb{T}$ is a reunion of intervals for $\mathbb{R}$ and the functions are not differentiable in the end points, we have $f^{\Delta} = f'_{-}$, $f^{\nabla} = f'_{+}$, $g^{\Delta} = g'_{-}$, $g^{\nabla} = g'_{+}$ in that points and so, we get an extended variant of Theorem 2.3 in* [6].

(iii) *We have:*

$$\int_a^b |t-s|^2 \diamond_\alpha s = \int_a^t (t-s)^2 \diamond_\alpha s + \int_t^b (s-t)^2 \diamond_\alpha s.$$

The function $s \mapsto (t-s)^2$ is nonincreasing on $[a,t]$ and $s \mapsto (s-t)^2$ is nondecreasing on $[t,b]$. By Remark 3.1, there exist an $\alpha_1 \in [0,1]$ and an $\alpha_2 \in [0,1]$ such that

$$\int_a^t (t-s)^2 \diamond_\alpha s \le \frac{(t-a)^3}{3} \qquad \textit{for all } \alpha \le \alpha_1$$

and

$$\int_a^t (s-t)^2 \diamond_\alpha s \le \frac{(b-t)^3}{3} \qquad \textit{for all } \alpha \ge \alpha_2.$$

Thus

$$\left| f(t)g(t) - \frac{1}{b-a}\left[g(t)\int_a^b f(s)\diamond_\alpha s + f(t)\int_a^b g(s)\diamond_\alpha s \right] + \frac{1}{b-a}\int_a^b f(s)g(s)\diamond_\alpha s \right|$$
$$\le \frac{1}{b-a}[\alpha\|f^{\Delta}\|_\infty + (1-\alpha)\|f^{\nabla}\|_\infty][\alpha\|g^{\Delta}\|_\infty + (1-\alpha)\|g^{\nabla}\|_\infty]\left[\frac{(t-a)^3+(b-t)^3}{3}\right],$$

for all $\alpha \in [\alpha_2, \alpha_1]$.
These α_1 and α_2 depend on the graininess of the time scale $\mathbb{T}$.

(iv) *If $\alpha = 1$ and $\alpha = 0$, then one can find the delta variant and the nabla variant, respectively, of Theorem 3.2 for time scales.*

(v) *If both f^{Δ} and f^{∇} attain their maximum value in the same point then we have*

$$\left| f(t)g(t) - \frac{1}{b-a}\left[g(t)\int_a^b f(s)\diamond_\alpha s + f(t)\int_a^b g(s)\diamond_\alpha s \right] + \frac{1}{b-a}\int_a^b f(s)g(s)\diamond_\alpha s \right|$$
$$\le \frac{1}{b-a}\|f^{\diamond_\alpha}\|_\infty \|g^{\diamond_\alpha}\|_\infty \int_a^b |t-s|^2 \diamond_\alpha s.$$

Remark 3.6. *Taking the diamond-α integral on both sides of (16) with respect to t, (from a to b), and using the properties of the modulus we get (after simplification by $(b-a)$):*

$$\left| \frac{1}{b-a}\int_a^b f(t)g(t)\diamond_\alpha t - \left(\frac{1}{b-a}\int_a^b f(t)\diamond_\alpha t\right)\left(\frac{1}{b-a}\int_a^b g(t)\diamond_\alpha t\right) \right|$$
$$\le \frac{1}{b-a}[\alpha\|f^{\Delta}\|_\infty + (1-\alpha)\|f^{\nabla}\|_\infty][\alpha\|g^{\Delta}\|_\infty + (1-\alpha)\|g^{\nabla}\|_\infty]$$
$$\cdot \int_a^b \int_a^b |t-s|^2 \diamond_\alpha s \diamond_\alpha t.$$

The last inequality is a Čebyšev type inequality. For $\mathbb{T} = \mathbb{R}$, we retrieve the well known Čebyšev inequality (*see, for example* [6], *Remark 2.4*).

References

[1] R. P. Agarwal, M. Bohner; *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999), 3-22.
[2] M. Bohner and A. Peterson; *Dynamic Equations on Time Scales, An introduction with Applications*, Birkhäuser, Boston, 2001.
[3] C. P. Niculescu and L.-E. Persson, *Convex functions and their applications. A contemporary approach*, Springer, 2005.
[4] S. Hilger; *Analysis on measure chains-a unified approach to continuous and discrete calculus*, Results Math. 35 (1990), 18-56.
[5] Moulay Rchid Sidi Ammi, Rui A. C. Ferreira and Delfim F. M. Torres, *Diamond-α Jensen's Inequality on Time Scales and Applications*, http://arXiv.org:0712.1680, 2007.
[6] B. G. Pachpatte, A note on Ostrowski like inequalities, *J. Inequal. Pure and Appl. Math.*, **6** (4) (2005), Art. 114.
[7] A. M. Ostrowski, Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmitelwert, *Comment. Math. Helv.*, **10** (1938), 226–227.

University of Craiova, Department of Mathematics, Craiova RO-200585, Romania
E-mail address: `c.dinu@yahoo.com`