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Convergence and Well-posedness Analysis of a Nonlinear Elliptic System

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Abstract. We consider two real Hilbert spaces X and Y and a nonlinear elliptic system governed by two sets of constraints $K \subset X$ and $W \subset Y$. We prove that, under appropriate assumptions, the system has a unique solution $(u, \varphi) \in K \times W$. Then, we provide necessary and sufficient conditions which guarantee the convergence of an arbitrary sequence $(u_n, \varphi_n) \in X \times Y$ to the solution (u, φ). The proofs are based on standard results on elliptic variational inequalities, various estimates and arguments of convex analysis. Next, we introduce two concepts of well-posedness for the system, compare them, and derive the corresponding well-posedness results. Our results can be applied in the study of various problems arising and Mechanics and Physics. To provide an example we consider a mathematical model which describes the frictionless unilateral contact of a piezoelectric body with an insulated foundation.

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1. Introduction

Nonlinear systems arise in the study of various mathematical models which describe a large number of physical settings. References in the field are [21, 22], for instance, where various existence and uniqueness results have been obtained. Comprehensive references concerning variational analysis of problems arising in Physics and Mechanics are the books [3, 6, 7, 13, 17]. Results on the numerical approach of nonlinear systems systems, including error estimates for semidiscrete schemes, discrete schemes and/or numerical simulations, have been provided in [9, 11, 12, 18], for instance.

Besides the existence and uniqueness of the solution, convergence results represent an important topic in the study of nonlinear problems and, in particular, in the study of nonlinear systems. Examples of convergence results abound in Nonlinear Analysis and Mechanics. The convergence of the solution of a penalty problem to the solution of the original one as the penalty parameter converge to zero, the convergence of the solution of a discrete problem to the solution of the continuous problem as the discretization parameter converges, the convergence of the solution of a frictional

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contact model to the solution of a frictionless one as the coefficient of friction vanishes, and the convergence of a viscoelastic problem to the solution of an elastic problem as the viscosity tensor vanishes represent just some simple examples, among many others. References on this topic include [1, 3, 4, 12, 14]. Nevertheless, in most of these references, only sufficient conditions which guarantee the corresponding convergence results have been considered.

The problem of establishing *convergence criteria*, i.e., necessary and sufficient conditions for convergence, is an important topic which, at the best of our knowledge, is widely open. The reason is that such criteria depend on the structure of the problem and the assumptions on the data and, therefore, each criteria is obtained by using specific functional arguments, which have to be adapted from case to case. Results on this direction have been obtained in [2, 23, 24] where convergence criteria have been obtained for elliptic quasivariational inequalities, elliptic hemivariational inequalities and history-dependent variational inequalities, respectively. There, besides the statement and the proof of convergence criteria, well-posedness results have been obtained and various applications have been given.

The well-posedness concepts of nonlinear problems depend on the problem considered, vary from author to author, and even from paper to paper. References in the field are the papers [8, 10, 15, 16] as well as the books [5, 19, 20], for instance. Nevertheless, most of these concepts are based on two main ingredients: the existence and uniqueness of the solution to the corresponding problem and the convergence to it of a special class of sequences, the so-called approximating sequences.

In this current paper we continue our research started in [2, 23, 24]. More precisely, the aim of this paper, described below, is three fold. The first one is to study a nonlinear system in real Hilbert spaces, governed by two sets of constraints and four bilinear forms. Thus, besides the unique solvability of the system, we provide a convergence criterion to its solution. We then use this criterion to introduce two well-posedness concepts in the study of the system and we prove that, under the considered assumptions, the second concept is the best one which can be introduced in the study this problem. This represents our second aim in this paper. Our third aim is to provide an application of these abstract results in the study of a contact model with unilateral constraints. This allows us to prove the continuous dependence of the solution with respect to the data. Contact problems involving deformable bodies abound in industry and everyday life. Because of the importance of contact processes in structural and mechanical systems, a considerable effort has been put into their modeling and analysis, and the literature in the field is extensive. References include the books [3, 6, 7, 20, 21, 22], for instance. The numerical analysis of the contact problems, including numerical simulations and examples arising in engineering sciences, can be found in [12, 14, 17, 18].

The rest of the manuscript is structured as follows. In Section 2 we introduce the nonlinear system we consider, list the assumptions on the data and prove an existence and uniqueness result, Theorem 2.1. Then, in Section 3 we state our main abstract result, Theorem 3.1. It represents a convergence criterion to the solution. We apply this abstract result in the Section 4 which is dedicated to the well-posedness analysis of the system. Finally, in Section 5 we introduce a mathematical model of piezoelectic contact. It is stated in a variational formulation and it describes the frictionless contact of an electro-elastic body with an insulator obstacle. We use our abstract results in Sections 2, 3 and 4 in the analysis of this model. In this way prove its unique weak solvability and we deduce a continuous dependence result for the solution.

2. The system

Let X and Y be two real Hilbert spaces, endowed with the inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, respectively. We denote by $\|\cdot\|_X$ and $\|\cdot\|_Y$ the associated norms on these spaces. Let $a : X \times X \to \mathbb{R}$, $b : Y \times Y \to \mathbb{R}$, $c : Y \times X \to \mathbb{R}$ and $d : X \times Y \to \mathbb{R}$ be four bilinear forms, $K \subset X$, $W \subset Y$, $f \in X$ and $q \in Y$. With these data we consider the following nonlinear system.

Problem $\mathcal{P}$. *Find two elements $u \in K$ and $\varphi \in W$ such that*

$$a(u, v-u) + c(\varphi, v-u) \geq (f, v-u)_X \quad \forall\, v \in K, \tag{1}$$

$$b(\varphi, \psi-\varphi) + d(u, \psi-\varphi) \geq (q, \psi-\varphi)_Y \quad \forall\, \psi \in W. \tag{2}$$

In the study of Problem $\mathcal{P}$ we consider the following assumptions on the data.

$$K \text{ is a nonempty closed convex subset on X.} \tag{3}$$

$$W \text{ is a nonempty closed convex subset on Y.} \tag{4}$$

$$\begin{aligned} &a : X \times X \to \mathbb{R} \text{ is a bilinear continuous coercive form with} \\ &\qquad \text{constant } m_a > 0, \text{ that is } a(v,v) \geq m_a \|v\|_X^2 \ \ \forall\, v \in X. \end{aligned} \tag{5}$$

$$\begin{aligned} &b : Y \times Y \to \mathbb{R} \text{ is a bilinear continuous coercive form with} \\ &\qquad \text{constant } m_b > 0, \text{ that is } b(\psi,\psi) \geq m_b \|\psi\|_Y^2 \ \ \forall\, \psi \in Y. \end{aligned} \tag{6}$$

$$c : Y \times X \to \mathbb{R} \text{ is a bilinear continuous form.} \tag{7}$$

$$d : X \times Y \to \mathbb{R} \text{ is a bilinear continuous form.} \tag{8}$$

$$\left\{ \begin{array}{l} d(u,\varphi) + c(\varphi,u) \geq -\alpha(\|u\|_X^2 + \|\varphi\|_Y^2) \ \ \forall\, u \in X,\ \varphi \in Y, \\ \text{with some } \alpha \geq 0 \text{ such that } \alpha < \min\{m_a, m_b\}. \end{array} \right. \tag{9}$$

Moreover, we recall that

$$f \in X, \tag{10}$$

$$q \in Y. \tag{11}$$

The unique solvability of Problem $\mathcal{P}$ is given by the following result.

Theorem 2.1. *Assume (3)–(11). Then, there exists a unique solution $(u,\varphi) \in K \times W$ to Problem $\mathcal{P}$.*

Proof. We use standard arguments on elliptic variational inequalities. To this end we consider the product space $Z = X \times Y$, endowed with the canonical inner product

$$(z, z')_Z = (u, v)_X + (\varphi, \psi)_Y \quad \forall\, z = (u,\varphi),\ z' = (v,\psi) \in Z$$

and the associated norm $\|\cdot\|_Z$. Moreover, we introduce the set $J = K \times W \subset Z$, the element $p = (f, q) \in Z$ and the form $e : Z \times Z \to \mathbb{R}$ defined by

$$e(z, z') = a(u, v)_X + b(\varphi, \psi) + c(\varphi, v) + d(u, \psi) \quad \forall\, z = (u, \varphi),\ z' = (v, \psi) \in Z. \quad (12)$$

Then, using assumptions (3)–(8) it is easy to see that J is a nonempty closed convex subset of the space Z and e is a bilinear continuous form on Z. Moreover,

$$e(z, z) \geq (\min\{m_a, m_b\} - \alpha)\|z\|_Z^2 \quad \forall\, z = (u, \varphi) \in Z$$

and, therefore, (9) shows that the form e is coercive, with constant

$$m = \min\{m_a, m_b\} - \alpha > 0. \quad (13)$$

Therefore, using (10), (11) and a standard existence and uniqueness result (see Theorem 2.1 in [21], for instance), we deduce the existence of a unique element $z = (u, \varphi) \in Z$ such that

$$z \in J, \quad e(z, z' - z) \geq (p, z' - z)_Z \quad \forall\, z' = (v, \psi) \in J. \quad (14)$$

We now succesively use inequality (14) with $z' = (v, \varphi)$ and $w = (u, \psi)$ where v and ψ are arbitrary elements in K and W, respectively. Then, we combine the resulting inequalities with definition (12), to see that the element $z = (u, \varphi)$ satisfies the inequalities (1) and (2). This proves the existence of the solution to Problem $\mathcal{P}$.

The uniqueness follows from the uniqueness of the solution to inequality (14). Indeed, if (u, φ) and $(\widetilde{u}, \widetilde{\varphi})$ represent two solution of Problem $\mathcal{P}$, then it is easy to see that the elements $z = (u, \varphi)$ and $\widetilde{z} = (\widetilde{u}, \widetilde{\varphi})$ are solution to inequality (14). We now use the uniqueness part in Theorem 2.1 to deduce that $z = \widetilde{z}$ which implies that $u = \widetilde{u}$, $\varphi = \widetilde{\varphi}$ and concludes the proof. □

We end this section with the following remark on the smallness assumption (9).

Remark 2.1. Assumption (7) shows that there exists a constant $M_c > 0$ such that $|c(\varphi, u)| \leq M_c\|\varphi\|_Y\|u\|_X$ for all $u \in X$, $\varphi \in Y$. This implies that $c(\varphi, u) \geq -M_c\|\varphi\|_Y\|u\|_X$ and, therefore,

$$c(\varphi, u) \geq -M_c\|\varphi\|_Y\|u\|_X \geq -\frac{M_c}{2}\left(\|\varphi\|_Y^2 + \|u\|_X^2\right) \quad \forall\, u \in X,\ \varphi \in Y. \quad (15)$$

A similar argument reveals that there exists a constant $M_d > 0$ such that

$$d(u, \varphi) \geq -M_d\|u\|_X\|\varphi\|_Y \geq -\frac{M_d}{2}\left(\|u\|_X^2 + \|\varphi\|_Y^2\right) \quad \forall\, u \in X,\ \varphi \in Y. \quad (16)$$

We now add the inequalities (15) and (16) to deduce that

$$d(u, \varphi) + c(\varphi, u) \geq -\frac{M_c + M_d}{2}\left(\|u\|_X^2 + \varphi\|_Y^2\right) \quad \forall\, u \in X,\ \varphi \in Y. \quad (17)$$

Finally, inequality (17) shows that condition (9) is satisfied if, for instance,

$$\frac{M_c + M_d}{2} < \min\{m_a, m_b\}. \quad (18)$$

Note that condition (18) represents only a sufficient condition which guarantees (9). Indeed, in the example we present in Section 5 inequality (9) holds with $\alpha = 0$ and, therefore, condition (18) could not be verified. This shows that (18) is not a necessary condition for having (9).

3. A convergence criterion

In this section we state and prove a convergence criterion in the study of Problem $\mathcal{P}$. To this end, we assume (3)–(11), consider an arbitrary sequence $\{(u_n, \varphi_n)\} \subset X \times Y$ and we denote by $(u, \varphi) \in X \times Y$ the solution of Problem $\mathcal{P}$ provided by Theorem 2.1. We use the symbol "$\to$" to indicate the convergence in various Hilbert spaces that will be specified, except in the case when these convergences take place in $\mathbb{R}$. All the limits are considered as $n \to \infty$, even if we do not mention it explicitly. For a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ which converges to zero we use the short hand notation $0 \le \varepsilon_n \to 0$ and we denote by $dist(\boldsymbol{v}, K)$ the distance between the element $\boldsymbol{v} \in X$ and the set K, in the Hilbertian structure of the space X. Notation $dist(\psi, W)$ will have a similar meaning, in the Hilbertian structure of the space Y. Finally, below in this section C_i $(i = 1, 2, \ldots)$ will represent generic positive constants which could depend on the solution and on the problem data but do not depend on $n \in \mathbb{N}$.

Our main result in this section is the following.

Theorem 3.1. *Assume (3)–(11). Then the following statements are equivalent:*

$$u_n \to u \quad \text{in } X \quad \text{and} \quad \varphi_n \to \varphi \quad \text{in } Y. \tag{19}$$

$$\begin{cases} \textit{There exists a sequence } 0 \le \varepsilon_n \to 0 \textit{ such that} \\ \text{(a)} \quad dist(u_n, K) \le \varepsilon_n, \quad dist(\varphi_n, W) \le \varepsilon_n, \\ \text{(b)} \quad a(u_n, v - u_n) + c(\varphi_n, v - u_n) + \varepsilon_n(1 + \|v - u_n\|_X) \\ \qquad\qquad \ge (f, v - u_n)_X \quad \forall\, v \in K, \\ \text{(c)} \quad b(\varphi_n, \psi - \varphi_n) + d(u_n, \psi - \varphi_n) + \varepsilon_n(1 + \|\psi - \varphi_n\|_Y) \\ \qquad\qquad \ge (q, \psi - \varphi_n)_Y \quad \forall\, \psi \in W, \\ \text{for any } n \in \mathbb{N}. \end{cases} \tag{20}$$

Proof. Assume (19). Let $n \in \mathbb{N}$ and let δ_n be given by

$$\delta_n = \|u_n - u\|_X + \|\varphi_n - \varphi\|_Y. \tag{21}$$

Then,

$$dist(u_n, K) \le \delta_n, \qquad dist(\varphi_n, W) \le \delta_n \tag{22}$$

and, moreover,

$$\delta_n \to 0. \tag{23}$$

Let $v \in K$. We use inequality (1) to see that

$$\Big[a(u_n, v - u_n) + c(\varphi_n, v - u_n) + (f, v - u_n)_X\Big] + a(u, v - u) + c(\varphi, v - u)$$
$$\ge (f, v - u)_X + \Big[a(u_n, v - u_n) + c(\varphi_n, v - u_n) + (f, v - u_n)_X\Big]$$

and, therefore,

$$\begin{aligned} a(u_n, v - u_n) + c(\varphi_n, v - u_n) + \Big[a(u, v - u) - a(u_n, v - u_n)\Big] \\ + \Big[c(\varphi, v - u) - c(\varphi_n, v - u_n)\Big] + \Big[(f, v - u_n)_X - (f, v - u)_X\Big] \ge (f, v - u_n)_X. \end{aligned} \tag{24}$$

We now write

$$a(u, v-u) - a(u_n, v-u_n) = \Big[a(u, v-u) - a(u, v-u_n)\Big] + \Big[a(u, v-u_n) - a(u_n, v-u_n)\Big],$$

then we use assumption (5) and notation (21) to see that

$$a(u, v-u) - a(u_n, v-u_n) \le C_1\|u\|_X\delta_n + C_2\delta_n\|v-u_n\|_X. \tag{25}$$

A similar argument, based on assumption (7), shows that

$$c(\varphi, v-u) - c(\varphi_n, v-u_n) \le C_3\|\varphi\|_Y\delta_n + C_4\delta_n\|v-u_n\|_X \tag{26}$$

and, obviously,

$$(f, v-u_n)_X - (f, v-u)_X \le \|f\|_X\|u-u_n\|_X \le \|f\|_X\delta_n. \tag{27}$$

We now combine the inequalities (24)–(27) to deduce that

$$a(u_n, v-u_n) + c(\varphi_n, v-u_n) + (C_1\|u\|_X + C_3\|\varphi\|_Y + \|f\|_X)\delta_n + \\ +(C_2+C_4)\delta_n\|v-u_n\|_X \ge (f, v-u_n)_X$$

and, therefore,

$$a(u_n, v-u_n) + c(\varphi_n, v-u_n) + C_5\delta_n(1 + \|v-u_n\|_X) \ge (f, v-u_n)_X. \tag{28}$$

On the other hand, using inequality (2) and arguments similar to those used above we deduce that, for any $\psi \in W$, the following inequality holds:

$$b(\varphi_n, \psi-\varphi_n) + d(u_n, \psi-\varphi_n) + C_6\delta_n(1 + \|\psi-\varphi_n\|_Y) \ge (q, \psi-\varphi_n)_Y. \tag{29}$$

Finally, using (22), (23), (28) and (29) we find that condition (20) is satisfied, with

$$\varepsilon_n = \max\{\,\delta_n, C_5\delta_n, C_6\delta_n)\,\},$$

which concludes the first part of the proof.

Conversely, assume now that (20) holds. Denote by $P_K : X \to K$ and $P_W : Y \to W$ the projection operators on the sets K and W, respectively. Let $n \in \mathbb{N}$ and let $v_n = P_K u_n$, $w_n = u_n - P_K u_n$, $\psi_n = P_W\varphi_n$ and $\omega_n = \varphi_n - P_W\varphi_n$. Then, we have

$$v_n \in K, \qquad u_n = v_n + w_n, \qquad dist(u_n, K) = \|w_n\|_X \le \varepsilon_n, \tag{30}$$

$$\psi_n \in W, \qquad \varphi_n = \psi_n + \omega_n, \qquad dist(\varphi_n, W) = \|\omega_n\|_Y \le \varepsilon_n. \tag{31}$$

We use the regularities $u \in K$ and $v_n \in K$ to take $v = u$ in (20)(a) and $v = v_n$ in (1). Then, adding the resulting inequalities, we find that

$$a(u_n, u-u_n) + a(u, v_n-u) + c(\varphi_n, u-u_n) + c(\varphi, v_n-u) \\ +\varepsilon_n(1 + \|u_n-u\|_X) \ge (f, v_n-u_n)_X. \tag{32}$$

Next, we write

$$a(u_n, u-u_n) + a(u, v_n-u) = a(u_n-u, u-u_n) + a(u, v_n-u_n),$$

$$c(\varphi_n, u-u_n) + c(\varphi, v_n-u) = c(\varphi_n-\varphi, u-u_n) + c(\varphi, v_n-u_n),$$

substitute these equalities in (32) and use equality $u_n - v_n = w_n$, guaranteed by (30), to find that

$$a(u_n - u, u - u_n) - a(u, w_n) + c(\varphi_n - \varphi, u - u_n) - c(\varphi, w_n) + (f, w_n)_X$$
$$+\varepsilon_n(1 + \|u_n - u\|_X) \geq 0.$$

Then, using assumptions (5) and (7) on the bilinear forms a and c we deduce that

$$m_a\|u_n - u\|_X^2 \leq c(\varphi_n - \varphi, u - u_n) + (C_7\|u\|_X + C_8\|\varphi\|_Y + \|f\|_X)\|w_n\|_X \quad (33)$$
$$+\varepsilon_n(1 + \|u_n - u\|_X).$$

A similar argument, based on inequalities (20)(b), (2), (31) and the properties of the bilinear forms b and d, shows that

$$m_b\|\varphi_n - \varphi\|_Y^2 \leq d(u_n - u, \varphi - \varphi_n) + (C_9\|\varphi\|_Y + C_{10}\|u\|_X + \|q\|_Y)\|\omega_n\|_Y \quad (34)$$
$$+\varepsilon_n(1 + \|\varphi_n - \varphi\|_Y.)$$

We now add inequalities (33) and (34) and use the inequalities $\|w_n\|_X \leq \varepsilon_n$, $\|\omega_n\|_Y \leq \varepsilon_n$ in (30) and (31), respectively, to deduce that

$$m_a\|u_n - u\|_X^2 + m_b\|\varphi_n - \varphi\|_Y^2 + c(\varphi_n - \varphi, u_n - u) + d(u_n - u, \varphi_n - \varphi) \quad (35)$$
$$\leq C_{11}\varepsilon_n + \varepsilon_n(1 + \|u_n - u\|_X + \|\varphi_n - \varphi\|_Y).$$

Let $m > 0$ be the constant defined in (13). Then, inequality (35) combined with assumption (9) shows that

$$m(\|u_n - u\|_X^2 + \|\varphi_n - \varphi\|_Y^2) \leq C_{11}\varepsilon_n + \varepsilon_n(\|u_n - u\|_X + \|\varphi_n - \varphi\|_Y)$$

and, using the elementary inequality

$$\frac{1}{2}(x + y)^2 \leq x^2 + y^2, \quad (36)$$

valid for all x, $y \in \mathbb{R}$, we deduce that

$$\frac{m}{2}\Big(\|u_n - u\|_X + \|\varphi_n - \varphi\|_Y\Big)^2 \leq C_{11}\varepsilon_n + \varepsilon_n(\|u_n - u\|_X + \|\varphi_n - \varphi\|_Y).$$

We now use the inequality

$$x^2 \leq ax + b \quad \Longrightarrow \quad x \leq a + \sqrt{b} \quad \forall\, x,\, a,\, b \geq 0$$

to find that

$$\|u_n - u\|_X + \|\varphi_n - \varphi\|_Y \leq C_{12}\varepsilon_n + C_{13}\sqrt{\varepsilon_n}\,.$$

Finally, the convergence $\varepsilon_n \to 0$, guaranteed by assumption (20), reveals that $u_n \to u$ in X and $\varphi_n \to \varphi$ in Y, which concludes the proof. □

Remark 3.1. Note that Theorem 3.1 provides necessary and sufficient conditions for the convergence of any sequence $\{(u_n, \varphi_n)\} \subset X \times Y$ to the solution (u, φ) of Problem $\mathcal{P}$. Therefore, with the terminology used in the Introduction, it represents a convergence criterion in the study of this nonlinear system.

4. Well-posedness results

In this section, we introduce two well-posedness concepts in the study of Problem $\mathcal{P}$. To this end, as mentioned in the Introduction, we need to define a special class of sequences, the so-called approximating sequences, corresponding to each well-posedness concept. Therefore, we start with the following definition.

Definition 4.1. 1) A sequence $\{(u_n,\varphi_n)\}\subset X\times Y$ is said to be a $\mathcal{T}_1$-approximating sequence if there exists $0\le\varepsilon_n\to 0$ such that

$$\left\{\begin{array}{ll}\text{(a)}\ u_n\in K,\quad \varphi_n\in W,\\ \text{(b)}\ \ a(u_n,v-u_n)+c(\varphi_n,v-u_n)+\varepsilon_n\|v-u_n\|_X\ge(f,v-u_n)_X & \forall\, v\in K,\\ \text{(c)}\ \ b(\varphi_n,\psi-\varphi_n)+d(u_n,\psi-\varphi_n)+\varepsilon_n\|\psi-\varphi_n\|_Y\ge(q,\psi-\varphi_n)_Y & \forall\,\psi\in W\\ \text{for any } n\in\mathbb{N}.\end{array}\right. \tag{37}$$

2) A sequence $\{(u_n,\varphi_n)\}\subset X\times Y$ is said to be a $\mathcal{T}_2$-approximating sequence if it satisfies condition (20).

3) Problem $\mathcal{P}$ is said to be $\mathcal{T}_i$-well-posed ($i=1,2$) if it has a unique solution $(u,\varphi)\in X\times Y$ and every $\mathcal{T}_i$-approximating sequence $\{(u_n,\varphi_n)\}$ converges in $X\times Y$ to (u,φ), that is, (19) holds.

We now introduce the following notation, for $i=1,2$:

$$\mathcal{S}=\Big\{\,(u,\varphi)\in X\times Y\ :\ u_n\to u\ \text{ in } X\ \text{ and }\ \varphi_n\to\varphi\ \text{ in } Y\,\Big\},$$
$$\mathcal{S}_i=\Big\{\,\{(u_n,\varphi_n)\}\in X\in Y\ :\{(u_n,\varphi_n)\}\text{ is a }\mathcal{T}_i\text{-approximating sequence}\,\Big\}.$$

Our first result in this section is the following.

Theorem 4.1. *Assume (3)–(11). Then, Problem $\mathcal{P}$ is $\mathcal{T}_1$-well-posed.*

Proof. First, we use Theorem 2.1 to see that Problem $\mathcal{P}$ has a unique solution, denoted by (u,φ). Let $\{(u_n,\varphi_n)\}\subset X\times Y$ be a $\mathcal{T}_1$-approximating sequence and let $n\in\mathbb{N}$. We use Definition 4.1 1), take $v=u$ in (37)(b), then $\psi=\varphi$ in (37)(c) to find that

$$a(u_n,u-u_n)+c(\varphi_n,u-u_n)+\varepsilon_n\|u-u_n\|_X\ge(f,u-u_n)_X \tag{38}$$
$$b(\varphi_n,\varphi-\varphi_n)+d(u_n,\varphi-\varphi_n)+\varepsilon_n\|\varphi-\varphi_n\|_Y\ge(q,\varphi-\varphi_n)_Y. \tag{39}$$

Next, we test with $v=u_n$ and $\psi=\varphi_n$ in (1) and (2), respectively, to find that

$$a(u,u_n-u)+c(\varphi,u_n-u)\ge(f,u_n-u)_X, \tag{40}$$
$$b(\varphi,\varphi_n-\varphi)+d(u,\varphi_n-\varphi)\ge(q,\varphi_n-\varphi)_Y. \tag{41}$$

We now add inequalities (38)–(41) and use assumptions (5)–(8) to deduce that

$$\begin{aligned}m_a\|u_n-u\|_X^2+m_b\|\varphi_n-\varphi\|_Y^2+c(\varphi_n-\varphi,u_n-u)+d(u_n-u,\varphi_n-\varphi)\\ \le\varepsilon_n(\|u_n-u\|_X+\|\varphi_n-\varphi\|_Y).\end{aligned} \tag{42}$$

Let m be the constant defined by (13). Then, inequality (42) combined with assumption (9) shows that

$$m(\|u_n-u\|_X^2+\|\varphi_n-\varphi\|_Y^2)\le\varepsilon_n(\|u_n-u\|_X+\|\varphi_n-\varphi\|_Y)$$

and, using inequality (36), we deduce that

$$\|u_n - u\|_X + \|\varphi_n - \varphi\|_Y \leq \frac{2\varepsilon_n}{m}\,.$$

We now use the convergence $\varepsilon_n \to 0$, guaranteed by Definition 4.1 1), to deduce that $u_n \to u$ in X and $\varphi_n \to \varphi$ in Y. Theorem 4.1 is now a direct consequence of Definition 4.1 c). □

We now reinforce the statement of Theorem 4.1 with the following well-posedness result.

Theorem 4.2. *Assume* (3)–(11)*. Then, Problem* $\mathcal{P}$ *is* $\mathcal{T}_1$*- and* $\mathcal{T}_2$*-well-posed.*

Proof. First, we use Theorem 2.1 to see that Problem $\mathcal{P}$ has a unique solution (u,φ). Next, we use Definition 4.1 2) and Theorem 3.1 to see that $\mathcal{S}_2 = \mathcal{S}$ which implies that $\mathcal{S}_2 \subset \mathcal{S}$. Therefore, Definition 4.1 3) guarantees that Problem $\mathcal{P}$ is $\mathcal{T}_2$-well-posed. Finally, Definition 4.1 1) shows that $\mathcal{S}_1 \subset \mathcal{S}_2$ and, since $\mathcal{S}_2 = \mathcal{S}$, we obtain that $\mathcal{S}_1 \subset \mathcal{S}$. We conclude from here that Problem $\mathcal{P}$ is $\mathcal{T}_1$-well-posed, too. □

We turn now to the inclusion $\mathcal{S}_1 \subset \mathcal{S}_2$ used in the proof of Theorem 4.2. The one-dimensional example below shows that this inclusion is strict, i.e., $\mathcal{S}_2 \neq \mathcal{S}_1$.

Example 4.1. Consider the following nonlinear system: find $u \in X$ and $\varphi \in Y$ such that

$$u \in K,\ \int_0^1 u'(v'-u')dx + \int_0^1 \varphi'(v'-u')dx \geq \int_0^1 (2x+1)(v'-u')dx,\ \ \forall\, v \in K, \quad (43)$$

$$\varphi \in W,\ \int_0^1 \varphi'(\psi'-\varphi')dx - \int_0^1 u'(\psi'-\varphi')dx \geq \int_0^1 (1-2x)(\psi'-\varphi')dx,\ \ \forall\, \varphi \in W. \quad (44)$$

Here and below in this example we use the notation

$$X = \{\, v \in H^1(0,1) :\ v(0) = 0\,\},\ \ Y = \{\, \psi \in H^1(0,1) :\ \psi(0) = 0\,\},$$

$$K = \{\, v \in X\ :\ v(x) \in [0,1]\ \ \forall\, x \in [0,1]\,\},\ W = \{\, \varphi \in Y\ :\ \varphi(x) \in [0,1]\ \ \forall\, x \in [0,1]\,\}.$$

Moreover, the prime represents the derivative with respect the space variable x, that is $u' = \frac{du}{dx}$, for instance. It is well-known that X and Y are real Hilbert spaces endowed with the inner products

$$(u,v)_X = \int_0^1 u'v'\,dx, \qquad (\varphi,\psi)_Y = \int_0^1 \varphi'\psi'\,dx$$

for any $u,\ v \in X$, $\varphi,\ \psi \in Y$. Then, it is easy that the problem (43)–(44) is a particular form of Problem $\mathcal{P}$, with

$$a(u,v) = \int_0^1 u'v'\,dx, \quad b(\varphi,\psi) = \int_0^1 \varphi'\psi'\,dx,$$

$$c(\varphi,u) = \int_0^1 \varphi'u'\,dx, \quad d(u,\varphi) = -\int_0^1 u'\varphi'\,dx,$$

$$f(x) = x^2 + x, \quad q(x) = x - x^2 \quad \forall\, x \in [0,1].$$

Note that in this case conditions (3)–(11) are satisfied with $m_a = m_b = 1$ and $\alpha = 0$. Therefore, Theorem 2.1 guarantees to the unique solvability of problem (43)–(44). Moreover, it is easy to see that the solution of this problem is given by

$$u(x) = x^2, \qquad \varphi(x) = x \quad \forall\, x \in [0,1]. \tag{45}$$

Consider now the sequence $\{(u_n, \varphi_n)\} \in X \times Y$ defined by

$$u_n(x) = x^2 + \frac{x}{n}, \qquad \varphi_n(x) = x + \frac{x}{n} \qquad \forall\, x \in [0,1],\ n \in \mathbb{N}. \tag{46}$$

Then, it is easy to see that

$$u_n \to u \quad \text{in} \quad X \quad \text{and} \quad \varphi_n \to \varphi \quad \text{in} \quad Y. \tag{47}$$

Therefore, Theorem 3.1 and Definition 4.1 3) guarantee, that $\{(u_n, \varphi_n)\}$ is a $\mathcal{T}_2$-approximating sequence for problem (43)–(44). Nevertheless, $\{(u_n, \varphi_n)\}$ is not a $\mathcal{T}_1$-approximating sequence for the above problem since, for instance, $u_n(1) = 1 + \frac{1}{n}$ and, therefore, $u_n \notin K$. We conclude from above that the inclusion $\mathcal{S}_1 \subset \mathcal{S}_2$ is strict, as claimed. Moreover, we note that the convergence (47) cannot be obtained by invoking the $\mathcal{T}_1$-well-posed of Problem $\mathcal{P}$. It follows from here that the concept of $\mathcal{T}_2$-well-posedness is better than the concept of $\mathcal{T}_1$-well-posedness.

We end this section with the following comments on the equality $\mathcal{S}_2 = \mathcal{S}$ above.

Remark 4.1. We claim that among all the concepts which make Problem $\mathcal{P}$ well-posed, the $\mathcal{T}_2$-well-posedness concept in Definition 4.1 is optimal, in the sense that it uses the largest set of approximating sequences. Indeed, consider a different well-posedness concept, say the $\mathcal{T}$-well-posedness concept, defined by a set of $\mathcal{T}$-approximating sequences, denoted by $\mathcal{S}_{\mathcal{T}}$. Then, if Problem $\mathcal{P}$ is $\mathcal{T}$-well-posed we have $\mathcal{S}_{\mathcal{T}} \subset \mathcal{S}$, by definition. Now, since $\mathcal{S}_2 = \mathcal{S}$, we deduce that $\mathcal{S}_{\mathcal{T}} \subset \mathcal{S}_2$, which justifies our claim.

5. A piezoelectric contact problem

In this section we introduce a relevant mathematical model of contact for which our abstract results in Sections 3 and 4 can be applied. We start with some notation, then we present the contact model and its analysis.

Notation. Let $N \in \{2, 3\}$. We denote by $\mathbb{S}^N$ the space of second order symmetric tensors. We use " $\cdot$ " and "$\|\cdot\|$" to represent the inner product and the Euclidean norm on the spaces $\mathbb{R}^N$ and $\mathbb{S}^N$, respectively, that is

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \quad \forall\, \boldsymbol{u} = (u_i),\, \boldsymbol{v} = (v_i) \in \mathbb{R}^N,$$
$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\, \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

The zero elements of $\mathbb{R}^N$ and $\mathbb{S}^d$ will be denoted by $\mathbf{0}$.

Let $\Omega \subset \mathbb{R}^N$ be a smooth domain with boundary Γ and consider two partitions of Γ into three measurable sets Γ_1, Γ_2, Γ_3, on one hand, and Γ_a, Γ_b, Γ_3, on the other hand. Everywhere below we assume that meas $(\Gamma_1) > 0$ and meas $(\Gamma_a) > 0$. We denote by $\boldsymbol{\nu}$ the outward unitary normal at Γ and use the standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ. Typical examples are the spaces $L^2(\Omega)^N$, $L^2(\Gamma)^N$, $H^1(\Omega)^N$, equipped with their canonical Hilbertian structure. For an element

$\boldsymbol{v} \in H^1(\Omega)^N$ we use $\boldsymbol{\varepsilon}(\boldsymbol{v})$ to denote the symmetric part of the spatial gradient of $\boldsymbol{v}$, i.e.,

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T\big). \tag{48}$$

We now introduce the subspaces V and W of the spaces $H^1(\Omega)^N$ and $H^1(\Omega)$, respectively, defined as follows:

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^N : \boldsymbol{v} = \boldsymbol{0} \ \text{ a.e. on } \Gamma_1 \,\},$$

$$W = \{\ \psi \in H^1(\Omega) : \ \psi = 0 \ \text{ a.e. on } \Gamma_a \ \}.$$

Here, the conditions "$\boldsymbol{v} = \boldsymbol{0}$ a.e. on Γ_1" and "$\psi = 0$ a.e. on Γ_a" are understood in the sense of trace. Since meas$\,(\Gamma_1) > 0$ and meas$\,(\Gamma_a) > 0$, using Korn's inequality and Friedricks-Poincaré's inequality, it follows that the spaces V and W are Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \quad (\varphi, \psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi\, dx$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. We also recall that the Sobolev trace theorem imply that there exists a positive constant C, such that

$$\|\boldsymbol{v}\|_{L^2(\Omega)^N} \le C\,\|\boldsymbol{v}\|_V, \quad \|\boldsymbol{v}\|_{L^2(\Gamma)^N} \le C\,\|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V, \tag{49}$$

$$\|\psi\|_{L^2(\Omega)} \le C\|\psi\|_W, \quad \|\psi\|_{L^2(\Gamma)} \le C\|\psi\|_W \quad \forall\, \psi \in W. \tag{50}$$

Finally, we introduce the set

$$K = \{\, \boldsymbol{v} \in V : v_\nu \le g \ \text{ a.e. on } \Gamma_1 \,\} \tag{51}$$

where $v_\nu \in L^2(\Gamma)$ denotes the normal trace of the vector field $\boldsymbol{v}$ on the boundary, i.e., $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$.

Problem statement and physical interpretation. The problem we consider in this section is governed by the data $\mathcal{F}$, $\mathcal{P}$, $\boldsymbol{\beta}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, q_0, q_b, g which will be described below, and it is stated as follows.

Problem $\mathcal{Q}$. *Find two elements $\boldsymbol{u} \in V$ and $\varphi \in W$* such that

$$\begin{aligned} \boldsymbol{u} \in K, \quad & \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\, dx + \int_\Omega \mathcal{P}^\top \nabla\varphi \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))\, dx \\ & \ge \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u})\, d\Gamma \quad \forall\, \boldsymbol{v} \in K, \end{aligned} \tag{52}$$

$$\int_\Omega \boldsymbol{\beta}\nabla\varphi \cdot \nabla\psi\, dx - \int_\Omega \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \nabla\psi\, dx = \int_\Omega q_0\psi\, dx - \int_{\Gamma_b} q_b\psi\, d\Gamma \quad \forall\, \psi \in W. \tag{53}$$

Problem $\mathcal{Q}$ represents the variational formulation of a mathematical model which describes the equilibrium of a piezoelectic body which, in the reference configuration, occupies the domain Ω. Details on the modelling and analysis of piezoelectic contact problems can be found in [21]. There, a detailed description on the piezoelectic constitutive laws, the contact boundary conditions, the electrical boundary conditions and the variational formulation of the models is provided. For this reason, we restrict ourselves to the following brief description of the physical setting which leads to the mathematical model given by Problem $\mathcal{Q}$. First, the piezoelectirc body is assumed to be electro-elastic, is clamped on Γ_1, is submitted to action of surface tractions on Γ_2 and is in frictionless contact with a rigid insulated foundation on Γ_3. Next, the electric potential is assumed to vanish on Γ_a and a surface electric charge is prescribed on Γ_b. Notation $\mathcal{F}$, $\boldsymbol{\beta}$, $\mathcal{P}$, represent the elasticity tensor, the electric permittivity tensor and the piezoelectic tensor, respectively. Moreover, $\mathcal{P}^\top$ denotes the transpose of the tensor $\mathcal{P}$ and, therefore, the following inequality holds:

$$\mathcal{P}\boldsymbol{\sigma}\cdot\boldsymbol{v} = \boldsymbol{\sigma}\cdot\mathcal{P}^\top\boldsymbol{v} \qquad \forall\,\boldsymbol{\sigma}\in\mathbb{S}^N,\ \boldsymbol{v}\in\mathbb{R}^N. \tag{54}$$

In addition, $\boldsymbol{f}_0$ and q_0 denote the density of the body forces and electric charges while $\boldsymbol{f}_2$ and q_b represent the density of surface forces and electric charges assumed to act on Γ_2 and Γ_b, respectively. The unknown of the problems are the displacement field $\boldsymbol{u}$ and the electric potential φ. We recall equality (48) which shows that $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor and, finally, g represents the initial gap between the body and the potential contact surface Γ_3.

An existence and uniqueness result. In the study of Problem $\mathcal{Q}$ we assume that the tensors $\mathcal{F}$, $\mathcal{P}$ and $\boldsymbol{\beta}$ satisfy the following conditions.

$$\begin{cases} \mathcal{F}=(\mathcal{F}_{ijkl})\colon\Omega\times\mathbb{S}^N\to\mathbb{S}^N \text{ is such that:}\\ \text{(a) } \mathcal{F}_{ijkl}=\mathcal{F}_{klij}=\mathcal{F}_{jikl}\in L^\infty(\Omega),\ 1\le i,j,k,l\le d.\\ \text{(b) There exists } m_{\mathcal{F}}>0 \text{ such that}\\ \qquad \mathcal{F}\boldsymbol{\tau}\cdot\boldsymbol{\tau}\ge m_{\mathcal{F}}\|\boldsymbol{\tau}\|^2 \text{ for all } \boldsymbol{\tau}\in\mathbb{S}^N,\ \text{a.e. in } \Omega. \end{cases} \tag{55}$$

$$\begin{cases} \mathcal{P}=(\mathcal{P}_{ijk})\colon\Omega\times\mathbb{S}^N\to\mathbb{R}^N \text{ is such that}\\ \mathcal{P}_{ijk}=\mathcal{P}_{ikj}\in L^\infty(\Omega),\ 1\le i,j,k\le d. \end{cases} \tag{56}$$

$$\begin{cases} \boldsymbol{\beta}=(\beta_{ij})\colon\Omega\times\mathbb{R}^N\to\mathbb{R}^N \text{ is such that:}\\ \text{(a) } \beta_{ij}=\beta_{ji}\in L^\infty(\Omega),\ 1\le i,j\le d.\\ \text{(b) There exists } m_\beta>0 \text{ such that}\\ \qquad \boldsymbol{\beta}\boldsymbol{E}\cdot\boldsymbol{E}\ge m_\beta\|\boldsymbol{E}\|^2 \text{ for all } \boldsymbol{E}\in\mathbb{R}^N,\ \text{a.e. in } \Omega. \end{cases} \tag{57}$$

The rest of the data are such that:

$$\boldsymbol{f}_0\in L^2(\Omega)^N,\quad \boldsymbol{f}_2\in L^2(\Gamma_2)^N, \tag{58}$$

$$q_0\in L^2(\Omega),\qquad q_b\in L^2(\Gamma_b), \tag{59}$$

$$g>0. \tag{60}$$

Our main existence and uniqueness result in this section is the following.

Theorem 5.1. *Assume (55)–(60). Then Problem $\mathcal{Q}$ has a unique solution $(\boldsymbol{u},\varphi)\in V\times W$.*

Proof. We start by introducing the bilinear forms $a: V\times V\to\mathbb{R}$, $b: W\times W\to\mathbb{R}$, $c: W\times V\to\mathbb{R}$ and $d: V\times W\to\mathbb{R}$ defined by

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \qquad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in V,$$

$$b(\varphi,\psi) = \int_\Omega \boldsymbol{\beta}\nabla\varphi\cdot\nabla\psi\,dx \qquad \forall\,\varphi,\,\psi\in W,$$

$$c(\varphi,\boldsymbol{u}) = \int_\Omega \mathcal{P}^\top\nabla\varphi\cdot\boldsymbol{\varepsilon}(\boldsymbol{u})\,dx, \qquad \forall\,\varphi\in W,\ \boldsymbol{u}\in V,$$

$$d(\boldsymbol{u},\varphi) = -\int_\Omega \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\nabla\varphi\,dx \qquad \forall\,\boldsymbol{u}\in V,\ \varphi\in W.$$

Moreover, using assumptions (58) and (59), inequalities (49), (50) and Riesz's representation theorem, we consider the elements $\boldsymbol{f}\in V$ and $q\in W$ given by

$$(\boldsymbol{f},\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2}\boldsymbol{f}_2\cdot\boldsymbol{v}\,d\Gamma \qquad \forall\,\boldsymbol{v}\in V, \tag{61}$$

$$(q,\psi)_W = \int_\Omega q_0\psi\,dx - \int_{\Gamma_b} q_b\psi\,d\Gamma \qquad \forall\,\psi\in W. \tag{62}$$

Then, it is easy to see that Problem $\mathcal{Q}$ is equivalent with the problem of finding two elements $\boldsymbol{u}\in K$ and $\varphi\in W$ such that

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + c(\varphi,\boldsymbol{v}-\boldsymbol{u}) \geq (\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_V \qquad \forall\,\boldsymbol{v}\in K, \tag{63}$$

$$b(\varphi,\psi-\varphi) + d(\boldsymbol{u},\psi-\varphi) \geq (q,\psi-\varphi)_W \qquad \forall\,\varphi\in W. \tag{64}$$

We now use Theorem 2.1 with $X=V$ and $Y=W$ in order to prove the unique solvability of the system (63)–(64). To this end we use assumptions (55) and (57) to see that the form a satisfies condition (5) with $m_a=m_{\mathcal{F}}$ and the form b satisfies condition (6) with $m_b=m_\beta$. On the other hand, assumption (56) shows that the forms c and d satisfy conditions (7) and (8), respectively. Moreover, equality (54) implies that

$$c(\varphi,\boldsymbol{u}) + d(\boldsymbol{u},\varphi) = 0 \qquad \forall\,\boldsymbol{u}\in V,\ \varphi\in W,$$

which shows that condition (9) also holds, with $\alpha=0$. Finally, assumption (60) guarantees that the set (51) satisfies condition (3) and, obviously, conditions (4), (10) and (11) hold. Therefore, using Theorem 2.1, we deduce the existence of a unique couple of functions $(\boldsymbol{u},\varphi)\in K\times W$ which satisfies the nonlinear system (63)–(64). We end the proof by invoking the equivalence between this system and Problem $\mathcal{Q}$. □

A continuous dependence result. The well-posedness results in Section 4 can be used in order to deduce convergence results in the study of Problem $\mathcal{Q}$. Various examples can be considered but, for simplicity, we restrict below to presenting only a convergence result of the solution with respect to the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, q_0 and q_b. To this end, we assume in what follows (55)–(60) and we denote by $(\boldsymbol{u},\varphi)$ the solution of Problem $\mathcal{Q}$ obtained in Theorem 5.1. Moreover, for each $n\in\mathbb{N}$ we consider the functions $\boldsymbol{f}_{0n}$, $\boldsymbol{f}_{2n}$, q_{0n}, q_{bn} such that

$$\boldsymbol{f}_{0n}\in L^2(\Omega)^N, \quad \boldsymbol{f}_{2n}\in L^2(\Gamma_2)^N, \tag{65}$$

$$q_{0n}\in L^2(\Omega), \quad q_{bn}\in L^2(\Gamma_b), \tag{66}$$

together with the following problem.

Problem $\mathcal{Q}_n$. *Find two elements $\boldsymbol{u}_n \in V$ and $\varphi_n \in W$* such that

$$\boldsymbol{u}_n \in K, \quad \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_n)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}_n))\,dx + \int_\Omega \mathcal{P}^\top\nabla\varphi_n\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}_n))\,dx \tag{67}$$
$$\geq \int_\Omega \boldsymbol{f}_{0n}\cdot(\boldsymbol{v}-\boldsymbol{u}_n)\,dx + \int_{\Gamma_2}\boldsymbol{f}_{2n}\cdot(\boldsymbol{v}-\boldsymbol{u}_n)\,d\Gamma \quad \forall\,\boldsymbol{v}\in K,$$

$$\int_\Omega \boldsymbol{\beta}\nabla\varphi_n\cdot\nabla\psi\,dx - \int_\Omega \mathcal{P}\boldsymbol{\varepsilon}(\boldsymbol{u}_n)\cdot\nabla\psi\,dx = \int_\Omega q_{0n}\psi\,dx - \int_{\Gamma_b} q_{bn}\psi\,d\Gamma \quad \forall\,\psi\in W. \tag{68}$$

Then, using Theorem 2.1, again, we obtain the existence and uniqueness of the solution of Problem $\mathcal{Q}_n$, denoted by $(\boldsymbol{u}_n,\varphi_n)$. The link between the solutions of Problems $\mathcal{Q}_n$ and $\mathcal{Q}$ is given by the following result.

Theorem 5.2. *Assume* (55)–(60), (65), (66) *and, moreover, assume that*

$$\boldsymbol{f}_{0n}\to\boldsymbol{f}_0 \quad \text{in} \quad L^2(\Omega)^N, \qquad \boldsymbol{f}_{2n}\to\boldsymbol{f}_2 \quad \text{in} \quad L^2(\Gamma_2)^N, \tag{69}$$
$$q_{0n}\to q_0 \quad \text{in} \quad \in L^2(\Omega), \qquad q_{bn}\to q_n \quad \text{in} \quad L^2(\Gamma_b). \tag{70}$$

Then,

$$\boldsymbol{u}_n\to\boldsymbol{u} \quad \text{in} \quad V \quad \text{and} \quad \varphi_n\to\varphi \quad \text{in} \quad W. \tag{71}$$

Proof. Let $n\in\mathbb{N}$. First, we define the elements $\boldsymbol{f}_n\in V$ and $q\in W$ by equalities

$$(\boldsymbol{f}_n,\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0n}\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2}\boldsymbol{f}_{2n}\cdot\boldsymbol{v}\,d\Gamma \qquad \forall\,\boldsymbol{v}\in V, \tag{72}$$
$$(q_n,\psi)_W = \int_\Omega q_{0n}\psi\,dx - \int_{\Gamma_b} q_{bn}\psi\,d\Gamma \qquad \forall\,\psi\in W. \tag{73}$$

Then, using the notation in the proof of Theorem 5.1 it follows that

$$a(\boldsymbol{u}_n,\boldsymbol{v}-\boldsymbol{u}_n)+c(\varphi_n,\boldsymbol{v}-\boldsymbol{u}_n)\geq(\boldsymbol{f}_n,\boldsymbol{v}-\boldsymbol{u}_n)_V \qquad \forall\,\boldsymbol{v}\in K, \tag{74}$$
$$b(\varphi_n,\psi-\varphi_n)+d(\boldsymbol{u}_n,\psi-\varphi_n)\geq(q_n,\psi-\varphi_n)_W \qquad \forall\,\varphi\in W, \tag{75}$$

which implies that

$$a(\boldsymbol{u}_n,\boldsymbol{v}-\boldsymbol{u}_n)+c(\varphi_n,\boldsymbol{v}-\boldsymbol{u}_n)+(\boldsymbol{f}-\boldsymbol{f}_n,\boldsymbol{v}-\boldsymbol{u}_n)_V\geq(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_n)_V \qquad \forall\,\boldsymbol{v}\in K,$$
$$b(\varphi_n,\psi-\varphi_n)+d(\boldsymbol{u}_n,\psi-\varphi_n)+(q-q_n,\psi-\varphi_n)_W\geq(q,\psi-\varphi_n)_W \qquad \forall\,\varphi\in W$$

and, moreover,

$$a(\boldsymbol{u}_n,\boldsymbol{v}-\boldsymbol{u}_n)+c(\varphi_n,\boldsymbol{v}-\boldsymbol{u}_n)+\|\boldsymbol{f}-\boldsymbol{f}_n\|_V\|\boldsymbol{v}-\boldsymbol{u}_n\|_V\geq(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_n)_V \quad \forall\,\boldsymbol{v}\in K,$$
$$b(\varphi_n,\psi-\varphi_n)+d(\boldsymbol{u}_n,\psi-\varphi_n)+\|q-q_n\|_W\|\psi-\varphi_n\|_W\geq(q,\psi-\varphi_n)_W \quad \forall\,\varphi\in W.$$

The last two inequalities imply that

$$a(\boldsymbol{u}_n,\boldsymbol{v}-\boldsymbol{u}_n)+c(\varphi_n,\boldsymbol{v}-\boldsymbol{u}_n)+\varepsilon_n\|\boldsymbol{v}-\boldsymbol{u}_n\|_V\geq(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_n)_V \quad \forall\,\boldsymbol{v}\in K, \tag{76}$$
$$b(\varphi_n,\psi-\varphi_n)+d(\boldsymbol{u}_n,\psi-\varphi_n)+\varepsilon_n\|\psi-\varphi_n\|_W\geq(q,\psi-\varphi_n)_W \quad \forall\,\varphi\in W \tag{77}$$

where

$$\varepsilon_n=\max\{\,\|\boldsymbol{f}-\boldsymbol{f}_n\|_V,\|q-q_n\|_W\,\}. \tag{78}$$

Note that the definitions (61), (62), (72), (73) combined with the convergences (69), (70) imply that $\|\boldsymbol{f}-\boldsymbol{f}_n\|_V\to 0$, $\|q-q_n\|_W\to 0$ and, therefore, (78) yields

$$0\leq\varepsilon_n\to 0. \tag{79}$$

We now use the regularity $(\boldsymbol{u}_n, \varphi_n) \in K \times W$, the convergence (79), inequalities (76), (77), and Definition 4.1 1) to deduce that the sequence $\{(\boldsymbol{u}_n, \varphi_n)\}$ is a $\mathcal{T}_1$-approximating sequence for the nonlinear system (63), (64). Therefore, the $\mathcal{T}_1$-well-posedness of this system, guaranteed by Theorem 4.1, implies the convergence (71), which concludes the proof. □

We end this section with the following comments.

Remark 5.1. Theorem 5.2 shows the continuous dependence of the solution of Problem $\mathcal{Q}$ with respect to the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, q_0 and q_b. Besides the mathematical interest in the convergence (71), it is important from the physical point of view since it shows that the solution of the piezoelectic contact problem (52)–(53) depends continuosly on the density of body forces, surface tractions, electric charges and surface charges applied on the body.

Remark 5.2. The proof of Theorem 5.2 is based on the $\mathcal{T}_1$-well-posedness of the system (63)–(64), guaranteed by Theorem 4.1. Theorem 4.2 which, obviously, extends Theorem 4.1, can also be used in order to obtain additional convergence results. For instance, it can be used to provide continuous dependence results of the solution of Problem $\mathcal{Q}$ with respect to the elasticity tensor $\mathcal{F}$, the piezoelectic tensor $\mathcal{P}$, the electric permittivity tensor $\boldsymbol{\beta}$ and the initial gap g. The arguments in the proof are similar to those used in Theorem 5.2 and to those used in [20] and, therefore, we skip the details.

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