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On the Glivenko-Frink theorem for Hilbert algebras

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Abstract. The Glivenko-Frink theorem for pseudocomplemented distributive lattices and, more generally, for pseudocomplemented meet semilatices, states that the set of regular elements can be made into a Boolean algebra which is a homomorphic image of the original algebra. Buşneag has extended this theorem to bounded Hilbert algebras. In the present paper we work with a Hilbert algebra A which is not supposed to be bounded and prove that each principal order filter $[a,1]$ is a bounded Hilbert algebra (Theorem 1) whose regular elements form a Boolean algebra which is a homorphic image of $[a,1]$ (Theorem 2). Then (Theorem 3) for each element a of A we construct a Boolean-like algebra on A and a surjective homomorphism of type (2,2,1,0,0) from this algebra to the Boolean algebra obtained in Theorem 2 .

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Motivated by intuitionistic logic, Glivenko [1929] has proved that in a pseudocomplemented distributive lattice $(L,\vee,\wedge,^*,0)$, the set $R(L)=\{x\in L\mid x^{**}=x\}=\{x^*\mid x\in L\}$ of regular elements can be made into a Boolean algebra $(R(L),\sqcup,\wedge,^*,0,0^*)$, where $x\sqcup y=(x^*\wedge y^*)^*$. This famous theorem was generalized by Frink [1962] to pseudocomplemented meet semilattices, i.e., meet semilattices $(S,\wedge,0)$ such that $\forall x\ \exists x^*\ \forall y\quad x\wedge y=0\iff x\le y^*$; see e.g. Grätzer [1978], where a new proof is given. Theorems of this type have been also given for several other classes of algebras occurring in algebra of logic; see e.g. Torrens [2008] and the literature cited therein. As certain authors do, it seems appropriate to refer to these theorems as Glivenko-Frink(-like) theorems, the more so as this name makes a distinction from other important contributions of Glivenko to lattice theory, for which the reader is referred to Birkhoff [1961] or Grätzer [1978].

One of these theorems, not cited by Torrens, is due to Buşneag [1985], [2006], who proved that although a bounded Hilbert algebra $(A,\to,1,0)$ need not be a semilattice, the operation $x^*=x\to 0$ has certain properties similar to pseudocomplementation, including the Glivenko-Frink theorem.

In this Note we remark that in any Hilbert algebra A (not necessarily bounded), each principal order filter $[a,1]$ is a bounded Hilbert algebra (Theorem 2.1), and using the Buşneag theorem we obtain a family of Glivenko-Frink theorems (Theorem 2.2). Then we apply this technique to construct a family of Boolean-like structures on A, indexed with the elements of A. Moreover, for each $a\in A$ the map $x\mapsto x^{aa}$, where $x^a=x\to a$, is a surjective homomorphism of type (2,2,1,0,0) from the Boolean-like algebra A to the Boolean algebra constructed in Theorem 2.2. This is Theorem 3.1, which produces another family of Glivenko-Frink-like theorems.

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1. Prerequisites

The concept of a Hilbert algebra is the algebraic counterpart of positive implicative propositional calculus. Algebras dual to Hilbert algebras were introduced by Henkin [1950] under the name implicative models. It is Diego who, in his PhD Thesis presented in 1961, introduced and studied the concept which he called Hilbert algebra (following a suggestion of A. Monteiro); cf. Diego [1965,1966]. Nowadays Hilbert algebras are much studied for their intrinsic algebraic interest; see e.g. Buşneag [1985], [2006], Rasiowa [1974].

An algebra $(A, \rightarrow, 1)$ of type (2,0) is said to be a *Hilbert algebra* provided it satisfies the axioms

(1) $$x \rightarrow (y \rightarrow x) = 1 \ ,$$

(2) $$(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1 \ ,$$

(3) $$x \rightarrow y = 1 \text{ and } y \rightarrow x = 1 \text{ imply } x = y \ .$$

Every Hilbert algebra A becomes a poset with greatest element $(A \leq, 1)$ under the relation $\leq$ defined by

(4) $$x \leq y \Longleftrightarrow x \rightarrow y = 1 \ .$$

If, moreover, the poset A has a least element 0, the algebra A is said to be *bounded.*

It was shown by Buşneag [1985] (see also [2006]) that in a bounded Hilbert algebra A the operation * defined by

(5) $$x^* = x \rightarrow 0$$

has certain properties similar to pseudocomplementation, although A need not be a semilattice, and in particular the Glivenko-Frink theorem holds for the set

(6) $$R(A) = \{x \in A \mid x^{**} = x\}$$

of *regular elements* of A.

2. Boolean algebras in principal order filters

We refer the reader to Buşneag, Diego or Rasiowa (op.cit.) for the following properties, which will be needed in the sequel:

(7) $$x \rightarrow x = 1 \ ,$$

(8) $$1 \rightarrow x = x \ ,$$

(9) $$y \leq x \rightarrow y \ ,$$

(10) $$x \leq y \Longrightarrow y \rightarrow z \leq x \rightarrow z \ ,$$

(11) $$x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) \ ,$$

(12) $$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \ ,$$

(13) $$x \leq (x \rightarrow y) \rightarrow y \ .$$

Let us begin with a characterization of principal order filters.

Recall that a *deductive system* of a Hilbert algebra A is a subset $D \subseteq A$ such that (i) $1 \in D$, and (ii) if $x, x \rightarrow y \in D$ then $y \in D$. We will say that D is *bounded* provided it has a least element.

Proposition 2.1. *Principal order filters coincide with bounded deductive systems.*

Proof: Let us prove that $[a,1]$ is a bounded deductive system. For a is least element, $1 \in [a,1]$ and if $x, x \to y \in [a,1]$, then by applying (4), (11) and (8) we get

$$1 = a \to (x \to y) = (a \to x) \to (a \to y) = 1 \to (a \to y) = a \to y \ ,$$

hence $a \leq y$.

Conversely, let D be a deductive system with least element a. Then $a \leq d$ for every $d \in D$, showing that $D \subseteq [a,1]$. To prove that $[a,1] \subseteq D$, take $b \in [a,1]$. Then (4) implies $a \to b = 1 \in D$ and since $a \in D$, it follows that $b \in D$. □

To prove Theorem 2.1, we first note the following

Lemma 2.1. *For every $x, y \in A$, if $y \in [a,1]$, then $x \to y \in [a,1]$.*

Proof: By (9). □

Theorem 2.1. *$[a,1]$ is a subalgebra of $(A, \to, 1)$ and a bounded Hilbert algebra.*

Proof: The first statement follows from Lemma 2.1 and $1 \in [a,1]$. But it was shown by Diego [1965,1966] that the class of Hilbert algebras is equational, threfore the subalgebra $([a,1], \to, 1)$ is a bounded Hilbert algebra. □

The next Proposition generalizes results established by Buşneag [1985] (see also [2006]) for a bounded Hilbert algebra. The proof of properties (14)–(16) is the same as in op. cit., but we give it for the sake of self-containedness.

Proposition 2.2. I) *The assignment $x \mapsto x^a$ defines a map from A to $[a,1]$ which satisfies*

(14) $$x \leq y \Longrightarrow y^a \leq x^a \ ,$$

(15) $$x \leq x^{aa} \ ,$$

(16) $$x^{aaa} = x^a \ .$$

II) *The assignment $x \mapsto x^{aa}$ is a closure operator on A, for which the set $R_a(A)$ of fixed points is included in $[a,1]$.*

Proof: I) The fact that $x^a \in [a,1]$ follows from (9). Properties (14) and (15) are obtained by (10) and (13), respectively. Taking $x := x^a$ in (15) we get $x^a \leq x^{aaa}$, while (15) implies $x^{aaa} \leq x^a$ via (14).

II) The fact that $x \mapsto x^{aa}$ is a closure operator follows from (14) and (15) by a well-known argument. It is also well known that (14) and (15) imply $R_a(A) = \{x^a \mid x \in A\}$, whence the second part of II) follows via I). □

Taking into account Theorem 2.1 and using also the identity

(17) $$x \to y^a = y \to x^a \ ,$$

which follows by (12), the Glivenko-Frink-like theorem of Buşneag amounts to the following structure for $R_a(A) \subseteq [a,1]$:

Theorem 2.2. I) *A Boolean algebra $(R_a(A), \sqcap_a, \sqcup_a, {}^a, a, 1)$ is defined by the operations*

(18) $$x^a = x \to a \ ,$$

(19) $$x \sqcap_a y = (x \to y^a)^a = (y \to x^a)^a \ ,$$

(20) $$x \sqcup_a y = (x^a \sqcap_a y^a)^a = x^a \to y = y^a \to x \ .$$

II) *The mapping*

(21) $$\varphi : [a,1] \longrightarrow R_a(A)$$

defined by $\varphi(x) = x^{aa}$ *is a surjective homomorphism of bounded Hilbert algebras.*

3. Boolean-like algebras

Beside (18), we introduce the following operations on A:

$$x \wedge_a y = x^{aa} \sqcap_a y^{aa} = (x^{aa} \to y^a)^a = (y^{aa} \to x^a)^a \,, \tag{22}$$

$$x \vee_a y = x^{aa} \sqcup_a y^{aa} = x^{aa} \to y^a = y^{aa} \to x^a \,, \tag{23}$$

and we are going to study their properties.

For every *Boolean term f*, that is, a term of type (2,2,1,0,0), we denote by f^A and f^R the term function generated by f in the algebra $(A, \wedge_a, \vee_a, {}^a, a, 1)$ and in the Boolean algebra $(R_a(A), \sqcap_a, \sqcup_a, {}^a, a, 1)$, respectively.

Proposition 3.1. *For every Boolean term f which is not a variable we have the identity*

$$f^A(x_1, \ldots, x_n) = f^R(x_1^{aa}, \ldots, x_n^{aa}) \,. \tag{24}$$

PROOF: By algebraic induction. The constant terms satisfy $(0)^A = a = (0)^R$ and $(1)^A = 1 = (1)^R$. For the terms $x \wedge y$ and $x \vee y$ the property is checked by (22) and (23). Now suppose the terms g and h are not variables and satisfy (24). Then

$$\begin{aligned}(g \wedge h)^A(x_1, \ldots, x_n) &= g^A(x_1, \ldots, x_n) \wedge_a h^A(x_1, \ldots, x_n) \\ &= g^R(x_1^{aa}, \ldots, x_n^{aa}) \wedge_a h^R(x_1^{aa}, \ldots, x_n^{aa}) \\ &= (g^R(x_1^{aa}, \ldots, x_n^{aa}))^{aa} \sqcap_a (h^R(x_1^{aa}, \ldots, x_n^{aa}))^{aa} \\ &= g^R(x_1^{aa}, \ldots, x_n^{aa}) \sqcap_a h^R(x_1^{aa}, \ldots, x_n^{aa}) = (g \wedge h)^R(x_1^{aa}, \ldots, x_n^{aa})\end{aligned}$$

and a similar proof holds for $g \vee h$, while

$$\begin{aligned}(g')^A(x_1, \ldots, x_n) &= (g^A(x_1, \ldots, x_n))^a \\ &= (g^R(x_1^{aa}, \ldots, x_n^{aa}))^a = (g')^R(x_1^{aa}, \ldots, x_n^{aa}) \,.\end{aligned}$$

□

Proposition 3.2. *Suppose $f = g$ is a Boolean identity, where the Boolean terms f and g are not variables. Then the identity $f^A = g^A$ holds.*

PROOF: $f^A(x_1, \ldots, x_n) = f^R(x_1^{aa}, \ldots, x_n^{aa}) = g^R(x_1^{aa}, \ldots, x_n^{aa}) = g^A(x_1, \ldots, x_n)$. □

Corollary 3.1. *The operations $\wedge_a, \vee_a$ are commutative, associative, mutually distributive and satisfy the De Morgan laws.*

PROOF: For instance,

$$x \wedge_a (y \wedge_a z) = x^{aa} \sqcap_a (y^{aa} \sqcap_a z^{aa}) = (x^{aa} \sqcap_a y^{aa}) \sqcap_a z^{aa} = (x \wedge_a y) \wedge_a z \,.$$

□

Corollary 3.2. *For every Boolean term f such that $f(x_1, \ldots, x_n) = x_i$ (where $1 \leq i \leq n$) is a lattice identity, we have the identity*

$$f_A(x_1, \ldots, x_n) = x_i^{aa} \,. \tag{25}$$

PROOF: $f_A(x_1, \ldots, x_n) = f_R(x_1^{aa}, \ldots, x_n^{aa}) = x_i^{aa}$. □

Corollary 3.3. *The following identities hold:*

$$x \wedge_a x = x^{aa}, \;\; x \vee_a x = x^{aa} \,, \tag{26}$$

$$x \wedge_a (x \vee_a y) = x^{aa}, \quad x \vee_a (x \wedge_a y) = x^{aa} \ , \tag{27}$$

$$x \wedge_a 1 = x^{aa}, \quad x \vee_a a = x^{aa} \ . \tag{28}$$

Proposition 3.3. *For every Boolean term f such that* $f(x_1, \ldots, x_n) = 0$ *(such that* $f(x_1, \ldots, x_n) = 1$*) is a Boolean identity, we have the identity* $f_A(x_a, \ldots, x_n) = a$ *(the identity* $f(x_1, \ldots, x_n) = 1$*).*

Proof: $f_A(x_1, \ldots, x_n) = f_R(x_1^{aa}, \ldots, x_n^{aa}) = a$ and dually. □

Corollary 3.4. *The following identities hold:*

$$x \wedge_a a = a, \quad x \vee_a 1 = 1 \ , \tag{29}$$

$$x \wedge_a x^a = a, \quad x \vee_a x^a = 1 \ . \tag{30}$$

The Boolean likeness extends to the ordering as well.

The order relation in the Boolean algebra $R_a(A)$ is inherited from the ordering of $[a, 1]$, which, in its turn, is the restriction of the partial order of A. So, if $x, y \in R_a(A)$, then $x \le y \Longleftrightarrow x \sqcap_a y^a = a$.

Now for arbitrary $x, y \in A$ define

$$x \preceq y \Longleftrightarrow x^{aa} \le y^{aa} \ . \tag{31}$$

This relation is a quasi-order such that $x \le y \Longrightarrow x \preceq y$, its restriction on $R_a(A)$ is the partial oder $\le$ and which has a Boolean-like behaviour, as shown below.

$$x \wedge_a y \preceq x, y \preceq x \vee_a y \tag{32}$$

because $(x \wedge_a y)^{aa} = x \wedge_a y = x^{aa} \sqcap_a y^{aa} \le x^{aa}, y^{aa}$ and dually.

$$z \preceq x, y \Longrightarrow z \preceq x \wedge_a y \ \text{ and } \ x, y \preceq z \Longrightarrow x \vee_a y \preceq z \ . \tag{33}$$

For suppose $z \preceq x, y$. Then $z^{aa} \le x^{aa}, y^{aa}$, hence $z^{aa} \le x^{aa} \sqcap_a y^{aa} = x \wedge_a y = (x \wedge_a y)^{aa}$, and dually.

$$x \wedge_a y \le z \Longleftrightarrow x \preceq y^{aa} \to z^{aa} \ . \tag{34}$$

For $x \wedge_a y \le z \Longleftrightarrow x^{aa} \sqcap_a y^{aa} = (x^{aa} \sqcap_a y^{aa})^{aa} \le z^{aa} \Longleftrightarrow x^{aa} \le y^{aa} \to z^{aa}$. In particular, taking $z := a$ we get

$$x \wedge_a y = a \Longleftrightarrow x \preceq y^a \ . \tag{35}$$

Theorem 3.1. I) *The binary operations of the algebra* $(A, \wedge_a, \vee_a, ^a, a, 1)$ *are commutative, associative and mutually distributive, and properties* (26)–(34) *hold.*

II) *The mapping*

$$\varphi : (A, \wedge_a, \vee_a, ^a, a, 1) \longrightarrow (R_a(A), \sqcap_a, \sqcup_a, ^a, a, 1) \tag{36}$$

defined by $\varphi(x) = x^{aa}$ *is a surjective homomorphism.*

Proof: Part I) has already been proved. The surjectivity of φ is clear and

$$\varphi(x \wedge_a y) = (x \wedge_a y)^{aa} = (x^{aa} \sqcap_a y^{aa})^{aa} = x^{aa} \sqcap_a y^{aa} = \varphi(x) \sqcap_a \varphi(y)$$

and dually, while $\varphi(x^a) = x^{aaa} = (\varphi(x))^a$. □

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