Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 34(1), 2007, Pages 78–81
ISSN: 1223-6934

Some properties of the operation $x \sqcup y = (x \to y) \to ((y \to x) \to x)$ in a Hilbert algebra

Dan Dorin Taşcău

Abstract. The aim of this paper is to study some properties of Hilbert algebras relative to the operation $x \sqcup y = (x \to y) \to ((y \to x) \to x)$.

2000 Mathematics Subject Classification. 06F35, 03G25.
Key words and phrases. Hilbert algebra, deductive system, commutative Hilbert algebra.

1. Introduction

Hilbert algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value “true”. The theory of Hilbert algebras has been developed by A. Diego in [4] and the theory of Abbott algebras has been developed by Abbott in [1] and [2]. Both theories can be found in [9], which also contains the theory of implicational lattices.

2. Preliminaries

We recall some basic definitions and results that are necessary for this paper. For more details we refer to references.

Definition 2.1. A *Hilbert algebra* is an algebra $(H, \to, 1)$ of type (2,0) such that the following three axioms hold in H :
(a_1) $x \to (y \to x) = 1$;
(a_2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;
(a_3) If $x \to y = y \to x = 1$, then $x = y$.

If H is a Hilbert algebra, then the relation $x \leq y$ iff $x \to y = 1$ is a partial order on H which will be called the *natural ordering* of H. With respect to this ordering, 1 is the largest element of H.

In [4] it is proved that the Definition 2.1 is equivalent with:

Definition 2.2. A *Hilbert algebra* is an algebra $(H, \to, 1)$ of type (2,0) such that the following axioms hold in H :
(a_4) $x \to x = y \to y$;
(a_5) $(x \to x) \to x = x$;
(a_6) $x \to (y \to z) = (x \to y) \to (x \to z)$;
(a_7) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$.

Received: June 9, 2007.

So, we deduce that the class of Hilbert algebras forms a variety.

Following [1] and [2], an *implication algebra* is an algebra $(A, \to, 1)$ of type (2,0) satisfying the following axioms:

(a_8): $x \to x = 1$;
(a_9): $(x \to y) \to x = x$;
(a_{10}): $x \to (y \to z) = y \to (x \to z)$;
(a_{11}): $(x \to y) \to y = (y \to x) \to x$.

In [3] and [4] it is proved that if H is a Hilbert algebra and $x, y, z \in H$, then we have the following rules of calculus:

(c_1): $x \to 1 = 1$;
(c_2): $x \leq y \to x$;
(c_3): $x \to (y \to z) = y \to (x \to z)$;
(c_4): $x \leq (x \to y) \to y$;
(c_5): $((x \to y) \to y) \to y = x \to y$;
(c_6): $x \to y \leq (y \to z) \to (x \to z)$;
(c_7): If $x \leq y$, then $z \to x \leq z \to y$ and $y \to z \leq x \to z$.

For a Hilbert algebra H and $x, y \in H$ we define

$$x \sqcup y = (x \to y) \to ((y \to x) \to x).$$

In [3] and [4] the following rules of calculus are proved:

(c_8): $x, y \leq x \sqcup y$ and $x \sqcup y = y \sqcup x$;
(c_9): $x \sqcup x = x, x \sqcup 1 = 1$;
(c_{10}): $x \sqcup (x \to y) = 1$;
(c_{11}): $x \to (y \sqcup z) = (x \to y) \sqcup (x \to z)$;
(c_{12}): $(x \to y) \sqcup z = x \to (y \sqcup z)$;
(c_{13}): $(x \to y) \sqcup (y \to x) = 1$;
(c_{14}): $(x \to z) \sqcup (y \to z) = x \to (y \to z)$.

A Hilbert algebra H is called *commutative* if H verifies (a_{11}).

Clearly, every implication algebra is a Hilbert algebra (see [5], p. 527).

Theorem 2.1. *For a Hilbert algebra H the following are equivalent:*
(i) *H is commutative;*
(ii) *$(H, \to, 1)$ is an implication algebra.*

Proof. $(i) \Rightarrow (ii)$. We have to prove only (a_{11}). Indeed, for $x, y \in H$ we have $((x \to y) \to x) \to x = (x \to (x \to y)) \to (x \to y) = (x \to y) \to (x \to y) = 1$, hence $(x \to y) \to x \leq x$. Since by (c_1), $x \leq (x \to y) \to x$ we deduce that $x = (x \to y) \to x$.

$(ii) \Rightarrow (i)$. Let $x, y \in H$. By (a_7) we have $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y) \overset{(c_3)}{\Rightarrow} (y \to x) \to ((x \to y) \to x) = (x \to y) \to ((y \to x) \to y) \overset{(a_{10})}{\Rightarrow} (y \to x) \to x = (x \to y) \to y$, that is, H is a commutative algebra.

So, we obtain the last Corollary from [5]. ■

3. Some properties of operation $\sqcup$

Following (c_8), for $x, y \in H, x \sqcup y$ is an upper bound for x and y.

Theorem 3.1. *The following are equivalent:*
(i) *H is a $\vee-$ semilattice relative to $\sqcup$;*
(ii) *H is commutative.*

Proof. $(i) \Rightarrow (ii)$. Suppose that for every $x, y \in H, x \vee y$ exists and $x \vee y = x \sqcup y$. Since $x, y \leq (y \to x) \to x$ (by (c_2) and (c_4)) we deduce that $x \sqcup y = x \vee y \leq (y \to x) \to x$; since $(y \to x) \to x \leq x \sqcup y$ we deduce that $x \sqcup y = x \vee y = (y \to x) \to x$. Analogously $x \sqcup y = x \vee y = (x \to y) \to y$, hence $x \sqcup y = x \vee y = (y \to x) \to x = (x \to y) \to y$, that is, H is commutative.

$(i) \Rightarrow (ii)$. Suppose that H is commutative. For $x, y \in H$, clearly $x \sqcup y = (x \to y) \to ((y \to x) \to x) = (x \to y) \to ((x \to y) \to y) = (x \to y) \to y$, so to prove $x \vee y = x \sqcup y$, let $t \in H$ such that $x, y \leq t$. By (c_7) we deduce that $(x \to y) \to y \leq (t \to x) \to x = (x \to t) \to t = 1 \to t = t$, hence $x \sqcup y = (x \to y) \to y = x \vee y$. ■

Remark 3.1. *In* [5]*, p. 527, the author incorrectly mentions that " Theorem 3.3 in* [7] *claims that commutative Hilbert algebras are just those which are join semilattices w.r.t. the natural ordering". We recall the ennounce of Theorem 3.3 in* [7] *: A Hilbert algebra H is commutative iff it is a semilattice with respect to $\veebar$ (where for $x, y \in H, x \veebar y = (y \to x) \to x$) (not relative to the natural ordering!).*

For $a \in H$ we denote by $[a) = \{x \in H : a \leq x\}$. We have a similar result for Theorem 3.6. from [7] for the case of operation $\sqcup$:

Theorem 3.2. *A Hilbert algebra is commutative iff $[a) \cap [b) = [a \sqcup b)$, for all $a, b \in H$.*

Proof. $'' \Rightarrow''$. Suppose H is commutative and let $a, b \in H$. Since $a, b \leq a \sqcup b$, we deduce that $[a \sqcup b) \subseteq [a) \cap [b)$. If $x \in [a) \cap [b)$, then $a \leq x$ and $b \leq x$. Then $a \sqcup b = (a \to b) \to b \leq (x \to b) \to b = (b \to x) \to x = 1 \to x = x$, so $x \in [a \sqcup b)$ and we deduce that $[a) \cap [b) = [a \sqcup b)$.

$'' \Leftarrow''$Let $a, b \in H$ and suppose $[a) \cap [b) = [a \sqcup b)$. By $(c_2), (c_4)$ we deduce that $(a \to b) \to b \in [a) \cap [b) = [a \sqcup b)$, hence $a \sqcup b = (b \to a) \to ((a \to b) \to b) \leq (a \to b) \to b$, that is $a \sqcup b = (a \to b) \to b$. Analogously we deduce that $a \sqcup b = (b \to a) \to a$, hence $(a \to b) \to b = (b \to a) \to a$, that is, H is commutative. ■

4. New characterization for the maximal deductive systems

We recall that a subset D of a Hilbert algebra H is called a *deductive system* if

(a_{12})**:** $1 \in D$;

(a_{13})**:** If $x, x \to y \in D$, then $y \in D$.

We denote by $Ds(H)$ the set of all deductive systems of H.

For a Hilbert algebra H we denote by $Max(H)$ the set of all maximal deductive systems of H. For $D \in Ds(H)$ and $a \in H$, we denote by $D(a)$ the deductive system generated by $D \cup \{a\}$.

We recall ([3], [4]) that $D(a) = \{x \in H : a \to x \in D\}$.

Theorem 4.1. *For $D \in Ds(H)$ the following are equivalent:*

(i) *$D \in Max(H)$;*

(ii) *If $x, y \in H$ and $x \sqcup y \in D$, then $x \in D$ or $y \in D$.*

Proof. $(i) \Rightarrow (ii)$. Let $D \in Max(H)$ and suppose by contrary that there exist $x, y \in H$ such that $x \notin D$ and $y \notin D$. By the maximality of D we deduce that $D(x) = D(y) = H$, hence $x \to y, y \to x \in D$. From $x \sqcup y = (x \to y) \to ((y \to x) \to x) \in D$ we deduce that $x \in D$, a contradiction.

$(ii) \Rightarrow (i)$. Suppose that D is not maximal, that is, there existS $D' \in Ds(H), D' \neq H$ such that $D \subset D'$ (that is, there exists $x \in D' \backslash D$ and $y \in H \backslash D$). Since $x \sqcup (x \to y) = 1 \in D$ (by (c_{10})) and $x \notin D$, then $x \to y \in D \subset D'$, hence $x \to y \in D'$. Since $x \in D'$ we deduce that $y \in D'$, a contradiction. ■

Theorem 4.2. *For $M \in Ds(H), M \neq H$, the following are equivalent:*
(i) $M \in Max(H)$;
(ii) *If $x \notin M$ then $x \to y \in M$, for every $y \in M$.*

Proof. $(i) \Rightarrow (ii)$. Suppose $M \in Max(H)$ and $x \notin M$. Since $x \sqcup (x \to y) = 1 \in M$ for every $y \in H$, by Theorem 4.1 we deduce that $x \to y \in M$.

$(ii) \Rightarrow (i)$. Suppose that M is not maximal, that is, there exists $N \in Max(H)$ such that $M \subset N$ (hence there exists $x_0 \in N \backslash M$). Since N is proper there exists $y_0 \in H \backslash N$. Then $x_0 \to y_0 \in M \subset N \Rightarrow x_0 \to y_0 \in N \Rightarrow y_0 \in N$, a contradiction. ■

We recall that an element $r \in H$ is called *regular* if $(r \to x) \to r = r$, for every $x \in H$.

For $D \in Ds(H)$ and $a \in H \backslash D$ we say that *a is associated with D* or that *D is maximal relative to a*, if D is maximal with respect to the deductive systems which do not contain a, i.e. D is maximal in the set $\{D' \in Ds(H) : a \notin D'\}$.

We have (see [4], p. 23):

Theorem 4.3. *(A. Monteiro) For $D \in Ds(H)$ and $a \in H \backslash D$, the following are equivalent:*
(i)**:** *D is maximal relative to a;*
(ii)**:** *$a \notin D$ and $x \notin D \Rightarrow x \to a \in D$.*

Corollary 4.1. *If $r \in H$ is regular and $D \in Ds(H)$ is maximal relative to r, then $D \in Max(H)$.*

Proof. We have $r \notin D$ and suppose by contrary that D is not maximal, that is, there is, $D' \in Ds(H), D' \neq H$ such that $D \subset D'$. Then there exists $x_0 \in H \backslash D'$ and by the maximality of D relative to r we deduce that $r \in D'$, hence $r \to x_0 \notin D'$. Then $r \to x_0 \notin D$, hence by Theorem 4.3, $(r \to x_0) \to r = r \in D$, a contradiction. ■

Remark 4.1. *The above Corollary is stated in* [8] *(see Proposition 13.3), but without proof !*

References

[1] J. C. Abbott: *Semi-Boolean algebra*, Matem. Vestnik, 4 (19) (1967), 177-198.
[2] J. C. Abbott: *Implicational algebras*, Bull. Math. Soc. Sci. Math. Roumanie, 11 (59) (1967), 3-23.
[3] D. Buşneag: *Contribuţii la studiul algebrelor Hilbert,* Ph. D. Thesis, Univ. of Bucharest, 1985.
[4] A. Diego: *Sur les algebres de Hilbert*, Collection de Logique Math. Serie A, no. 21, (1966), 1-52.
[5] R. Halaš: *Remarks on commutative Hilbert algebras,* Math. Bohemica, no. 4, 127 (2002), 525-529.
[6] S. M. Hong, Y. B. Jun: *On a special class of Hilbert algebras*, Algebra Coll. 3:3 (1996), 285-288.
[7] Y. B. Jun: *Commutative Hilbert algebras*, Soochow Journal of Mathematics, vol. 22, no. 4 (1996), 477-484.
[8] H. Porta: *Sur quelques algebres de la logique*, Portugaliae Math. vol.40, fasc. 1 (1981), 210-238.
[9] H. Rasiowa: *An Algebraic Approach to Non-Classical Logics*, North-Holland Pub. Comp., Amsterdam, 1974.
[10] A. Torens: *On the role of the polynomial $(X \to Y) \to Y$ in some implicative algebras,* Zeitschr. f. math. Logik und Grundlagen d. Math. Bd. 34, (1988), 117-122.

(Dan Dorin Taşcău) National College "Tudor Vladimirescu" Tg. Jiu
Street June, 11, no. 92 A
Tg. Jiu - Gorj/Romania
E-mail address: dorintascau@yahoo.com