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The Problem of the Center for Quartic Differential Systems with an Affine Invariant Straight Line and the Line at Infinity of Maximal Multiplicity

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Abstract. Quartic differential systems with a center-focus critical point and non-degenerate infinity which admit an affine invariant straight line are studied. It is shown that, in this class, the maximal multiplicity of the line at infinity is six. Modulo the affine transformations and time rescaling the subclasses of systems with the line at infinity of multiplicity two, three,..., six are determined. Moreover, for quartic differential systems with an affine invariant straight line and the line at infinity of maximal multiplicity, the problem of the center is solved.

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1. Introduction

We consider a real polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are real and coprime polynomials in the variables x and y, $\dot{x} = dx/dt$, $\dot{y} = dy/dt$. The degree n of this polynomial differential system is the maximum degrees of the polynomials P and Q, $n = max\{degP(x, y), degQ(x, y)\}$. If $n = 2$ (respectively, $n = 3$, $n = 4$), then the system (1) is called a quadratic (respectively, a cubic, a quartic) differential system.

Assume that $n = 4$ and let (x_0, y_0) be a critical point for (1) with pure imaginary eigenvalues. Then via an affine transformation of coordinates and time rescaling the system (1) can be brought to the form

$$\begin{cases} \dot{x} = y + p_2(x, y) + p_3(x, y) + p_4(x, y) \equiv P(x, y), \\ \dot{y} = -(x + q_2(x, y) + q_3(x, y) + q_4(x, y)) \equiv Q(x, y), \end{cases} \tag{2}$$

where

$$p_i(x, y) = \sum_{j=0}^{i} a_{i-j,j} x^{i-j} y^j, \; q_i(x, y) = \sum_{j=0}^{i} b_{i-j,j} x^{i-j} y^j, \; i \in \{2, 3, 4\}$$

are homogeneous polynomials in x and y of degree i with real coefficients.

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Suppose that

$$yp_4(x,y) + xq_4(x,y) \not\equiv 0, \tag{3}$$

then the system (2) has at most five distinct critical points at infinity.

The origin $(0,0)$ is either a focus or a center for system (2), i.e. the trajectories can be spirals or closed in some neighborhood of a critical point $(0,0)$. The problem of distinguishing between a center and a focus (the problem of the center) is open for polynomial differential systems of degree $n \geq 3$.

The problem of the center was solved for some classes of cubic differential systems with invariant straight lines: at least three invariant straight lines [4], [5], [7]; two parallel invariant straight lines [6], [9]; the line at infinity of multiplicity four [10] and five [11]; three affine invariant straight lines of total multiplicity four [12]; the line at infinity and an affine real invariant straight line of total multiplicity four [15].

The center conditions were found for some classes of quartic differential systems with: the line at infinity of maximal multiplicity [16]; an affine invariant straight line of maximal multiplicity [19].

The goal of this paper is to solve the problem of the center for quartic differential systems (2) with an affine invariant straight line and the line at infinity of maximal multiplicity. The paper is organized as follows. In Section 2 we recall some known results concerning the existence and multiplicity of invariant straight lines. In Section 3 we prove that in the class of quartic systems, with a center-focus critical point and an affine invariant straight line, the maximal multiplicity of the line at infinity is six. We determine the subclasses of quartic differential systems (2) with an affine invariant straight line and the line at infinity of multiplicity two, three,...,six. In Section 4 we solve the problem of the center for quartic differential systems with an affine invariant straight line and the line at infinity of maximal multiplicity.

2. Multiplicity of invariant straight lines in quartic differential systems

We consider the quartic differential system (2) and the vector field

$$\mathbb{X} = P\left(x,y\right)\frac{\partial}{\partial x} + Q\left(x,y\right)\frac{\partial}{\partial y}$$

associated to system (2). One of the most important question to ask is if any non-singular trajectories of the system are contained in algebraic curves, for example, $f(x,y) = 0$, where $f(x,y)$ is a polynomial.

Definition 2.1. An algebraic curve $f(x,y) = 0$ in $\mathbb{C}^2$ with $f \in \mathbb{C}[x,y]$ is an invariant algebraic curve of a quartic system (2) if

$$\mathbb{X}(f) \equiv f(x,y)K_f(x,y), \tag{4}$$

for some polynomial $K_f \in \mathbb{C}[x,y]$, $\deg(K) \leq n-1$ called the cofactor of the invariant algebraic curve $f(x,y) = 0$.

Definition 2.2. Let $g,h \in \mathbb{C}[x,y]$ be relatively prime in the ring $\mathbb{C}[x,y]$. The function $f = \exp(g/h)$ is called an exponential factor of (2) if it satisfies the identity (4) for some polynomial $K_f \in \mathbb{C}[x,y]$, $\deg(K) \leq n-1$.

In particular, a straight line

$$l \equiv \alpha x + \beta y + \gamma = 0,\ \alpha, \beta, \gamma \in \mathbb{C}$$

is invariant for (2) if there exists a polynomial $K_l \in \mathbb{C}[x, y]$ such that

$$\alpha P(x,y) + \beta Q(x,y) \equiv (\alpha x + \beta y + \gamma) K_l(x,y),\ (x,y) \in \mathbb{R}^2. \tag{5}$$

In this paper we are interested in studying quartic differential systems with a center-focus critical point having multiple invariant straight lines.

Let us consider the homogeneous system associated with (2) of the form

$$\begin{cases} \dot{x} = yZ^3 + p_2(x,y)Z^2 + p_3(x,y)Z + p_4(x,y) \equiv P(x,y,Z), \\ \dot{y} = -(xZ^3 + q_2(x,y)Z^2 + q_3(x,y)Z + q_4(x,y)) \equiv Q(x,y,Z). \end{cases} \tag{6}$$

Denote $\mathbb{X}_\infty = P(x,y,Z)\frac{\partial}{\partial x} + Q(x,y,Z)\frac{\partial}{\partial y}$ and $\mathbb{E}_\infty = P \cdot \mathbb{X}_\infty(Q) - Q \cdot \mathbb{X}_\infty(P)$. The polynomial $\mathbb{E}_\infty$ is expressed as

$$\begin{aligned} \mathbb{E}_\infty = {} & A_2(x,y) + A_3(x,y)Z + A_4(x,y)Z^2 + A_5(x,y)Z^3 \\ & + A_6(x,y)Z^4 + A_7(x,y)Z^5 + A_8(x,y)Z^6 + A_9(x,y)Z^7 \\ & + A_{10}(x,y)Z^8 + A_{11}(x,y)Z^9, \end{aligned} \tag{7}$$

where $A_k(x,y)$, $k = 2, ..., 11$, are polynomials in x and y.

According to [3], the invariant straight line $\alpha x + \beta y + \gamma = 0$ has *multiplicity* ν, if ν is the greatest positive integer such that $(\alpha x + \beta y + \gamma)^\nu$ divides $\mathbb{E} = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$. For system (2), the line at infinity $Z = 0$ is said to have *multiplicity* ν if

$$A_2(x,y) \equiv 0, ..., A_\nu(x,y) \equiv 0,\ A_{\nu+1}(x,y) \not\equiv 0,$$

i.e. $\nu - 1$ is the greatest positive integer such that $Z^{\nu-1}$ divides $\mathbb{E}_\infty$. If $A_2(x,y) \not\equiv 0$, the line $Z = 0$ is said to have multiplicity one.

More information about the multiplicity of an invariant algebraic curve and, in particular, of the line at infinity, can be found in the works [3], [13].

Polynomial differential systems of degree n, $n \in \{3, 4\}$ with a center-focus type critical point and a multiple line at infinity were examined in [10], [11], and [18]. Cubic differential systems with multiple invariant straight lines, including the line at infinity, were investigated in [2], [13], [20].

Quartic differential systems (2) having an affine invariant straight line of maximal multiplicity were studied in [14], and the systems (2) with the line at infinity of maximal multiplicity were studied in [17]. It was established that the maximal multiplicity of an affine invariant straight line (or of the line at infinity) equals ten.

In this paper, we prove that the maximal multiplicity of the line at infinity $Z = 0$ in quartic differential systems (2) possessing an affine invariant straight line is six.

3. Quartic differential systems with an affine invariant straight line and a multiple line at infinity

Let the quartic system (2) have an affine real invariant straight line $l = 0$, i.e. the identity (5) holds. By a transformation of the form

$$x \to \omega(x\cos\varphi + y\sin\varphi),\ y \to \omega(y\cos\varphi - x\sin\varphi),\ \omega \neq 0,$$

we can make the line $l = 0$ to be $x = 1$. Then,

$$\begin{aligned} & a_{40} = -(a_{20} + a_{30}),\ a_{31} = -(1 + a_{11} + a_{21}), \\ & a_{22} = -(a_{02} + a_{12}),\ a_{13} = -a_{03},\ a_{04} = 0, \end{aligned} \tag{8}$$

and the system (2) can be written in the form

$$\begin{cases} \dot{x} = (1-x)(a_{20}x^2 + a_{20}x^3 + a_{30}x^3 + y + xy + a_{11}xy + x^2y + \\ \quad +a_{11}x^2y + a_{21}x^2y + a_{02}y^2 + a_{02}xy^2 + a_{12}xy^2 + a_{03}y^3) \equiv P(x,y), \\ \dot{y} = -(x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + \\ \quad +b_{03}y^3 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4) \equiv Q(x,y). \end{cases} \tag{9}$$

Denote by $m(Z)$ the multiplicity of the line at infinity $Z = 0$. Then taking into account (3), we determine the subclasses of systems $\{(9), (3)\}$ for which the line at infinity attains multiplicities two, three, ..., and six.

3.1. Systems (9) with $m(Z) \geq 2$. The line at infinity has multiplicity at least two if $A_2(x,y) \equiv 0$. The polynomial $A_2(x,y)$ looks as $A_2(x,y) = -A_{21}(x,y)A_{22}(x,y)$, where

$A_{21}(x,y) = x(b_{40}x^4 - (a_{20}+a_{30}-b_{31})x^3y - (1+a_{11}+a_{21}-b_{22})x^2y^2 - (a_{02}+a_{12}-b_{13})xy^3 - (a_{03}-b_{04})y^4)$, and

$A_{22}(x,y) = (a_{20}b_{31}+a_{30}b_{31}-b_{40}-a_{11}b_{40}-a_{21}b_{40})x^6 + 2(a_{20}b_{22}+a_{30}b_{22}-a_{02}b_{40}-a_{12}b_{40})x^5y + (3a_{20}b_{13}+3a_{30}b_{13}+b_{22}+a_{11}b_{22}+a_{21}b_{22}-a_{02}b_{31}-a_{12}b_{31}-3a_{03}b_{40})x^4y^2 + 2(2a_{20}b_{04}+2a_{30}b_{04}+b_{13}+a_{11}b_{13}+a_{21}b_{13}-a_{03}b_{31})x^3y^3 + (3b_{04}+3a_{11}b_{04}+3a_{21}b_{04}+a_{02}b_{13}+a_{12}b_{13}-a_{03}b_{22})x^2y^4 + 2(a_{02}+a_{12})b_{04}xy^5 + a_{03}b_{04}y^6$.

As $A_{21}(x,y) = yp_4(x,y) + xq_4(x,y) \not\equiv 0$, we assume that $A_{22}(x,y)$ vanishes identically. The solutions of the identity $A_{22}(x,y) \equiv 0$ leads to the following result:

Lemma 3.1. *The line at infinity has multiplicity at least two in the quartic system (9) if and only if the coefficients of (9) satisfy one of the following sets of conditions:*

$$a_{03} = 0,\ a_{12} = -a_{02},\ a_{21} = -1 - a_{11},\ a_{30} = -a_{20}; \tag{10}$$

$$a_{03} = 0,\ a_{12} = -a_{02},\ a_{21} = -1 - a_{11},\ b_{04} = b_{13} = b_{22} = b_{31} = 0; \tag{11}$$

$$\begin{gathered} a_{03} = 0,\ a_{12} = -a_{02},\ b_{04} = b_{13} = b_{22} = 0, \\ b_{40} = (a_{20}+a_{30})b_{31}/(1+a_{11}+a_{21}); \end{gathered} \tag{12}$$

$$\begin{gathered} a_{03} = b_{04} = b_{13} = 0,\ b_{31} = (1+a_{11}+a_{21})b_{22}/(a_{02}+a_{12}), \\ b_{40} = (a_{20}+a_{30})b_{22}/(a_{02}+a_{12}); \end{gathered} \tag{13}$$

$$\begin{gathered} b_{04} = 0,\ b_{22} = (a_{02}+a_{12})b_{13}/a_{03},\ b_{31} = (1+a_{11}+a_{21})b_{13}/a_{03}, \\ b_{40} = (a_{20}+a_{30})b_{13}/a_{03}. \end{gathered} \tag{14}$$

3.2. Systems (9) with $m(Z) \geq 3$. The line at infinity has multiplicity $m(Z)$ at least three if $\{A_2(x,y) \equiv 0,\ A_3(x,y) \equiv 0\}$. In each of the sets of equalities (10)-(14) in Lemma 3.1, the identity $A_3(x,y) \equiv 0$ leads, respectively, to the following series of conditions:

(10) $\Rightarrow A_3(x,y) \equiv 0 \Rightarrow$

$$a_{02} = 0,\ a_{11} = -1,\ a_{20} = 0; \tag{15}$$

$$a_{02} = 0,\ a_{11} = -1,\ b_{04} = b_{13} = b_{22} = b_{31} = 0; \tag{16}$$

$$a_{02} = 0,\ b_{04} = b_{13} = b_{22} = 0,\ b_{40} = a_{20}b_{31}/(1+a_{11}); \tag{17}$$

$$b_{04} = b_{13} = 0,\ b_{31} = (1+a_{11})b_{22}/a_{02},\ b_{40} = a_{20}b_{22}/a_{02}. \tag{18}$$

(11) $\Rightarrow A_3(x,y) \equiv 0 \Rightarrow$

$$a_{30} = -a_{20},\ a_{02} = 0,\ a_{11} = -1; \tag{19}$$

$$b_{03}=0,\ b_{12}=a_{02}b_{40}/(a_{20}+a_{30}),\ b_{21}=(1+a_{11})b_{40}/(a_{20}+a_{30}),\\ b_{30}=-a_{30}b_{40}/(a_{20}+a_{30}). \tag{20}$$

(12) $\Rightarrow A_3(x,y)\equiv 0 \Rightarrow$

$$b_{03}=0,\ b_{12}=a_{02}b_{31}/(1+a_{11}+a_{21}),\ b_{21}=-a_{21}b_{31}/(1+a_{11}+a_{21}),\\ b_{31}=-a_{20}-a_{30}; \tag{21}$$

$$b_{03}=0,\ b_{12}=a_{02}b_{31}/(1+a_{11}+a_{21}),\ b_{21}=-a_{21}b_{31}/(1+a_{11}+a_{21}),\\ b_{30}=-a_{30}b_{31}/(1+a_{11}+a_{21}). \tag{22}$$

(13) $\Rightarrow A_3(x,y)\equiv 0 \Rightarrow$

$$b_{03}=0,\ b_{12}=-a_{12}b_{22}/(a_{02}+a_{12}),\ b_{30}=-a_{30}b_{22}/(a_{02}+a_{12}),\\ b_{21}=-a_{21}b_{22}/(a_{02}+a_{12}); \tag{23}$$

$$b_{03}=0,\ \ b_{12}=-a_{12}b_{22}/(a_{02}+a_{12}),\\ b_{30}=(a_{02}b_{21}+a_{02}a_{11}b_{21}+a_{12}b_{21}+a_{11}a_{12}b_{21}+a_{02}a_{21}b_{21}+\\ +a_{12}a_{21}b_{21}+a_{21}b_{22}+a_{11}a_{21}b_{22}+a_{21}^2b_{22}-a_{02}a_{30}b_{22}-\\ -a_{12}a_{30}b_{22}+a_{02}b_{21}b_{22}+a_{12}b_{21}b_{22}+a_{21}b_{22}^2)(a_{02}+a_{12})^2,\\ a_{20}=-(a_{02}a_{30}+a_{12}a_{30}+b_{22}+a_{11}b_{22}+a_{21}b_{22}+b_{22}^2)/(a_{02}+a_{12}). \tag{24}$$

(14) $\Rightarrow A_3(x,y)\equiv 0 \Rightarrow$

$$a_{30}=-(a_{03}^2a_{20}-a_{03}b_{03}-a_{03}a_{11}b_{03}-a_{03}a_{21}b_{03}+a_{02}b_{03}^2+a_{12}b_{03}^2-\\ -b_{03}^3)/a_{03}^2,\quad b_{13}=-b_{03},\quad b_{21}=(-a_{02}a_{12}b_{03}-a_{12}^2b_{03}+\\ +a_{03}a_{21}b_{03}+a_{12}b_{03}^2+a_{02}a_{03}b_{12}+a_{03}a_{12}b_{12}-a_{03}b_{03}b_{12})/a_{03}^2,\\ b_{30}=-(a_{03}a_{12}b_{03}+a_{03}a_{11}a_{12}b_{03}+a_{03}^2a_{20}b_{03}+a_{03}a_{12}a_{21}b_{03}-\\ -a_{03}b_{03}^2-a_{03}a_{11}b_{03}^2-a_{02}a_{12}b_{03}^2-a_{12}^2b_{03}^2-a_{03}a_{21}b_{03}^2+a_{02}b_{03}^3+\\ +2a_{12}b_{03}^3-b_{03}^4-a_{03}^2b_{12}-a_{03}^2a_{11}b_{12}-a_{03}^2a_{21}b_{12}+a_{02}a_{03}b_{03}b_{12}+\\ +a_{03}a_{12}b_{03}b_{12}-a_{03}b_{03}^2b_{12})/a_{03}^3; \tag{25}$$

$$b_{13}=-b_{03},\ b_{12}=a_{12}b_{03}/a_{03},\ b_{21}=a_{21}b_{03}/a_{03},\ b_{30}=a_{30}b_{03}/a_{03}. \tag{26}$$

It is easy to see that the conditions {(10), (16)} and {(11), (19)} are the same.

Lemma 3.2. *In the quartic system* (9)*, the line at infinity has multiplicity at least three if and only if the coefficients of* (9) *fulfill one of the following sets of conditions:*
1) {(10), (15)}, *2)* {(10), (16)}, *3)* {(10), (17)}, *4)* {(10), (18)},
5) {(11), (20)}, *6)* {(12), (21)}, *7)* {(12), (22)}, *8)* {(13), (23)},
9) {(13), (24)}, *10)* {(14), (25)}, *11)* {(14), (26)}.

3.3. Systems (9) with $m(Z)\geq 4$. The multiplicity $m(Z)$ of the line at infinity is at least four if $\{A_2(x,y)\equiv 0,\ A_3(x,y)\equiv 0,\ A_4(x,y)\equiv 0\}$. Under conditions 1)-11) of Lemma 3.2, we solve the identity $A_4(x,y)\equiv 0$, yielding, respectively:

{(10), (15)} $\Rightarrow A_4(x,y)\equiv 0 \Rightarrow$

$$b_{04}=b_{13}=b_{22}=b_{40}=0. \tag{27}$$

{(10), (16)} $\Rightarrow A_4(x,y)\equiv 0 \Rightarrow$

$$b_{40}=a_{20}(a_{20}+b_{21}),\ b_{03}=b_{12}=0. \tag{28}$$

{(10), (17)} $\Rightarrow A_4(x,y)\equiv 0 \Rightarrow$

$$b_{03}=0,\ b_{12}=-(1+a_{11}),\\ b_{30}=a_{20}((1+a_{11})(a_{20}+b_{21})-b_{31})/(1+a_{11})^2. \tag{29}$$

$\{(10), (18)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} b_{03} = -a_{02},\ b_{22} = -(1+a_{11})(1+a_{11}+b_{12}) + a_{02}(a_{20}+b_{21}), \\ b_{30} = a_{20}(1+a_{11}+b_{12})/a_{02}. \end{gathered} \tag{30}$$

$\{(11), (20)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} b_{02} = -a_{02}b_{40}/(a_{20}+a_{30}),\ b_{11} = -a_{11}b_{40}/(a_{20}+a_{30}), \\ b_{20} = -a_{20}b_{40}/(a_{20}+a_{30}). \end{gathered} \tag{31}$$

$\{(12), (21)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} b_{02} = a_{02}(a_{20}+a_{30})/(1+a_{11}+a_{21}), \\ b_{11} = a_{11}(a_{20}+a_{30})/(1+a_{11}+a_{21}), \\ b_{20} = a_{20}(a_{20}+a_{30})/(1+a_{11}+a_{21}), \\ b_{30} = a_{30}(a_{20}+a_{30})/(1+a_{11}+a_{21}). \end{gathered} \tag{32}$$

$\{(12), (22)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} b_{02} = -a_{02}b_{31}/(1+a_{11}+a_{21}),\ b_{11} = -a_{11}b_{31}/(1+a_{11}+a_{21}), \\ b_{20} = -a_{20}b_{31}/(1+a_{11}+a_{21}). \end{gathered} \tag{33}$$

$\{(13), (23)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} b_{02} = -a_{02}b_{22}/(a_{02}+a_{12}),\ b_{11} = -a_{11}b_{22}/(a_{02}+a_{12}), \\ b_{20} = -a_{20}b_{22}/(a_{02}+a_{12}); \end{gathered} \tag{34}$$

$$\begin{gathered} a_{30} = ((1+a_{11})(1+a_{11}+2a_{21}) - 4(a_{02}+a_{12})a_{20} + a_{21}^2)/(4(a_{02}+a_{12})), \\ b_{02} = a_{02}(1+a_{11}+a_{21})/(2(a_{02}+a_{12}), \\ b_{11} = a_{11}(1+a_{11}+a_{21})/(2(a_{02}+a_{12}), \\ b_{22} = -(1+a_{11}+a_{21})/2. \end{gathered} \tag{35}$$

$\{(13), (24)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} a_{30} = (-a_{12} - 2a_{11}a_{12} - a_{11}^2a_{12} - 2a_{12}a_{21} - 2a_{11}a_{12}a_{21} - \\ -a_{12}a_{21}^2 + 4a_{02}^2b_{21} + 8a_{02}a_{12}b_{21} + 4a_{12}^2b_{21})/(4(a_{02}+a_{12})^2), \\ b_{11} = (a_{02}a_{11} + a_{02}a_{11}^2 + a_{11}a_{12} + a_{11}^2a_{12} + a_{02}a_{11}a_{21} + a_{12}a_{21} + \\ +2a_{11}a_{12}a_{21} + a_{12}a_{21}^2 - 2a_{02}a_{12}b_{21} - 2a_{12}^2b_{21})/(2(a_{02}+a_{12})^2)), \\ b_{02} = a_{02}(1+a_{11}+a_{21})/(2(a_{02}+a_{12})),\quad b_{22} = -(1+a_{11}+a_{21})/2; \end{gathered} \tag{36}$$

$$\begin{gathered} b_{02} = a_{02}(1+a_{11}+a_{21})/(2(a_{02}+a_{12})), \\ b_{11} = a_{11}(1+a_{11}+a_{21}))/(2(a_{02}+a_{12})), \\ b_{21} = a_{21}(1+a_{11}+a_{21})/(2(a_{02}+a_{12})), \\ b_{22} = -(1+a_{11}+a_{21})/2; \end{gathered} \tag{37}$$

$$\begin{gathered} b_{21} = -a_{21}b_{22}/(a_{02}+a_{12}),\ b_{02} = -a_{02}b_{22}/(a_{02}+a_{12}), \\ b_{20} = b_{22}(a_{02}a_{30} + a_{12}a_{30} + b_{22} + a_{11}b_{22} + a_{21}b_{22} + b_{22}^2)/(a_{02}+a_{12})^2, \\ b_{11} = -(a_{02}a_{12}b_{21} + a_{12}^2b_{21} + a_{02}a_{11}b_{22} + a_{11}a_{12}b_{22} + \\ +a_{12}a_{21}b_{22})/(a_{02}+a_{12})^2. \end{gathered} \tag{38}$$

$\{(14),\ (25)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} a_{21} = -(a_{03} + a_{03}a_{11} - 2a_{02}b_{03} - 2a_{12}b_{03} + 3b_{03}^2)/a_{03}, \\ a_{20} = (a_{02}a_{03}b_{02} + a_{03}a_{12}b_{02} - a_{02}^2 b_{03} + a_{03}b_{03} + a_{03}a_{11}b_{03} - a_{02}a_{12}b_{03} - \\ -3a_{03}b_{02}b_{03} + 2a_{02}b_{03}^2)/a_{03}^2, \quad b_{12} = -(a_{03}b_{02} - a_{02}b_{03} - a_{12}b_{03})/a_{03}, \\ b_{20} = (-a_{03}^2 b_{02} - a_{03}^2 a_{11}b_{02} - a_{02}a_{03}a_{12}b_{02} - a_{03}a_{12}^2 b_{02} + a_{03}^2 b_{02}^2 + \\ +a_{02}a_{03}b_{03} + a_{02}^2 a_{12}b_{03} - a_{03}a_{11}a_{12}b_{03} + a_{02}a_{12}^2 b_{03} + 2a_{02}a_{03}b_{02}b_{03} + \\ +5a_{03}a_{12}b_{02}b_{03} - 3a_{02}^2 b_{03}^2 + a_{03}b_{03}^2 + 3a_{03}a_{11}b_{03}^2 - 5a_{02}a_{12}b_{03}^2 - \\ -7a_{03}b_{02}b_{03}^2 + 6a_{02}b_{03}^3 + a_{02}a_{03}^2 b_{11} + a_{03}^2 a_{12}b_{11} - 2a_{03}^2 b_{03}b_{11})/a_{03}^3; \end{gathered} \tag{39}$$

$$\begin{gathered} a_{21} = -(a_{03} + a_{03}a_{11} - 2a_{02}b_{03} - 2a_{12}b_{03} + 3b_{03}^2)/a_{03}, \quad b_{12} = a_{12}b_{03}/a_{03}, \\ b_{02} = a_{02}b_{03}/a_{03}, \ b_{20} = (-a_{02}a_{11}b_{03} - a_{11}a_{12}b_{03} + a_{03}a_{20}b_{03} + 2a_{11}b_{03}^2 + \\ +a_{02}a_{03}b_{11} + a_{03}a_{12}b_{11} - 2a_{03}b_{03}b_{11})/a_{03}^2; \end{gathered} \tag{40}$$

$$b_{12} = a_{12}b_{03}/a_{03}, \ b_{02} = a_{02}b_{03}/a_{03}, \ b_{11} = a_{11}b_{03}/a_{03}, \ b_{20} = a_{20}b_{03}/a_{03}. \tag{41}$$

$\{(14),\ (26)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} a_{21} = -(a_{03} + a_{03}a_{11} - 2a_{02}b_{03} - 2a_{12}b_{03} + 3b_{03}^2)/a_{03}, \\ a_{30} = -(a_{03}^2 a_{20} - a_{02}b_{03}^2 - a_{12}b_{03}^2 + 2b_{03}^3)/a_{03}^2, \\ b_{02} = a_{02}b_{03}/a_{03}, \quad b_{20} = (-a_{02}a_{11}b_{03} - a_{11}a_{12}b_{03} + a_{03}a_{20}b_{03} + \\ +2a_{11}b_{03}^2 + a_{02}a_{03}b_{11} + a_{03}a_{12}b_{11} - 2a_{03}b_{03}b_{11})/a_{03}^2; \end{gathered} \tag{42}$$

$$b_{02} = a_{02}b_{03}/a_{03}, \ b_{20} = a_{20}b_{03}/a_{03}, \ b_{11} = a_{11}b_{03}/a_{03}. \tag{43}$$

The set of conditions $\{(12),\ (21),\ (32)\}$ is a particular case of $\{(12),\ (22),\ (33)\}$. Similarly, the set $\{(13),\ (24),\ (37)\}$ is a particular case of $\{(13),\ (23),\ (35)\}$; the set $\{(13),\ (24),\ (38)\}$ is a particular case of $\{(13),\ (23),\ (34)\}$; and the set $\{(14),\ (25),\ (41)\}$ is a particular case of $\{(14),\ (26),\ (43)\}$. Moreover, the sets of conditions $\{(14),\ (25),\ (40)\}$ and $\{(14),\ (26),\ (43)\}$ are identical.

Lemma 3.3. *In the quartic system* (9)*, the line at infinity is of multiplicity at least four if and only if the coefficients of* (9) *obey one of the following sets of conditions:*
1) $\{(10),\ (15),\ (27)\}$, *2)* $\{(10),\ (16),\ (28)\}$, *3)* $\{(10),\ (17),\ (29)\}$,
4) $\{(10),\ (18),\ (30)\}$, *5)* $\{(11),\ (20),\ (31)\}$, *6)* $\{(12),\ (22),\ (33)\}$,
7) $\{(13),\ (23),\ (34)\}$, *8)* $\{(13),\ (23),\ (35)\}$, *9)* $\{(13),\ (24),\ (36)\}$,
10) $\{(14),\ (25),\ (39)\}$, *11)* $\{(14),\ (26),\ (42)\}$, *12)* $\{(14),\ (26),\ (43)\}$.

3.4. Systems (9) **with** $m(Z) \geq 5$**.** Under conditions 1) – 12) of Lemma 3.3 we solve the identity $A_5(x,y) \equiv 0$. We obtain the following results for multiplicity:

$\{(10),\ (15),\ (27)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$b_{30} = b_{12} = b_{03} = 0. \tag{44}$$

$\{(10),\ (16),\ (28)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$\begin{gathered} b_{02} = -(2a_{20} + b_{21})/(a_{20} + b_{21}), \\ b_{11} = (2a_{20}^3 + 2a_{20}^2 b_{21} + 2a_{20}b_{30} + b_{21}b_{30})/(a_{20}(a_{20} + b_{21})). \end{gathered} \tag{45}$$

$\{(10),\ (17),\ (29)\} \Rightarrow A_5(x,y) \not\equiv 0$ because $(a_{11} + 1) \neq 0$.
$\{(10),\ (18),\ (30)\} \Rightarrow A_5(x,y) \not\equiv 0$, because $a_{02} \neq 0$.
$\{(11),\ (20),\ (31)\} \Rightarrow A_5(x,y) \not\equiv 0$ because $(a_{20} + a_{30}) \neq 0$.
$\{(12),\ (22),\ (33)\} \Rightarrow A_5(x,y) \not\equiv 0$ because $(1 + a_{11} + a_{21}) \neq 0$.
$\{(13),\ (23),\ (34)\} \Rightarrow A_5(x,y) \not\equiv 0$ because $(a_{02} + a_{12}) \neq 0$.

$\{(13), (23), (35)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$a_{02} = -a_{12}(1+b_{20}),\ a_{20} = 0,\ a_{21} = -(1+a_{11}). \tag{46}$$

$\{(13), (24), (36)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$a_{02} = -a_{12}(1+b_{20}),\ a_{21} = -(1+a_{11}),\ b_{21} = 0. \tag{47}$$

$\{(14), (25), (39)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$a_{11} = 2b_{02} - 1,\ a_{12} = -a_{02} + 3b_{03},\ b_{11} = 2b_{03}(2a_{03}b_{02} - a_{02}b_{03})/a_{03}^2; \tag{48}$$

$$\begin{array}{c} a_{11} = (-a_{02}a_{03}^2 - a_{03}^2a_{12} - a_{03}b_{03} + 3a_{03}^2b_{03} - a_{02}a_{12}b_{03} - a_{12}^2b_{03}+ \\ +4a_{02}b_{03}^2 + 5a_{12}b_{03}^2 - 6b_{03}^3)/(a_{03}b_{03}), \quad b_{02} = a_{02}b_{03}/a_{03}, \\ b_{11} = (-a_{02}a_{03}^2 - a_{03}^2a_{12} + 3a_{03}^2b_{03} - a_{02}a_{12}b_{03} - a_{12}^2b_{03} + 4a_{02}b_{03}^2+ \\ +5a_{12}b_{03}^2 - 6b_{03}^3)/a_{03}^2; \end{array} \tag{49}$$

$$\begin{array}{c} a_{11} = (-a_{02}a_{03} + 2a_{02}^2b_{03} - 2a_{03}b_{03})/(a_{02}a_{03}),\ a_{12} = -a_{02} + 3b_{03}, \\ b_{11} = -2(-a_{02}^2 + 2a_{03})b_{03}^2)/(a_{02}a_{03}^2),\ b_{02} = (a_{02}^2 - a_{03})b_{03}/(a_{02}a_{03}); \end{array} \tag{50}$$

$$a_{12} = -a_{02},\ b_{02} = b_{11} = b_{03} = 0. \tag{51}$$

$\{(14), (26), (42)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$\begin{array}{c} a_{20} = -(a_{02}a_{03}^2 + a_{03}^2a_{12} - 3a_{03}^2b_{03} + a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 3a_{02}b_{03}^2- \\ -5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2, \quad a_{11} = (-a_{02}a_{03}^2 - a_{03}^2a_{12} - a_{03}b_{03} + 3a_{03}^2b_{03}- \\ -a_{02}a_{12}b_{03} - a_{12}^2b_{03} + 4a_{02}b_{03}^2 + 5a_{12}b_{03}^2 - 6b_{03}^3)/(a_{03}b_{03}), \\ b_{11} = (1+a_{11})b_{03}/a_{03}; \end{array} \tag{52}$$

$$a_{12} = -a_{02},\ b_{11} = b_{03} = 0. \tag{53}$$

$\{(14), (26), (43)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$a_{12} = -a_{02},\ a_{21} = -(1+a_{11}),\ a_{30} = -a_{20},\ b_{03} = 0. \tag{54}$$

The sets of conditions $\{(13), (24), (36), (47)\}$ and $\{(13), (24), (35), (46)\}$ are the same. The set of conditions $\{(14), (25), (39), (51)\}$ is a particular case of the set $\{(14), (26), (42), (53)\}$. The set of conditions $\{(14), (26), (43), (54)\}$ coincides with the set of conditions $\{(14), (26), (42), (53)\}$.

Lemma 3.4. *The line at infinity of the quartic system* (9) *has multiplicity at least five if and only if the coefficients of* (9) *verify one of the following sets of conditions:*
1) $\{(10), (15), (27), (44)\}$, *2)* $\{(10), (16), (28), (45)\}$,
3) $\{(13), (23), (35), (46)\}$, *4)* $\{(14), (25), (39), (48)\}$,
5) $\{(14), (25), (39), (49)\}$, *6)* $\{(14), (25), (39), (50)\}$,
7) $\{(14), (26), (42), (52)\}$, *8)* $\{(14), (26), (42), (53)\}$.

3.5. Systems (9) with $m(Z) = 6$. The line at infinity $Z = 0$ has multiplicity at least six if, in each of cases 1)−8) of Lemma 3.4, the identity $A_6(x,y) \equiv 0$ holds. Following the approach used in the previous case and taking (3) into account, we examine each case separately:

$\{(10), (15), (27), (44)\} \Rightarrow A_6(x,y) = b_{31}x^4y(b_{20}x^2 - 2y^2 - b_{02}y^2) \equiv 0 \Rightarrow$

$$b_{20} = 0,\ b_{02} = -2. \tag{55}$$

$$\begin{array}{c} \{(10), (16), (28), (45)\} \Rightarrow A_6(x,y) = a_{20}x^5((3a_{20}^2b_{20} + 4a_{20}b_{20}b_{21} + b_{20}b_{21}^2- \\ -a_{20}^2b_{30} - a_{20}b_{21}b_{30} - b_{30}^2)x^2 + (12a_{20}^3 + 16a_{20}^2b_{21} + 4a_{20}b_{21}^2+ \\ +8a_{20}b_{30} + 2b_{21}b_{30})xy - 3a_{20}(3a_{20} + b_{21})y^2)/(a_{20} + b_{21}) \equiv 0 \Rightarrow \end{array}$$

$$b_{21} = -3a_{20},\ b_{30} = 0. \tag{56}$$

$\{(13), (23), (35), (46)\} \Rightarrow A_6(x, y) \equiv 0$ contradicts the conditions (3).
$\{(14), (25), (39), (48)\} \Rightarrow A_6(x, y) \not\equiv 0$ because $a_{03} \neq 0$.
$\{(14), (25), (39), (49)\} \Rightarrow A_6(x, y) \not\equiv 0$ because $a_{03}b_{03} \neq 0$.
$\{(14), (25), (39), (50)\} \Rightarrow A_6(x, y) \not\equiv 0$ because $a_{02}a_{03} \neq 0$.
$\{(14), (26), (42), (52)\} \Rightarrow A_6(x, y) \not\equiv 0$ because $a_{03}b_{03} \neq 0$.
$\{(14), (26), (42), (53)\} \Rightarrow$ the identity $A_6(x, y) \equiv 0$ contradicts the conditions (3).

Lemma 3.5. *For the quartic system (9), the line at infinity has multiplicity at least six if and only if the coefficients of (9) satisfy one of the following sets of conditions: 1)* $\{(10), (15), (27), (44), (55)\}$, *2)* $\{(10), (16), (28), (45), (56)\}$.

Next we prove that the maximal multiplicity of the line at infinity in quartic differential systems (2) possessing an affine invariant straight line is six.

In the conditions 1) of Lemma 3.5 we have

$A_7(x, y) = b_{31}x^5y + (b_{21} + 3b_{31})x^3y^3$ and $yp_4(x, y) + xq_4(x, y) = b_{31}x^4y$.

It is easy to see that the identity $A_7(x, y) \equiv 0$ contradicts the conditions (3). Hence, the multiplicity of the line at infinity $Z = 0$ is exactly six.

In the conditions 2) of Lemma 3.5 we obtain

$A_7(x, y) = -a_{20}x^3(a_{20}(2b_{20} - 1)x^3 + 2(3a_{20}^2 + b_{20})x^2y + 6a_{20}xy^2 + y^3)$ and $yp_4(x, y) + xq_4(x, y) = -2a_{20}^2x^5$. It is clear that the identity $A_7(x, y) \equiv 0$ is incompatible with the conditions (3). Consequently, the line at infinity $Z = 0$ has multiplicity exactly six.

In this manner, we have proved the following theorem.

Theorem 3.6. *In the class of quartic differential systems with a center-focus critical point, a real affine invariant straight line, and non-degenerate infinity, the maximal multiplicity of the line at infinity is six.*

4. Centers in quartic systems with an affine invariant straight line and the line at infinity of maximal multiplicity

An approach to the problem of the center for differential systems (2) is to study the local integrability of the system in some neighborhood of the critical point $(0, 0)$. It is known from Poincaré and Lyapunov [1] that a critical point $(0, 0)$ is a center for system (2) if and only if the system has in some neighborhood of $(0, 0)$ a nonconstant analytic first integral $F(x, y) = C$ or an analytic integrating factor of the form $\mu(x, y) = 1 + \sum_{k=1}^{\infty} \mu_k(x, y)$, where μ_k are homogeneous polynomials of degree k.

If a first integral or an integrating factor can be constructed in the form

$$\Phi_1^{\alpha_1} \cdots \Phi_s^{\alpha_s}, \tag{57}$$

where Φ_j, $1 \leq j \leq p$ are invariant algebraic curves and Φ_j, $p + 1 \leq j \leq s$ are exponential factors, then the system (2) is called Darboux integrable.

Let $F(x, y) = x^2 + y^2 + F_3(x, y) + \cdots + F_n(x, y) + \cdots$ be a function such that

$$\frac{\partial F}{\partial x}P(x, y) + \frac{\partial F}{\partial y}Q(x, y) \equiv \sum_{j=1}^{\infty} L_j(x^2 + y^2)^{j+1}, \tag{58}$$

where $F_k(x, y) = \sum_{i+j=k} f_{ij}x^iy^j$, $f_{0j} = 0$ if j is even.

In (58) L_j are polynomials in the coefficients of system (2) called the Lyapunov (focus) quantities. For example, the first Lyapunov quantity looks as $L_1 = (a_{12} - a_{02}a_{11} - a_{11}a_{20} + 3a_{30} + 2a_{02}b_{02} - 3b_{03} + b_{02}b_{11} - 2a_{20}b_{20} + b_{11}b_{20} - b_{21})/4$.

The origin is a fine focus of *order* m if $L_k = 0$, $k = \overline{1, m-1}$ and $L_m \neq 0$; at most m small amplitude limit cycles can bifurcate from a fine focus of order m (see [8]).

Theorem 4.1. *The critical point* $(0,0)$ *is a center for system* (2) *if and only if all the Lyapunov quantities vanish (*$L_k = 0,\ k = \overline{1, \infty}$*).*

In what follows, we solve the problem of the center for system (9) with invariant straight line $x - 1 = 0$ under conditions 1) and 2) of Lemma 3.5, i.e., when the line at infinity $Z = 0$ has maximal multiplicity.

Lemma 4.2. *The following two sets of conditions are sufficient conditions for the origin to be a center of system* (9)*:*

(i) $a_{20} = a_{02} = a_{03} = a_{12} = a_{21} = a_{30} = 0,\ a_{11} = -1,\ b_{04} = b_{13} = b_{22} = b_{40} = 0,$ $b_{30} = b_{12} = b_{03} = b_{20} = 0,\ b_{02} = -2,\ b_{11} = b_{31},\ b_{21} = -2b_{31};$

(ii) $a_{02} = a_{03} = a_{12} = a_{21} = 0,\ a_{30} = -a_{20},\ a_{11} = -1,\ b_{04} = b_{13} = b_{22} = b_{31} = 0,$ $b_{40} = -2a_{20}^2,\ b_{03} = b_{12} = 0,\ b_{21} = -3a_{20},\ b_{30} = 0,\ 2b_{02} = -1,\ b_{11} = 2a_{20}.$

Proof. In the case (i), the quartic system (9) has the form

$$\begin{aligned} \dot{x} &= (1 - x)y \equiv P(x, y), \\ \dot{y} &= -x - b_{31}xy + 2y^2 + 2b_{31}x^2y - b_{31}x^3y \equiv Q(x, y), \end{aligned} \tag{59}$$

where $b_{31} \neq 0$. Applying the identity (4), it is easy to show that the system (59) has the following exponential factors:

$$\Phi_1 = \exp(x),\ \Phi_2 = \exp(x^2),\ \Phi_3 = \exp(3y - b_{31}x^3),$$

$$\Phi_4 = \exp(12xy + 4b_{31}x^3 - 3b_{31}x^4),\ \Phi_5 = \exp(20x^2y + 5b_{31}x^4 - 4b_{31}x^5)$$

with cofactors, respectively

$$K_1 = y(1 - x),\ K_2 = 2xy(1 - x),\ K_3 = 3(b_{31}x^2y - b_{31}xy + 2y^2 - x),$$

$$K_4 = 12(xy^2 - x^2 + y^2),\ K_5 = 20x(2y^2 - x^2).$$

Using the Darboux integrability method [6], we can construct for system (59) a Darboux integrating factor of the form (57)

$$\mu(x, y) = (x - 1)^3 \exp((b_{31}(4b_{31}x^5 - 5b_{31}x^2(2 - 4x + 3x^2) - 20(x - 1)^2y))/20).$$

In the case (ii), the quartic system (9) takes the form

$$\begin{aligned} \dot{x} &= (1 - x)(a_{20}x^2 + y) \equiv P(x, y), \\ \dot{y} &= (-2x - 2b_{20}x^2 - 4a_{20}xy + y^2 + 6a_{20}x^2y + 4a_{20}^2x^4)/2 \equiv Q(x, y), \end{aligned} \tag{60}$$

where $a_{20} \neq 0$. It is easy to check that the divergence vanishes identically

$$div(P, Q) = P_x(x, y) + Q_y(x, y) \equiv 0$$

in a neighborhood of a critical point $(0, 0)$. The first integral of the system is

$$x^2 + y^2 + \frac{2}{3}b_{20}x^3 - xy^2 + 2a_{20}x^2y - 2a_{20}x^3y - \frac{4}{5}a_{20}^2x^5 = C.$$

□

Lemma 4.3. *Let the quartic differential system (2) have an affine invariant straight line and the line at infinity of the maximal multiplicity six. Then the critical point* $(0, 0)$ *is a center if and only if the first two Lyapunov quantities vanish.*

Proof. We compute the first two Lyapunov quantities L_1 and L_2 assuming that one of the conditions of Lemma 3.5 holds.

In the case 1) the first two Lyapunov quantities are

$$L_1 = -(2b_{11} + b_{21})/4,$$

$$L_2 = (498b_{11} + 46b_{11}^3 + 225b_{21} + 23b_{11}^2 b_{21} - 48b_{31})/96.$$

The equations $L_1 = 0, L_2 = 0$ yield $b_{11} = b_{31}, b_{21} = -2b_{31}$ and we obtain the set of conditions (i) of Lemma 4.2.

In the case 2) the first two Lyapunov quantities vanish. We are in the conditions (ii) of Lemma 4.2. □

Taking into account Lemmas 4.2 and 4.3, we give the necessary and sufficient conditions for the origin to be a center in the following theorem.

Theorem 4.4. *The critical point* $(0, 0)$ *is a center for quartic differential system (2), with one invariant straight line* $x = 1$ *and the line at infinity* $Z = 0$ *of maximal multiplicity six, if and only if one of the sets of conditions (i), (ii) holds.*

References

[1] V.V. Amel'kin, N.A. Lukashevich, A.P. Sadovskii, *Non-linear oscillations in the systems of second order*, Belarusian University Press, Minsk, 1982 (in Russian).

[2] C. Bujac, D. Schlomiuk, N. Vulpe, Cubic differential systems with invariant straight lines of total multiplicity seven and four real distinct infinite singularities, *Electronic Journal of Differential Equations* **2021** (2021), no. 83, 1-110.

[3] C. Christopher, J. Llibre, J.V. Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, *Pacific Journal of Mathematics* **329** (2007), no. 1, 63-117.

[4] D. Cozma, A. Şubă, Conditions for the existence of four invariant straight lines in a cubic differential system with a singular point of a center or a ficus type, *Bulletin of Academy of Sciences of the Republic of Moldova, Mathematics* **3** (1993), 54-62.

[5] D. Cozma, *Integrability of cubic systems with invariant straight lines and invariant conics*, Ştiinţa, Chişinău, 2013.

[6] D. Cozma, Darboux integrability of a cubic differential system with two parallel invariant straight lines, *Carpathian J. Math.* **38** (2022), no. 1, 129-137.

[7] J. Llibre, On the centers of cubic polynomial differential systems with four invariant straight lines, *Topological Methods in Nonlinear Analysis* **55** (2020), no. 2, 387-402.

[8] V.G. Romanovski, D.S. Shafer, *The center and cyclicity problems: a computational algebra approach*, Birkhäuser, Boston, Basel, 2009.

[9] A.P. Sadovskii, T.V. Shcheglova, Solution of the center-focus problem for a nine-parameter cubic system, *Differential Equations* **47** (2011), no. 2, 208-223.

[10] A. Şubă, Real cubic differential systems with a linear center and multiple line at infinity, *Acta et Commentationes, Exact and Natural Sciences* **12** (2021), no. 2, 50-62.

[11] A. Şubă, Centers of cubic differential systems with the line at infinity of maximal multiplicity, *Acta et Commentationes, Exact and Natural Sciences* **14** (2022), no. 2, 38-46.

[12] A. Şubă, S. Turuta, Solution of the problem of the center for cubic differential systems with the line at infinity and an affine real invariant straight line of total algebraic multiplicity five, *Bulletin of Academy of Sciences of the Republic of Moldova, Mathematics* **90** (2019), no. 2, 13-40.

[13] A. Şubă, O. Vacaraş, Cubic differential systems with an invariant straight line of maximal multiplicity, *Annals of the University of Craiova, Mathematics and Computer Science Series* **42** (2015), no. 2, 427-449.

[14] A. Şubă, O. Vacaraş, Quartic differential systems with an invariant straight line of maximal multiplicity, *Bulletin of Academy of Sciences of the Republic of Moldova, Mathematics* **86** (2018), no. 1, 76-91.

[15] A. Şubă, O. Vacaraş, Center problem for cubic differential systems with the line at infinity and an affine real invariant straight line of total multiplicity four, *Bukovinian Math. Journal* **9** (2021), no. 2, 35-52.

[16] A. Şubă, O. Vacaraş, Quartic differential systems with a non-degenerate monodromic critical point and multiple line at infinity, *Acta et Commentationes, Exact and Natural Sciences* **16** (2023), no. 2, 25-34.

[17] O. Vacaraş, Maximal multiplicity of the line at infinity for quartic differential systems, *Acta et Commentationes, Exact and Natural Sciences* **6** (2018), no. 2, 68-75.

[18] O. Vacaraş, Quartic differential systems with a center-focus critical point and an affine invariant straight line of maximal multiplicity, *Proceedings of the International Conference IMCS-60*, Chişinău, October 10 – 13, 2024, 208-211.

[19] O. Vacaraş, Center problem for quartic differential systems with an affine invariant straight line of maximal multiplicity, *Acta et Commentationes, Exact and Natural Sciences* **18** (2024), no. 2, 38-54.

[20] N. Vulpe, J. Llibre, Planar cubic polynomial differential systems with the maximum number of invariant straight lines, *Rocky Mountain J. Math.* **36** (2006), no. 4, 1301-1373.

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