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Some estimates on the Hermite-Hadamard inequality through quasi-convex functions

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Abstract. In this paper we establish some estimates of the right hand side of a Hermite-Hadamard type inequality in which some quasi-convex functions are involved. We also point out some applications of our results to give estimates for the approximation error of the integral $\int_a^b f(x)\,dx$ in the trapezoidal formula. Next, we extend our initial results to functions of several variables.

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1. Introduction and main results

The classical *Hermite-Hadamard inequality* provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}\,. \tag{1}$$

The history of this inequality begins with the papers of Ch. Hermite [7] and J. Hadamard [6] in the years 1883-1893 (see, e.g. C. P. Niculescu and L.-E. Persson [10] and the references therein for some historical notes on the Hermite-Hadamard inequality). Inequality (1) has triggered a huge amount of interest over the years. We just remember the recent studies in [1, 2, 3, 4, 5, 8, 9] on that topic.

An interesting problem related with the Hermite-Hadamard inequality is the precision in the Hermite-Hadamard inequality. In that context, we remember that the left Hermite-Hadamard inequality can be estimated by the *inequality of Ostrowski*,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \cdot M \cdot (b-a)\,, \tag{2}$$

while the right Hermite-Hadamard inequality can be estimated by the *inequality of Iyengar*,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{M\cdot(b-a)}{4} - \frac{1}{4\cdot M\cdot(b-a)}\cdot(f(b)-f(a))^2\,, \tag{3}$$

where by M we denoted the *Lipschitz constant*, i.e. $M = \sup\{|\frac{f(x)-f(y)}{x-y}|;\ x \ne y\}$.

The goal of this paper is to establish an inequality of type (3) in the case when $f : [a, b] \to \mathbb{R}$ is a differentiable mapping on (a, b) for which either $|f'|$ is a quasi-convex function or $|f'|^{p/(p-1)}$ is a quasi-convex function for a real number $p > 1$. We

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remember that the notion of quasi-convex function generalizes the notion of convex function. More exactly, a function $f : [a, b] \to \mathbb{R}$ is said *quasi-convex* on $[a, b]$ if

$$f((1-\lambda)\cdot x + \lambda \cdot y) \leq \sup\{f(x), f(y)\}, \quad \forall\, x, y \in [a, b],\ \forall\, \lambda \in [0, 1]\,.$$

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex. In that context, we point out an elementary example. The function $g : [-2, 2] \to \mathbb{R}$,

$$g(t) = \begin{cases} 1, & \text{for} \quad t \in [-2, -1] \\ t^2, & \text{for} \quad t \in (-1, 2]\,, \end{cases}$$

is not a convex function on $[-2, 2]$ but it is a quasi-convex function on $[-2, 2]$.

On the other hand, we will show that our results can be used in order to give estimates for the approximation error of the integral $\int_a^b f(x)\ dx$ in the trapezoidal formula. Finally, we will remark that our estimates of type (3) can be extended to functions of several variables.

The main results of this paper are given by the following theorems.

Theorem 1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f^{'}|$ is quasi-convex on $[a, b]$ then the following inequality holds true*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)\cdot \sup\{|f^{'}(a)|, |f^{'}(b)|\}}{4}\,. \tag{4}$$

Theorem 2. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f^{'}|^{p/(p-1)}$ is quasi-convex on $[a, b]$ then the following inequality holds true*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{(b-a)}{2(p+1)^{1/p}} \cdot [\sup\{|f^{'}(a)|^{p/(p-1)}, |f^{'}(b)|^{p/(p-1)}\}]^{(p-1)/p}\,. \tag{5}$$

Remark. The results of Theorems 1 and 2 generalize the results of Theorems 2.2 and 2.3 in [2].

2. Proof of the main results

In order to prove Theorems 1 and 2 we point out an useful auxiliary result.

Lemma 2.1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $f' \in L^1(a, b)$ then the following equality holds true*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\ dx = \frac{b-a}{2}\int_0^1 (1-2t)f^{'}(ta+(1-t)b)\ dt. \tag{6}$$

Proof. Integrating by parts we have

$$I = \int_0^1 (1-2t)f^{'}(ta+(1-t)b)\ dt = \frac{f(ta+(1-t)b)}{a-b}(1-2t)\Big|_0^1 + 2\int_0^1 \frac{f(ta+(1-t)b)}{a-b}\ dt\,.$$

Next, by the change of variable $x = ta + (1-t)b$ we obtain

$$\mathrm{I} = \frac{f(a)+f(b)}{b-a} - \frac{2}{(b-a)^2}\int_a^b f(x)dx\,. \tag{7}$$

Thus, we deduce that equality (6) holds true. The proof of Lemma 2.1 is complete. □

Proof of Theorem 1. By Lemma 2.1, the fact that $|f^{'}|$ is quasi-convex on (a,b) and some elementary integration calculus we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &= \left|\frac{b-a}{2}\int_0^1 (1-2t)f^{'}(ta+(1-t)b)dt\right| \\ &\le \frac{b-a}{2}\int_0^1 |(1-2t)||f^{'}(ta+(1-t)b)|dt \\ &\le \frac{b-a}{2}\int_0^1 |(1-2t)|\sup\{|f^{'}(a)|,|f^{'}(b)|\}dt \\ &= \frac{(b-a)\sup\{|f^{'}(a)|,|f^{'}(b)|\}}{2}\int_0^1 |(1-2t)|dt \\ &= \frac{(b-a)\sup\{|f^{'}(a)|,|f^{'}(b)|\}}{4}.\end{aligned}$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. First, we point out that

$$\int_0^1 |1-2t|^p\,dt=\int_0^{1/2}(1-2t)^p\,dt+\int_{1/2}^1 (2t-1)^p\,dt=2\int_0^{1/2}(1-2t)^p\,dt=\frac{1}{p+1}\,.$$

Next, using the above information, Lemma 2.1 and the Hölder's inequality we obtain

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{b-a}{2}\int_0^1 |(1-2t)||f^{'}(ta+(1-t)b)|\,dt \\ &\le \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{1/p}\left(\int_0^1 |f^{'}(ta+(1-t)b)|^q\,dt\right)^{1/q} \\ &\le \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{1/p}\left(\int_0^1 \sup\{|f^{'}(a)|^q,|f^{'}(b)|^q\}\,dt\right)^{1/q} \\ &= \frac{b-a}{2(p+1)^{1/p}}\cdot[\sup\{|f^{'}(a)|^q,|f^{'}(b)|^q\}]^{1/q},\end{aligned}$$

where $1/p+1/q=1$ (or, $q=p/(p-1)$).

The proof of Theorem 2 is complete. □

3. Applications of Theorems 1 and 2 to the trapezoidal formula

Assume Δ is a division of the interval $[a,b]$ such that

$$\Delta:\qquad a=x_0<x_1<...<x_{n-1}<x_n=b\,.$$

For a given function $f:[a,b]\to\mathbb{R}$ we consider the trapezoidal formula

$$T(f,\Delta)=\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\,. \tag{8}$$

It is well known that if f is twice differentiable on (a,b) and $M=\sup_{x\in(a,b)}|f^{''}(x)|<\infty$ then

$$\int_a^b f(x)dx=T(f,\Delta)+E(f,\Delta)\,, \tag{9}$$

where $E(f, \Delta)$ is the approximation error of the integral $\int_a^b f(x)dx$ by the trapezoidal formula and satisfies,

$$|E(f, \Delta)| \leq \frac{M}{12} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 . \tag{10}$$

Clearly, if the function f is not twice differentiable or the second derivative is not bounded on (a, b), then (10) does not hold true. In that context, the following results are important in order to obtain some estimates of $E(f, \Delta)$.

Corollary 1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f^{'}|$ is quasi-convex on $[a, b]$ then for each division Δ of the interval $[a, b]$ we have,*

$$|E(f, \Delta)| \leq \frac{\sup\{|f^{'}(a)|, |f^{'}(b)|\}}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 . \tag{11}$$

Proof. We apply Theorem 1 on the sub-intervals $[x_i, x_{i+1}]$, $i = \overline{0, n-1}$ given by the division Δ. Adding from $i = 0$ to $i = n - 1$ we deduce

$$|T(f, \Delta) - \int_a^b f(x)dx| \leq \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \sup\{|f^{'}(x_i)|, |f^{'}(x_{i+1})|\} . \tag{12}$$

On the other hand, for each $x_i \in [a, b]$ there exists $\alpha_i \in [0, 1]$ such that $x_i = \alpha_i a + (1 - \alpha_i)b$. Since $|f^{'}|$ is quasi-convex we deduce

$$|f^{'}(x_i)| \leq \sup\{|f^{'}(a)|, |f^{'}(b)|\}, \quad \forall\, i = \overline{0, n} .$$

Thus,

$$\sup\{|f^{'}(x_i)|, |f^{'}(x_{i+1})|\} \leq \sup\{|f^{'}(a)|, |f^{'}(b)|\}, \quad \forall\, i = \overline{0, n-1}. \tag{13}$$

Relations (12) and (13) imply that relation (11) holds true. Thus, Corollary 1 is completely proved. □

A similar method as that used in the proof of Corollary 1 but based on Theorem 2 shows that the following result is valid.

Corollary 2. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f^{'}|^{p/(p-1)}$ is quasi-convex on $[a, b]$ then for each division Δ of the interval $[a, b]$ we have,*

$$|E(f, \Delta)| \leq \frac{\sup\{|f^{'}(a)|, |f^{'}(b)|\}}{2(p + 1)^{1/p}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2.$$

4. An extension of Theorem 1

The aim of this section is to extend the result of Theorem 1 to functions of several variables.

In this section we will denote by U an open and convex set of $\mathbb{R}^N$ $(N \geq 1)$.

We say that a function $f : U \to \mathbb{R}$ is *quasi-convex* on U if

$$f((1 - \lambda) \cdot x + \lambda \cdot y) \leq \sup\{f(x), f(y)\}, \quad \forall\, x, y \in U,\ \forall\, \lambda \in [0, 1] .$$

The following auxiliary result holds true.

Lemma 4.1. *Assume $f : U \to \mathbb{R}$ is a function. Then f is quasi-convex on U if and only if for every $x,\ y \in U$ the function $\varphi : [0,1] \to \mathbb{R}$, $\varphi(t) = f((1-t)x + ty)$ is quasi-convex on $[0,1]$.*

Proof. "$\Leftarrow$". Let $x, y \in U$ be fixed. Assume that $\varphi : [0,1] \to \mathbb{R}$, $\varphi(t) = f((1-t)x + ty)$ is quasi-convex on $[0,1]$.

Let $\lambda \in [0,1]$ be arbitrary, but fixed. Clearly, $\lambda = (1-\lambda)\cdot 0 + \lambda \cdot 1$ and thus,

$$\begin{aligned} f((1-\lambda)x + \lambda y) = \varphi(\lambda) &= \varphi((1-\lambda)\cdot 0 + \lambda\cdot 1) \\ &\le \sup\{\varphi(0), \varphi(1)\} = \sup\{f(x), f(y)\}\,. \end{aligned}$$

It follows that f is quasi-convex on U.

"$\Rightarrow$".Assume that f is quasi-convex on U. Let $x, y \in U$ be fixed and define $\varphi : [0,1] \to \mathbb{R}$, $\varphi(t) = f((1-t)x + ty)$. We show that φ is quasi-convex on $[0,1]$.

Let $t_1, t_2 \in [0,1]$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} \varphi((1-\lambda)t_1 + \lambda t_2) &= f([1-(1-\lambda)t_1 - \lambda t_2]x + [(1-\lambda)t_1 - \lambda t_2]y]) \\ &= f((1-\lambda)((1-t_1)x + t_1 y) + \lambda((1-t_2)x + t_2 y)) \\ &\le \sup\{f((1-t_1)x + t_1 y), f((1-t_2)x + t_2 y)\} \\ &= \sup\{\varphi(t_1), \varphi(t_2)\}\,. \end{aligned}$$

We deduce that φ is quasi-convex on $[0,1]$.

The proof of Lemma 4.1 is complete. □

Using the above lemma we will prove an extension of Theorem 1 to functions of several variables.

Proposition 1. *Assume $f : U \to \mathbb{R}^+$ is a quasi-convex function on U. Then for any $x, y \in U$ and any $a, b \in (0,1)$ the following inequality holds true*

$$\begin{aligned} |\frac{1}{2}\cdot\int_0^a f((1-s)x+sy)ds \quad &+ \quad \frac{1}{2}\cdot\int_0^b f((1-s)x+sy)ds - \\ &\quad \frac{1}{b-a}\cdot\int_a^b \left(\int_0^s f((1-\theta)x+\theta y)d\theta\right) ds| \\ &\le \quad \frac{b-a}{4}\cdot\sup\{f((1-a)x+ay), f((1-b)x+by)\}\,. \end{aligned} \tag{14}$$

Proof. We fix $x, y \in U$ and $a, b \in (0,1)$ with $a < b$. Since f is quasi-convex, by Lemma 4.1 it follows that the function

$$\varphi : [0,1] \to \mathbb{R}, \quad \varphi(t) = f((1-t)x + ty)\,,$$

is quasi-convex on $[0,1]$.

Define $\Phi : [0,1] \to \mathbb{R}$,

$$\Phi(t) = \int_0^t \varphi(s)\, ds = \int_0^t f((1-s)x + sy)\, ds\,.$$

Obviously, $\Phi'(t) = \varphi(t)$ for all $t \in (0,1)$.

Since $f(U) \subset \mathbb{R}^+$ it results that $\varphi \ge 0$ on $[0,1]$ and thus, $\Phi' \ge 0$ on [0,1].

Applying Theorem 1 to the function Φ we obtain

$$\left|\frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(s)ds\right| \le \frac{b-a}{4}\sup\{\Phi'(a), \Phi'(b)\}\,,$$

and we deduce that relation (14) holds true.

Proposition 1 is completely proved. □

Remark. We point out that a similar result as those of Proposition 1 can be stated by using Theorem 2.

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