Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 34, 2007, Pages 100–106
ISSN: 1223-6934

# Manifolds and Symplectic Manifolds Trajectories Generation and Visualization

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Abstract. Manifolds visualization was a challenging subject in order interpret the data and to have a better understanding of system dynamics. Different mathematical functions combined with geometry and color have been used to define the shapes for a successful visualization. In this paper the visualization of different manifolds types is discussed. Trajectories visualization in the case of a symplectic manifold via a Poisson integrator method for the free rigid body is also provided and compared with other classical numerical results.

*2000 Mathematics Subject Classification.* Primary 60J05; Secondary 60J20.
*Key words and phrases.* Numerical Algorithms, System Dynamics, Visualization, Modeling.

## 1. Introduction

Visualization techniques are indispensable tools for the understanding of dynamical systems, the modeling and visualization techniques being a challenging subject for the researchers.

So far, a lot of dynamical systems turned out to be Lie-Poisson or Hamilton-Poisson systems. They contain the Euler equations of the free rigid body, the Maxwell-Poisson equations from plasma physics, the magnetohydrodynamic equations and many others Abraham, Marsden and Ratiu[3], Marsden[5] and Puta[12]. In the last time, a lot of technics that approximate conventional mechanical systems using discreet methods have been developed. From these technics we choose the numerical integrators in order to approximate the solution. If they preserve the symplectic structure on the phase space of the system, then they are usually called symplectic algorithms. Details regarding symplectic algorithms, structures and manifolds may be found in Marsden[5], McLahlan and Scovel[7] and the references included there.

Theoretically, any manifold can be visualized using numerical methods. However, numerical solutions are not always very successful, having a bad behavior near singularities. Instead of numerical solutions parametric representation of manifold's geometry may be used, which is more simpler and more practical. Parametric representations of shapes have been discussed in [8], and are very appropriate selections when handling complex shapes in three dimensional dimensions.

Since computer visualization is very important for understanding geometrical shapes, the associated graphics is a useful tool for researching manifolds and their properties. The study of the system dynamics, and the visualization of the associated trajectories may cover some essential properties, many times hidden within manifolds equations.

*Received*: 10 October 2007.

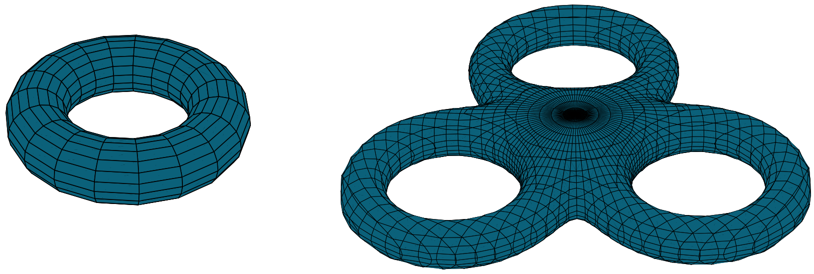

FIGURE 1. Torus and triple torus

The work in the present paper is dedicated to the manifolds visualization. For this purpose, a parametric representation, the most straightforward method of function-based shape modeling, was used to visualize some classical manifolds. Geometry and color, as well as different textures, may be included.

An example of trajectory visualization via a Poisson integrator method is provided and compared with other numerical results, obtained by using Runge-Kutta and Euler method.

## 2. Manifolds and Manifolds Visualization

In a general manifold theory there are many defined manifold types, such as differentiable manifolds, topological manifolds and complex manifold. A differential manifold is a topological manifold where the notions of continuity and differentiation applies. The differentiation in this case means that for every non-empty intersection, the transitions functions are diffeomorphisms (bijective and differentiable) from $R^n$ to $R^n$. For the next part, local coordinates, open subset, chart, atlas are supposed to be well known notions.

A 3D manifold can be represented either explicitly or implicitly (atlas or set of points), and so the visualization of such of 3D objects can be envisioned either explicitly or implicitly. Sometimes, such objects are difficult to visualize (the case of complex functions) because their representation is inside a four-dimensional space.

There are two basic ways to visualize a manifold, by numerically solving the associated equation and using the Marching Cubes algorithm [4], or by a classical parametric representation. The firs case applies when the manifold can be describe by an algebraic equation with integers coefficients

$$F(x_1, x_2, ..., x_n) = \sum_n C_{g_1,...,g_n} x_1^{g_1} ... x_n^{g_n} = \xi \tag{1}$$

where the sum is finite and the exponents are natural numbers. In the second case, the manifold may be described using a parametric representation such as

$$\Pi(x_1, ..., x_n) = \begin{cases} y_1 = F_1(x_1, ..., x_n) \\ . \\ . \\ . \\ y_n = F_n(x_1, ..., x_n) \end{cases}$$

While the first case may require a lot of computational power the second one is usually very simple and fast (but a parametric representation may be sometimes

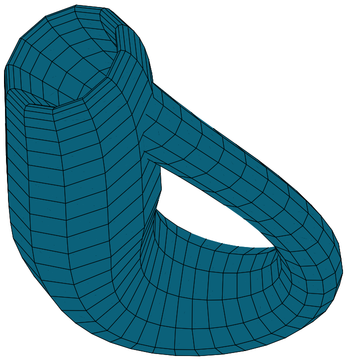

FIGURE 2. Klein Bottle

very difficult to obtain). Defining shapes with parametric representation is the most straightforward method of function-based shape modeling. Besides defining geometry, functions can be used for creating sophisticated colors and textures. Parametric functions for mapping texture images, or simple texture mapping, is a common approach implemented in many software and hardware systems Peachey [11], Wyvill *et al.* [21].

Some classical manifolds (a torus, a triple torus and the Klein bottle) using a parametric representation are shown in Fig. 1 and Fig. 2.

## 3. Poisson integrators of the free rigid body

The Euler equations of the free rigid body can be written as

$$\begin{cases} \dot{m}_1 = a_1 m_2 m_3 \\ \dot{m}_2 = a_2 m_1 m_3 \\ \dot{m}_3 = a_3 m_1 m_2 \end{cases} \tag{2}$$

with a canonical Poisson realization $(R^3, \Pi, H)$, where

$$\begin{aligned} \Pi &= \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \\ H &= \frac{1}{2}\left[\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right] \end{aligned}$$

A Casimir of the configuration is given by

$$C = \frac{1}{2}\left[x_1^2 + x_2^2 + x_3^2\right]$$

One can observe that

$$H = H_1 + H_2 + H_3$$

where

$$H_1 = \frac{1}{2I_1} m_1^2, H_2 = \frac{1}{2I_2} m_2^2, H_3 = \frac{1}{2I_3} m_3^2$$

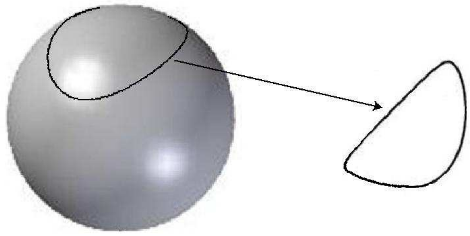

FIGURE 3. Trajectory using Poisson Algorithm

The dynamics of $X_{H_1}$ is given by $m = \Pi \cdot \nabla H_1$ or equivalent

$$\begin{cases} \dot{m_1} = 0 \\ \dot{m_2} = -\dfrac{m_1 m_3}{I_1} \\ \dot{m_3} = \dfrac{m_1 m_2}{I_1} \end{cases}$$

One can obtain

$$\begin{bmatrix} \dot{m_1} \\ \dot{m_2} \\ \dot{m_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dfrac{m_1(0)}{I_1} \\ 0 & \dfrac{m_1(0)}{I_1} & 0 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

with the solution

$$\begin{bmatrix} m_1(t) \\ m_2(t) \\ m_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dfrac{tm_1(0)}{I_1} \\ 0 & \dfrac{tm_1(0)}{I_1} & 0 \end{bmatrix} \begin{bmatrix} m_1(0) \\ m_2(0) \\ m_3(0) \end{bmatrix}$$

In a similar way the integral curves of $X_{H_2}$ and $X_{H_3}$ can be written as

$$\begin{bmatrix} m_1(t) \\ m_2(t) \\ m_3(t) \end{bmatrix} = \begin{bmatrix} \cos\dfrac{tm_2(0)}{I_2} & 0 & -\sin\dfrac{tm_2(0)}{I_2} \\ 0 & 1 & 0 \\ \sin\dfrac{tm_2(0)}{I_2} & 0 & \cos\dfrac{tm_2(0)}{I_2} \end{bmatrix} \begin{bmatrix} m_1(0) \\ m_2(0) \\ m_3(0) \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} m_1(t) \\ m_2(t) \\ m_3(t) \end{bmatrix} = \begin{bmatrix} \cos\dfrac{tm_3(0)}{I_3} & \sin\dfrac{tm_3(0)}{I_3} & 0 \\ -\sin\dfrac{tm_3(0)}{I_3} & \cos\dfrac{tm_3(0)}{I_3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_1(0) \\ m_2(0) \\ m_3(0) \end{bmatrix} \tag{4}$$

The Hamilton-Poisson system given by Eq.(2) is linear separable, and via (3) and (4), the first order Poisson integrator (the explicit Poisson integrators appear for the first time in the papers of McLachlan[6], Reich[16] and Puta[14]) $P_1(t)$ can be written

FIGURE 4. Trajectories using Poisson, Runge-Kutta and Euler Algorithms

as,

$$\begin{cases} m_1^{k+1} = m_1^k \cos B \cos C + m_2^k (\sin C \cos A + \sin A \cos C \sin B) \\ +m_3^k (\sin A \sin C - \cos A \cos C \sin B) \\ m_2^{k+1} = -m_1^k \sin C \cos B + m_2^k (\sin C \cos A - \sin A \sin B \sin C) \\ +m_3^k (\sin A \cos C + \cos A \sin C \sin B) \\ m_3^{k+1} = m_1^k \sin B - m_2^k \sin A \cos B + m_3^k \cos A \cos B \end{cases} \tag{5}$$

where

$$A = \frac{m_1(0)}{I_1} t; B = \frac{m_2(0)}{I_2} t; C = \frac{m_3(0)}{I_3} t.$$

Since computer visualization is very important for understanding geometrical shapes, the associated graphics is a useful tool for researching manifolds and their properties. The study of the system dynamics, and the visualization of the associated trajectories may cover some essential properties, many times hidden within manifolds equations. Such trajectories are shown in Fig. 3, the case of Poisson integrators for the free rigid body (for more details see Puta and Dupac [15]). Similar trajectories shown in Fig. 4 have been obtained when using Runge-Kutta and Euler method.

## 4. Conclusions

In this paper different parametric representations for manifold visualization have been used. Geometry and color, as well as different textures combinations may also be included.

It was demonstrated that trajectory visualization may cover some essential properties, many times hidden within manifolds equations. Furthermore, an example of trajectory visualization via a Poisson integrator method is provided and compared with other classical numerical results.

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