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Inverse Spectral Problem for a Singular Bessel-Type Sturm–Liouville Operator on a Finite Interval

R. F. Efendiev

Abstract. This paper studies an inverse spectral problem for a singular Bessel-type Sturm–Liouville operator on a finite interval with a complex periodic potential and Robin boundary condition. A regular solution near the singular endpoint is constructed, and Jost-type solutions are introduced to analyze the spectral properties of the non-self-adjoint operator. The resolvent is constructed explicitly, and it is shown that the spectrum consists of isolated eigenvalues of finite multiplicity. The main result establishes that the spectral data uniquely determine the potential, and a constructive reconstruction procedure is provided, together with an illustrative example.

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1. Introduction

Consider the singular Sturm–Liouville differential expression of Bessel type

$$\ell u := -u''(x) + \left(\frac{\nu^2 - \frac{1}{4}}{x^2} + q(x)\right) u(x), \qquad x \in (0, a), \tag{1}$$

where $\nu \in \mathbb{C}$ satisfies

$$\operatorname{Re} \nu > \frac{1}{2},$$

and $q(x)$ is a complex-valued potential defined on $(0, a)$.

Throughout this paper, we assume that the potential $q(x)$ admits the Fourier representation

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \qquad \sum_{n=1}^{\infty} |q_n| < \infty. \tag{2}$$

This condition ensures that $q(x) \in L_1(0, a)$ and defines a periodic complex-valued potential.

The presence of the inverse-square term

$$\frac{\nu^2 - \frac{1}{4}}{x^2}$$

introduces a strong Bessel-type singularity at the endpoint $x = 0$. Consequently, the differential expression generates a singular Sturm–Liouville operator.

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We define the linear operator L in $L^2(0,a)$ generated by the differential expression

$$Lu = \ell u. \tag{3}$$

The domain of the operator L is defined by

$$D(L) = \left\{ \begin{array}{l} u \in L^2(0,a) : u, u' \in AC_{loc}(0,a), lu \in L^2(0,a) \\ u(x) = O\left(x^{\nu+\frac{1}{2}}\right), (x \to 0), u'(a) + hu(a) = 0 \end{array} \right\} \tag{4}$$

where $h \in \mathbb{C}$ is a fixed constant.

The condition near $x = 0$ determines the admissible class of solutions corresponding to the regular solution of the Bessel-type singularity.

The aim of this work is to study the inverse spectral problem for the operator L, namely, to reconstruct the potential $q(x)$ from the corresponding spectral data.

Inverse spectral problems for Sturm–Liouville operators play a fundamental role in modern mathematical physics, spectral theory, and applied analysis. Such problems consist in recovering the coefficients of a differential operator from spectral data. These questions arise naturally in quantum mechanics, wave propagation, geophysics, and signal processing.

Inverse spectral and inverse scattering problems for Sturm–Liouville operators play a fundamental role in modern mathematical physics, spectral theory, and applied analysis. These problems consist of recovering coefficients of differential operators from spectral data and arise naturally in quantum mechanics, wave propagation, geophysics, and signal processing. A systematic presentation of these results and the corresponding reconstruction methods can be found in the monographs of Freiling and Yurko [1] and Yurko [2].

When the operator contains a singularity at one of the endpoints, the spectral analysis becomes significantly more complicated. In particular, operators containing the inverse–square term exhibit a Bessel-type singularity at the endpoint $x = 0$. Such operators arise naturally in radial Schrödinger equations and in problems involving cylindrical or spherical symmetry.

One of the earliest works addressing inverse spectral problems for singular Sturm–Liouville operators on a finite interval is the paper by Guillot and Ralston [3], where the inverse spectral theory was studied for a singular operator on $[0,1]$. Subsequently, Zhornitskaya and Serov [4] investigated inverse eigenvalue problems for a singular Sturm-Liouville operator on $(0,1)$ and established reconstruction results for operators related to radial Schrödinger equations.

A major advance in the theory of inverse problems for Bessel-type operators was obtained by Albeverio, Hryniv and Mykytyuk [5], who studied inverse spectral problems for Bessel operators on $(0,1)$ and provided a complete description of the admissible spectral data together with constructive reconstruction procedures. Closely related results for radial Schrödinger operators on a finite interval were obtained by Serier [6].

Further developments include numerical approaches to inverse spectral problems for Bessel operators, studied by Hryniv [7]. More recently, Bondarenko [8] extended the theory to matrix Sturm–Liouville equations with Bessel-type singularities on finite intervals. In addition, Xu and coauthors [9, 10] investigated inverse problems for perturbed Bessel operators with discontinuities and partial spectral data. Furthermore, spectral and inverse spectral problems for differential operators with discontinuities, impulses, and almost periodic coefficients have been extensively studied in recent years [11, 12, 13, 14, 15, 16].

Thus, the inverse problem for a singular Sturm-Liouville operator with a Bessel-type singularity on a finite interval belongs to an actively developing research area. The present work contributes to this direction by studying an inverse problem for a Bessel-type operator with a complex periodic potential and Robin boundary condition at the right endpoint.

The endpoint $x = 0$ is singular due to the presence of the inverse-square term in (3).

The unperturbed equation in relation to (3) is given by:

$$-u''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2} u(x) = \lambda^2 u(x). \tag{5}$$

This equation has two linearly independent solutions of the form

$$x^{\frac{1}{2}+\nu}, \qquad x^{\frac{1}{2}-\nu}.$$

Since $\operatorname{Re}\nu > \frac{1}{2}$, only the function $x^{\frac{1}{2}+\nu}$ belongs to $L^2(0, a)$ near $x = 0$. Therefore, admissible solutions of equation (14) must satisfy the regular asymptotic condition

$$u(x, \lambda) \sim x^{\nu+\frac{1}{2}}, \qquad x \to 0. \tag{6}$$

This condition uniquely determines the regular solution near the singular endpoint.

At the right endpoint $x = a$, which is regular, we impose the Robin boundary condition

$$u'(a) + hu(a) = 0, \qquad h \in \mathbb{C}. \tag{7}$$

Such boundary conditions naturally arise in applications involving partially absorbing or reactive boundaries, including diffusion processes, wave propagation, and quantum mechanical models.

The boundary condition (7) determines the realization of the operator L in the Hilbert space $L^2(0, a)$.

The main contributions of this paper can be summarized as follows:

- We investigate a singular Sturm–Liouville operator of Bessel type on a finite interval $(0, a)$ with a complex-valued periodic potential and Robin boundary condition at the regular endpoint. The presence of the inverse-square term produces a strong singularity at $x = 0$, which significantly complicates the spectral analysis.
- We construct the regular solution near the singular endpoint using a Frobenius-type expansion and establish its convergence under natural conditions summability assumptions on the Fourier coefficients of the potential.
- We introduce Jost-type solutions for the perturbed Bessel operator and derive their analytic representation. These solutions allow us to define the spectral coefficient and analyze spectral properties of the non-self-adjoint operator.
- We investigate the spectral properties of the operator and construct the resolvent explicitly using Green's function. In particular, we prove that the resolvent operator is compact in $L^2(0, a)$, which implies that the spectrum consists only of isolated eigenvalues of finite multiplicity.
- We formulate and solve the inverse spectral problem for the singular Bessel-type Sturm–Liouville operator. We show that the spectral data uniquely determine

the Fourier coefficients of the potential and provide a constructive reconstruction procedure.

- We present an explicit example illustrating the reconstruction algorithm and demonstrate the practical implementation of the inverse problem solution.

Thus, the paper provides a complete framework including construction of fundamental solutions, spectral analysis, resolvent construction, and a constructive solution of the inverse spectral problem for a non-self-adjoint Bessel-type Sturm–Liouville operator with periodic complex potential.

1.1. Spectral Problem. The spectral problem associated with the operator L consists in studying solutions of equation

$$Lu = \lambda^2 u. \tag{8}$$

Values of λ for which nontrivial solutions exist are called eigenvalues of the operator L. The corresponding solutions are eigenfunctions of the operator.

2. Fundamental Solutions

In this section, we construct fundamental solutions of equation (8) and investigate their basic properties.

2.1. Regular Solution at the Singular Endpoint.

Definition 2.1. The solution $\varphi(x, \lambda)$ of equation (8) satisfying

$$\varphi(x, \lambda) \sim x^{\nu+\frac{1}{2}}, \qquad x \to 0, \tag{9}$$

is called the regular solution.

We construct a regular solution of equation (8) in the form of a Frobenius-type series near the singular point $x = 0$.

To construct a solution near the singular point $x = 0$, we apply the Frobenius method. We first seek a solution in the form

$$u(x, \lambda) = x^{\nu+\frac{1}{2}} \sum_{m=0}^{\infty} a_m x^m, \tag{10}$$

where the coefficients $\{a_m\}_{m=0}^{\infty}$ are to be determined.

Substituting the expansion (10) into equation (8) and equating coefficients of like powers of x, we obtain recurrence relations for the coefficients a_m. Without loss of generality, we normalize the solution by choosing $a_0 = 1$.

Thus, using the Frobenius method, we obtain a regular solution of equation (8) in the form

$$\phi(x, \lambda) = x^{\nu+\frac{1}{2}} \left(1 + \sum_{m=2}^{\infty} a_m(\lambda) x^m \right), \tag{11}$$

where $\operatorname{Re} \nu > \frac{1}{2}$.

Under this condition, the series in (11) converges in a neighborhood of the origin and defines an analytic solution of equation (8).

Substituting into equation (8), we obtain

$$a_0 = 1, \qquad a_1 = 0,$$

and for $m \geq 2$

$$a_m(\lambda) = \frac{\sum_{j=0}^{m-2} c_j a_{m-2-j}(\lambda) - \lambda^2 a_{m-2}(\lambda)}{m(m+2\nu)}.$$

Lemma 2.1. *Let $\Re\nu > \frac{1}{2}$, and assume that q is analytic in a neighborhood of $x = 0$, with expansion*

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx} = \sum_{j=0}^{\infty} c_j x^j.$$

Then equation

$$-u''(x) + \left(\frac{\nu^2 - \frac{1}{4}}{x^2} + q(x)\right) u(x) = \lambda^2 u(x)$$

admits a solution of the form

$$\phi(x,\lambda) = x^{\nu+\frac{1}{2}} \sum_{m=0}^{\infty} a_m(\lambda) x^m, \qquad a_0 = 1,$$

where the coefficients satisfy

$$a_1 = 0,$$

and for $m \geq 2$,

$$a_m(\lambda) = \frac{\sum_{j=0}^{m-2} c_j a_{m-2-j}(\lambda) - \lambda^2 a_{m-2}(\lambda)}{m(m+2\nu)}.$$

The series converges in a neighborhood of $x = 0$ and defines the regular solution satisfying

$$\phi(x,\lambda) \sim x^{\nu+\frac{1}{2}}, \qquad x \to 0.$$

3. Jost-type Solutions

Let us extend the potential (2) by zero to the half-line:

$$Q(x) = \begin{cases} q(x), & 0 < x < a \\ 0, & x > a \end{cases} \tag{12}$$

Then for $x > a$ the equation becomes

$$u''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2} u(x) = \lambda^2 u(x), \tag{13}$$

whose fundamental solutions are given by Jost-type functions

$$f^+(x,\lambda) = \sqrt{x} H_\nu^{(1)}(\lambda x),$$
$$f^-(x,\lambda) = \sqrt{x} H_\nu^{(2)}(\lambda x).$$

Theorem 3.1. *Let $Q(x)$ be of the form (12). Then the equation*

$$u''(x) + \frac{\nu^2 - \frac{1}{4}}{x^2} u(x) + Q(x)u(x) = \lambda^2 u(x), \tag{14}$$

has a solution of the form

$$F^\pm(x,\lambda) = \begin{cases} f^\pm(x,\lambda), & 0 < x < a \\ e^{ix\lambda}, & x > a \end{cases}$$

where

$$\begin{aligned} f^{\pm}(x,\lambda) &= \psi_\nu^{(\pm)}(x,\lambda)\,\eta^{(\pm)}(x,\lambda) \\ &= \psi_\nu^{(\pm)}(x,\lambda)\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{n\pm 2\lambda}\,e^{i\alpha x}\right) \\ &= \sqrt{x}\,H_\nu^{(1,2)}(\lambda x)\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{n\pm 2\lambda}\,e^{i\alpha x}\right), \end{aligned} \tag{15}$$

and the numbers $V_{n\alpha}$ *(*$n\in\mathbb{N}$, $\alpha>n$*) are determined by the recurrent relations*

$$\alpha(\alpha-n)V_{n\alpha}-\sum_{s=n}^{\alpha-1}q_{\alpha-s}V_{ns}=0, \tag{16}$$

$$q_\alpha-\alpha\sum_{n=1}^{\alpha}V_{n\alpha}=0. \tag{17}$$

The series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n+1}^{\infty}\alpha(\alpha-n)\,|V_{n\alpha}|,\qquad \sum_{n=1}^{\infty}n\,|V_{nn}| \tag{18}$$

are converging.

Lemma 3.2. *Assume that the potential is extended by zero for* $x>a$*, and let* $f^+(x,\lambda)$ *and* $f^-(x,\lambda)$ *be the Jost-type solutions of the extended equation satisfying*

$$f^+(x,\lambda)\sim\sqrt{x}\,H_\nu^{(1)}(\lambda x),\qquad f^-(x,\lambda)\sim\sqrt{x}\,H_\nu^{(2)}(\lambda x),\qquad x\to\infty.$$

Then their Wronskian is independent of x *and is given by*

$$W[f^+(x,\lambda),f^-(x,\lambda)]=-\frac{4i}{\pi}.$$

Proof. Since the differential equation does not contain a first–derivative term, the Wronskian of any two solutions is independent of x. Therefore, it is sufficient to evaluate the Wronskian in the limit as $x\to\infty$.

Using the asymptotic relations

$$f^+(x,\lambda)\sim\sqrt{x}\,H_\nu^{(1)}(\lambda x),\qquad f^-(x,\lambda)\sim\sqrt{x}\,H_\nu^{(2)}(\lambda x),$$

we obtain

$$W[f^+(x,\lambda),f^-(x,\lambda)]=W\left[\sqrt{x}H_\nu^{(1)}(\lambda x),\sqrt{x}H_\nu^{(2)}(\lambda x)\right].$$

Let $U_j(x)=\sqrt{x}\,y_j(\lambda x)$. Then the Wronskian satisfies

$$W_x[U_1,U_2]=\lambda x\,W_z[y_1,y_2]\big|_{z=\lambda x}.$$

Taking $y_1(z)=H_\nu^{(1)}(z)$ and $y_2(z)=H_\nu^{(2)}(z)$, and using the known identity

$$W_z\left[H_\nu^{(1)}(z),H_\nu^{(2)}(z)\right]=-\frac{4i}{\pi z},$$

we obtain

$$W[f^+(x,\lambda),f^-(x,\lambda)]=\lambda x\left(-\frac{4i}{\pi\lambda x}\right)=-\frac{4i}{\pi}.$$

Thus, the Wronskian of the Jost solutions is constant and given by

$$W[f^+(x,\lambda), f^-(x,\lambda)] = -\frac{4i}{\pi}.$$

□

Let us introduce the functions

$$\begin{aligned} f_{\nu n}(x) &:= \lim_{\lambda\to\frac{n}{2}} (n-2\lambda)\, f^-(x,\lambda) \\ &= \lim_{\lambda\to\frac{n}{2}} (n-2\lambda)\, x^{\frac{1}{2}} H_\nu^{(2)}(\lambda x) \left(1 + \sum_{s=1}^{\infty}\sum_{\alpha=s}^{\infty} \frac{V_{s\alpha}}{s-2\lambda} e^{i\alpha x}\right) \\ &= x^{\frac{1}{2}} H_\nu^{(2)}\left(\frac{n}{2}x\right) \sum_{\alpha=n}^{\infty} V_{n\alpha}\, e^{i\alpha x}. \end{aligned} \tag{19}$$

Remark 3.1. From Lemma 1, it follows that the functions

$$f^+(x,\lambda),\, f^-(x,\lambda)$$

are linearly independent.

Note that the Wronskian of the functions $f_{\nu n}(x)$ and $f^+\left(x, \frac{n}{2}\right)$ is equal to zero. Hence, these functions are linearly dependent. Therefore, we obtain

$$f_{\nu n}(x) = S_n^\nu\, f^+\left(x, \frac{n}{2}\right). \tag{20}$$

Next, we use the asymptotic formulas for the Hankel functions as $z \to \infty$:

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}, \tag{21}$$

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}. \tag{22}$$

Hence,

$$\frac{H_\nu^{(1)}(z)}{H_\nu^{(2)}(z)} \sim e^{2i\left(z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)} = e^{2iz}\, e^{-i\pi\left(\nu+\frac{1}{2}\right)}.$$

Setting $z = \frac{n}{2}x$ and letting $x \to \infty$, we obtain

$$H_\nu^{(1)}\left(\frac{n}{2}x\right) \approx e^{-i\pi\left(\nu+\frac{1}{2}\right)} e^{inx} H_\nu^{(2)}\left(\frac{n}{2}x\right).$$

Finally, from (20), it follows that

$$S_n^\nu = e^{-i\pi\left(\nu+\frac{1}{2}\right)} V_{nn}. \tag{23}$$

In particular, for the regular solution $\varphi(x,\lambda)$ we have

$$\varphi(x,\lambda) = A(\lambda) f^+(x,\lambda) + B(\lambda) f^-(x,\lambda), \tag{24}$$

Since φ satisfies the Robin condition at $x = a$, we obtain

$$\varphi'(a,\lambda) + h\,\varphi(a,\lambda) = 0. \tag{25}$$

Substituting the connection formula into this relation gives

$$0 = A(\lambda)\big(f^{+\prime}(a,\lambda) + h\, f^+(a,\lambda)\big) + B(\lambda)\big(f^{-\prime}(a,\lambda) + h\, f^-(a,\lambda)\big). \tag{26}$$

Therefore the coefficients $A(\lambda)$ and $B(\lambda)$ are not independent; they satisfy

$$A(\lambda)\,\Phi^{+}(\lambda) + B(\lambda)\,\Phi^{-}(\lambda) = 0, \tag{27}$$

where

$$\Phi^{\pm}(\lambda) := f^{\pm\prime}(a,\lambda) + h\,f^{\pm}(a,\lambda). \tag{28}$$

Assuming $\Phi^{+}(\lambda) \neq 0$, we obtain

$$\frac{A(\lambda)}{B(\lambda)} = -\frac{\Phi^{-}(\lambda)}{\Phi^{+}(\lambda)} = -\frac{f^{-\prime}(a,\lambda) + h\,f^{-}(a,\lambda)}{f^{+\prime}(a,\lambda) + h\,f^{+}(a,\lambda)}. \tag{29}$$

Thus, the spectral coefficient can be defined by

$$S(\lambda) := \frac{A(\lambda)}{B(\lambda)} = -\frac{f^{-\prime}(a,\lambda) + h\,f^{-}(a,\lambda)}{f^{+\prime}(a,\lambda) + h\,f^{+}(a,\lambda)}. \tag{30}$$

Equivalently, introducing the boundary form

$$U_a(y) := y'(a) + h\,y(a), \tag{31}$$

the above formula can be written compactly as

$$S(\lambda) = -\frac{U_a(f^{-})}{U_a(f^{+})}. \tag{32}$$

Hence the Robin boundary condition at the right endpoint directly determines the connection between the f^{-} and f^{+} components of the regular solution. Moreover, the eigenvalues of the original boundary value problem are precisely the zeros of

$$\Delta(\lambda) := \varphi'(a,\lambda) + h\,\varphi(a,\lambda), \tag{33}$$

and, in terms of the Jost basis, this characteristic function takes the form

$$\Delta(\lambda) = A(\lambda)\,\Phi^{+}(\lambda) + B(\lambda)\,\Phi^{-}(\lambda). \tag{34}$$

For eigenvalues, one has $\Delta(\lambda) = 0$, which is exactly the compatibility condition above.

4. Spectrum and Construction of the Resolvent

This section provides an investigation of the spectral properties of the singular Bessel-type Sturm–Liouville operator defined by (8) and constructs the corresponding resolvent operator.

Definition 4.1. Values λ^2 for which

$$\Delta(\lambda) = 0$$

are called eigenvalues of the boundary value problem, and the corresponding solutions $\varphi(x,\lambda)$ are eigenfunctions.

Equivalently, the spectrum of the operator L is

$$\sigma(L) = \{\lambda^2 : \Delta(\lambda) = 0\}. \tag{35}$$

Let $f^{+}(x,\lambda)$ and $f^{-}(x,\lambda)$ denote the solutions constructed in Section 3.
Define the solution satisfying the boundary condition at $x = a$ by

$$\psi(x,\lambda) = \Phi^{-}(\lambda) f^{+}(x,\lambda) - \Phi^{+}(\lambda) f^{-}(x,\lambda). \tag{36}$$

Then

$$\psi'(a,\lambda) + h\psi(a,\lambda) = 0,$$

therefore $\psi(x,\lambda)$ satisfies the boundary condition at $x=a$.

Let $\varphi(x,\lambda)$ be the regular solution at $x=0$ and $\psi(x,\lambda)$ be the solution satisfying the boundary condition at $x=a$.

Define the Wronskian

$$W(\lambda)=W[\varphi,\psi]=\varphi(x,\lambda)\psi'(x,\lambda)-\varphi'(x,\lambda)\psi(x,\lambda). \tag{37}$$

Since equation contains no first derivative term, the Wronskian is independent of x.

The Green function is defined as

$$G(x,t;\lambda)=\begin{cases}\dfrac{\varphi(x,\lambda)\psi(t,\lambda)}{W(\lambda)}, & x<t,\\ \dfrac{\varphi(t,\lambda)\psi(x,\lambda)}{W(\lambda)}, & x>t.\end{cases} \tag{38}$$

and satisfies

$$(L-\lambda^2)G(x,t;\lambda)=\delta(x-t). \tag{39}$$

The resolvent operator is defined by

$$R(\lambda^2)=(L-\lambda^2 I)^{-1}. \tag{40}$$

For $f\in L_2(0,a)$ the resolvent acts as

$$(R(\lambda^2)f)(x)=\int_0^a G(x,t;\lambda)f(t)dt. \tag{41}$$

Hence,

$$(R(\lambda^2)f)(x)=\int_0^x \frac{\varphi(t,\lambda)\psi(x,\lambda)}{W(\lambda)}f(t)dt \tag{42}$$

$$+\int_x^a \frac{\varphi(x,\lambda)\psi(t,\lambda)}{W(\lambda)}f(t)dt. \tag{43}$$

Theorem 4.1. *For $\lambda^2\notin\sigma(L)$, the resolvent operator admits the integral representation*

$$(R(\lambda^2)f)(x)=\int_0^a G(x,t;\lambda)f(t)dt, \tag{44}$$

where $G(x,t;\lambda)$ is the Green function constructed above.

Let us note some basic properties of the resolvent operator :

(1) $R(\lambda^2)$ is analytic outside the spectrum
(2) Poles of the resolvent coincide with eigenvalues
(3) Residues of the resolvent determine spectral projections

Near an eigenvalue λ_n we have

$$R(\lambda^2)=\frac{P_n}{\lambda^2-\lambda_n^2}+R_0(\lambda), \tag{45}$$

where P_n is the spectral projection operator.

Theorem 4.2. *Assume that $\Re\nu>\frac{1}{2}$ and q defines by (2). Then the operator L has a compact resolvent. Consequently, its spectrum consists only of isolated eigenvalues of finite algebraic multiplicity, and the only possible accumulation point of the spectrum is infinity.*

Proof. The proof is divided into several steps.

Step 1: Homogeneous equation and the characteristic function. For each fixed $\lambda \in \mathbb{C}$, the differential equation

$$-u''(x) + \left(\frac{\nu^2 - \frac{1}{4}}{x^2} + q(x)\right) u(x) = \lambda^2 u(x) \tag{46}$$

has, up to a constant factor, a unique solution $\varphi(x, \lambda)$ satisfying the regular asymptotic condition at $x = 0$. Therefore a nontrivial solution satisfying both boundary conditions exists if and only if

$$U_a(\varphi) = \varphi'(a, \lambda) + h\varphi(a, \lambda) = 0,$$

that is, if and only if $\Delta(\lambda) = 0$.

Step 2: Analyticity of the characteristic function. By the construction of the regular solution near the singular endpoint, $\varphi(x, \lambda)$ depends analytically on λ for each fixed $x \in (0, a]$. Hence $\Delta(\lambda)$ is an entire function. Therefore its zeros form a discrete subset of the λ-plane unless $\Delta \equiv 0$. The latter is impossible, because for generic λ the Robin boundary condition at $x = a$ is not satisfied by the regular solution. Thus the set

$$\{\lambda \in \mathbb{C} : \Delta(\lambda) = 0\}$$

is discrete in $\mathbb{C}$.

Step 3: Construction of the resolvent for non-spectral points.

Fix λ such that $\Delta(\lambda) \neq 0$. Let $\varphi(x, \lambda)$ be the regular solution at $x = 0$. Let $\psi(x, \lambda)$ be any solution of the same differential equation satisfying the same differential equation that satisfies the boundary condition at the right endpoint $x = a$.

$$U_a(\psi) := \psi'(a, \lambda) + h\psi(a, \lambda) = 0. \tag{47}$$

Since $\Delta(\lambda) \neq 0$, the functions φ and ψ are linearly independent. Hence their Wronskian

$$W(\lambda) := W[\varphi, \psi](x) = \varphi(x, \lambda)\psi'(x, \lambda) - \varphi'(x, \lambda)\psi(x, \lambda) \tag{48}$$

is a nonzero constant, independent of x, because the differential equation contains no first-derivative term.

From (38) by the standard Green-function calculation, we obtain that $G(\cdot, t; \lambda)$ satisfies the differential equation for $x \neq t$, satisfies the regular condition at $x = 0$, satisfies the Robin condition at $x = a$, is continuous at $x = t$, and its derivative has the jump relation

$$\frac{\partial}{\partial x} G(x, t; \lambda)\Big|_{x=t+0} - \frac{\partial}{\partial x} G(x, t; \lambda)\Big|_{x=t-0} = 1. \tag{49}$$

Therefore, for every $f \in L_2(0, a)$, the function

$$u(x) := \int_0^a G(x, t; \lambda) f(t)\, dt \tag{50}$$

belongs to $D(L)$ and satisfies

$$(L - \lambda^2 I)u = f. \tag{51}$$

Hence $(L - \lambda^2 I)^{-1}$ exists for every λ such that $\Delta(\lambda) \neq 0$.

Step 4: Compactness of the resolvent. We claim that the integral operator

$$(R(\lambda^2)f)(x) := \int_0^a G(x,t;\lambda)f(t)\,dt$$

is compact in $L_2(0,a)$. Indeed, on every compact subset away from the diagonal, the kernel $G(x,t;\lambda)$ is continuous. Near the singular endpoint $x=0$, the regular solution satisfies

$$\varphi(x,\lambda) = O\left(x^{\nu+\frac{1}{2}}\right), \qquad x \to 0,$$

with $\Re\nu > \frac{1}{2}$, hence

$$|\varphi(x,\lambda)| \le Cx^{\Re\nu+\frac{1}{2}}.$$

The second solution $\psi(x,\lambda)$ is locally bounded near $x=0$. Therefore

$$|G(x,t;\lambda)| \le C \begin{cases} x^{\Re\nu+\frac{1}{2}}, & x \le t, \\ t^{\Re\nu+\frac{1}{2}}, & t \le x, \end{cases}$$

and consequently

$$|G(x,t;\lambda)|^2 \le C\min\left\{x^{2\Re\nu+1},\, t^{2\Re\nu+1}\right\}. \tag{52}$$

Since $\Re\nu > \frac{1}{2}$, we have $2\Re\nu+1 > 2$, and thus

$$\int_0^a \int_0^a |G(x,t;\lambda)|^2\,dx\,dt < \infty. \tag{53}$$

Hence $G(\cdot,\cdot;\lambda) \in L_2((0,a)\times(0,a))$, so the resolvent is a Hilbert–Schmidt operator and therefore compact.

Step 5: Spectral consequence. It is a classical result of operator theory that a closed operator with compact resolvent has a purely discrete spectrum; that is, every spectral point is an eigenvalue of finite algebraic multiplicity and the spectrum has no finite accumulation point (see, e.g.,[17]). Therefore, the spectrum of L consists only of isolated eigenvalues of finite multiplicity, and the only possible accumulation point is infinity.

It is a standard theorem of operator theory that an operator with compact resolvent has purely discrete spectrum: every spectral point is an eigenvalue of finite algebraic multiplicity, and the spectrum has no finite accumulation point. Therefore the spectrum of L consists only of isolated eigenvalues of finite multiplicity, and the only possible accumulation point is infinity. □

5. Inverse Problem

In this section, we formulate and solve the inverse spectral problem for the singular Bessel-type Sturm–Liouville operator introduced in Section 1. The goal is to reconstruct the potential $q(x)$ from spectral data associated with the boundary value problem.

Definition. The spectral data of the non-self-adjoint Bessel-type Sturm–Liouville operator are defined as

$$\mathcal{S} = \left\{S(\lambda),\ \pm\frac{n}{2}, n \in N\right\}$$

where $S(\lambda)$ is the spectral coefficient defines in (32), $\pm\frac{n}{2}, n \in N$ denotes the corresponding resonance.

Inverse Problem Given the spectral data

$$S(\lambda) := \frac{A(\lambda)}{B(\lambda)} = -\frac{f^{-\prime}(a,\lambda) + h\, f^{-}(a,\lambda)}{f^{+\prime}(a,\lambda) + h\, f^{+}(a,\lambda)} \ \textit{and} \ \pm \frac{n}{2}, n \in N$$

reconstruct the potential $q(x)$.

We begin by showing that the diagonal coefficients V_{nn}, $n \in \mathbb{N}$, can be uniquely recovered from the spectral data $S(\lambda)$.

Using relation (23), the spectral coefficient admits the representation (30).

$$S(\lambda) = -\frac{f^{-\prime}(a,\lambda) + h\, f^{-}(a,\lambda)}{f^{+\prime}(a,\lambda) + h\, f^{+}(a,\lambda)}.$$

Taking the limit as $\lambda \to \frac{n}{2}$, we obtain

$$\begin{aligned}\lim_{\lambda\to\frac{n}{2}} (n-2\lambda)S(\lambda) &= \lim_{\lambda\to\frac{n}{2}} (n-2\lambda)\frac{f^{-\prime}(a,\lambda) + h\, f^{-}(a,\lambda)}{f^{+\prime}(a,\lambda) + h\, f^{+}(a,\lambda)} \\ &= \frac{\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)f^{-\prime}(a,\lambda) + h\,\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)f^{-}(a,\lambda)}{f^{+\prime}\left(a,\frac{n}{2}\right) + h\, f^{+}\left(a,\frac{n}{2}\right)} = S_n^{\nu}.\end{aligned}$$

Since

$$S_n^{\nu} = e^{-i\pi\left(\nu+\frac{1}{2}\right)} V_{nn},$$

the coefficients V_{nn}, $n \in \mathbb{N}$, are uniquely determined by the spectral data.

Once the diagonal coefficients V_{nn} are known, the recursive relations

$$\alpha(\alpha-n)V_{n\alpha} = \sum_{s=n}^{\alpha-1} q_{\alpha-s}V_{ns}, \qquad \alpha > n,$$

allow one to determine all coefficients $V_{n\alpha}$ successively.

Assume that $q_1, \dots, q_{m-1}$ and all coefficients $V_{n\beta}$ with $\beta < m$ are already determined. Then the above relation uniquely determines V_{nm} for $n < m$. Since V_{mm} is known from the spectral data, the identity

$$q_m = m \sum_{n=1}^{m} V_{nm}$$

yields the coefficient q_m.

Proceeding inductively, all Fourier coefficients q_m, $m \geq 1$, are uniquely recovered. Consequently, the potential is reconstructed in the form

$$q(x) = \sum_{m=1}^{\infty} q_m e^{imx}.$$

Theorem 5.1. *The spectral data $S(\lambda)$ uniquely and constructively determine the potential $q(x)$.*

6. Illustrative Example

In this section, we present an explicit example illustrating the reconstruction procedure for the inverse problem of the singular Bessel-type Sturm–Liouville operator considered in the previous sections.

6.1. Model Problem. Consider the boundary value problem

$$-u''(x)+\left(\frac{\nu^2-\frac{1}{4}}{x^2}+q(x)\right)u(x)=\lambda^2u(x),\quad x\in(0,a), \tag{54}$$

with boundary conditions

$$u(x,\lambda)\sim x^{\nu+\frac{1}{2}},\quad x\to 0, \tag{55}$$

$$u'(a,\lambda)+hu(a,\lambda)=0. \tag{56}$$

Let us choose a simple periodic potential of the form

$$q(x)=q_1e^{ix}, \tag{57}$$

where $q_1\in\mathbb{C}$ is a constant. This choice satisfies the Fourier representation

$$q(x)=\sum_{n=1}^{\infty}q_ne^{inx}, \tag{58}$$

with

$$q_1\neq 0,\quad q_n=0,\quad n\geq 2.$$

Thus, the assumptions of the inverse problem formulation are satisfied.

6.2. Jost Solutions. For $x>a$, the equation reduces to the unperturbed Bessel equation

$$-u''(x)+\frac{\nu^2-\frac{1}{4}}{x^2}u(x)=\lambda^2u(x), \tag{59}$$

whose fundamental solutions are

$$f_+(x,\lambda)=\sqrt{x}H_\nu^{(1)}(\lambda x), \tag{60}$$

$$f_-(x,\lambda)=\sqrt{x}H_\nu^{(2)}(\lambda x). \tag{61}$$

Inside the interval $(0,a)$, the Jost solutions admit the representation

$$f_\pm(x,\lambda)=\sqrt{x}H_\nu^{(1,2)}(\lambda x)\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{n\pm 2\lambda}e^{i\alpha x}\right). \tag{62}$$

For the chosen potential, only coefficients

$$V_{11},\ V_{12},\ \dots$$

are nonzero.

6.3. Determination of Spectral Data. From the inverse problem formulation, the spectral coefficient is defined as

$$S(\lambda)=\frac{A(\lambda)}{B(\lambda)}. \tag{63}$$

Using the Wronskian representation, we obtain

$$S(\lambda)=\frac{W[\phi(x,\lambda),f_-(x,\lambda)]}{W[f_+(x,\lambda),\phi(x,\lambda)]}. \tag{64}$$

Taking limits near the poles yields

$$\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)S(\lambda)=-S_n^\nu. \tag{65}$$

Hence,

$$S_n^\nu = e^{-i\pi\left(\nu+\frac{1}{2}\right)} V_{nn}. \tag{66}$$

Thus,

$$V_{11} = e^{i\pi\left(\nu+\frac{1}{2}\right)} S_1^\nu. \tag{67}$$

Therefore, the spectral coefficient uniquely determines the coefficient V_{11}.

6.4. Reconstruction of Fourier Coefficient. Using the recursive relation

$$q_m = m \sum_{n=1}^{m} V_{nm}, \tag{68}$$

for $m = 1$ we obtain

$$q_1 = V_{11}. \tag{69}$$

Thus, the reconstructed potential takes the form

$$q(x) = V_{11} e^{ix}. \tag{70}$$

6.5. Numerical Illustration. Let

$$\nu = 1, \qquad a = \pi, \qquad h = 0.$$

Assume the spectral data

$$S_1^\nu = 0.5e^{i}. \tag{71}$$

Then

$$V_{11} = e^{i\pi\left(\frac{3}{2}\right)} \cdot 0.5e^{i} = -0.5ie^{i}. \tag{72}$$

Therefore, the reconstructed potential is

$$q(x) = -0.5ie^{ix}. \tag{73}$$

6.6. Conclusion. This example demonstrates the reconstruction procedure:

(1) Compute spectral coefficient
(2) Determine coefficients V_{nn}
(3) Recover coefficients $V_{n\alpha}$ recursively
(4) Reconstruct Fourier coefficients
(5) Obtain potential $q(x)$

Thus, the inverse problem for the singular Bessel-type Sturm-Liouville operator is constructive and practically implementable.

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(R. F. Efendiev) Department of Mathematics and Computer Science, Baku Engineering University, Baku AZ0102, Azerbaijan