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A note on the GBS Bernstein's approximation formula

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Abstract. The Bernstein's GBS (Generalized Boolean Sum) is revisited and it's remainder term is expressed in terms of bivariate divided differences. When the approximated function is two times differentiable on $]0,1[\times]0,1[$, an upper bound estimation for the remainder term is given. A sufficient condition for the convergence of the sequence of the pseudopolynomials is also established.

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1. Introduction

Let's denote $\mathbb{N} = \{1, 2, \dots\}$ and $N_0 = N \cup \{0\}$. It is well known ([12]) that the classical Bernstein's operator is defined for any $f \in C([0,1])$ and any $x \in [0,1]$ by:

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1}$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{2}$$

are the fundamental Bernstein's polynomials, $k \in \{0, 1, \dots, n\}$.
There are well known many extensions of operator (1)) to the case of bivariate functions. All of these generalizations are in fact essentially based on the method of parametric extensions ([9]). Let us to describe shortly this method in the concret case of operators (1).

Suppose that $f \in C([0,1] \times [0,1])$ is given, $(x, y) \in [0,1] \times [0,1]$ and $m, n \in \mathbb{N}$. Then, the parametric extensions of (1) are defined respectively by:

$$\left(B_m^x f\right)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k}{m}, y\right) \tag{3}$$

and

$$\left(B_n^y f\right)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(x, \frac{j}{n}\right). \tag{4}$$

Considering the operators (3) and (4), two kinds of bivariate Bernstein's operators can be defined.

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The first of them, known as the "Bernstein's bivariate operator" was obtained by considering the so called "tensorial product of parametrical extensions" ([9]) and it is defined by

$$(B_{m,n}f)(x,y)=\sum_{k=0}^{m}\sum_{j=0}^{n}p_{m,k}(x)p_{n,j}(y)f\left(\frac{k}{m},\frac{j}{n}\right). \tag{5}$$

It easy to observe that $B_{m,n}=B_m^xB_n^y$.

The second generalization is the so called "GBS operator of Bernstein type" and it is defined for any $f\in C([0,1]\times[0,1])$, any $m,n\in\mathbb{N}$ and any $(x,y)\in[0,1]\times[0,1]$, by:

$$\begin{aligned}&(U_{m,n}f)(x,y)\\&=\sum_{k=0}^{m}\sum_{j=0}^{n}p_{m,n}(x)p_{n,j}(y)\left\{f\left(\frac{k}{m},y\right)+f\left(x,\frac{j}{n}\right)-f\left(\frac{k}{m},\frac{j}{n}\right)\right\}.\end{aligned} \tag{6}$$

Note that the operators (6) were first considered by E. Dobrescu and I. Matei ([8]). In fact, the operator (6) is the "boolean sum" of parametrical extensions (3) and (4), i.e.

$$U_{m,n}=B_m^x+B_n^y-B_{mn}. \tag{7}$$

C. Badea and C. Cottin were the firsts who introduced the term of "generalized boolean sum operator". General results concerning the approximation properties of operator (6) were obtained by H. Gonska, C. Badea and I. Badea ([1]).

In this paper we are dealing with the approximation formula

$$f=U_{m,n}f+R_{m,n}f, \tag{8}$$

where $f:[0,1]\times[0,1]\to\mathbb{R}$ is a real function and $m,n\in\mathbb{N}$. More exactly, we shall express the remainder term of (8) in terms of bivariate divided differences.

In the following, we recall some well-known results.

Let $I,J\subset\mathbb{R}$ be intervals, $p,q\in\mathbb{N}_0$ and $f:I\times J\to\mathbb{R}$ be given function. Then, the following identities are true (see [6])

$$\begin{aligned}&\begin{bmatrix}x_0,x_1\\y_0,y_1\end{bmatrix};f\Bigg]\\&=\frac{1}{(x_1-x_0)(y_1-y_0)}\left(f(x_1,y_1)-f(x_0,y_1)-f(x_1,y_0)+f(x_0,y_0)\right)\end{aligned} \tag{9}$$

for any $(x_0,y_0),(x_1,y_1)\in I\times J$,

$$\begin{aligned}\left[\begin{matrix}x_0,x_1,\dots,x_p\\y_0,y_1,\dots,y_q\end{matrix};f\right]&=\frac{1}{(x_p-x_0)(y_q-y_0)}\Bigg(\left[\begin{matrix}x_1,x_2,\dots,x_p\\y_1,y_2,\dots,y_q\end{matrix};f\right]\\&-\left[\begin{matrix}x_0,x_1,...,x_{p-1}\\y_1,y_2,...,y_q\end{matrix};f\right]-\left[\begin{matrix}x_1,x_2,...,x_p\\y_0,y_1,...,y_{q-1}\end{matrix};f\right]+\left[\begin{matrix}x_0,x_1,...,x_{p-1}\\y_0,y_1,...,y_{q-1}\end{matrix};f\right]\Bigg)\end{aligned} \tag{10}$$

and

$$\left[\begin{matrix}x_0,x_1,\dots,x_p\\y_0,y_1,\dots,y_q\end{matrix};f\right]=\left[\begin{matrix}x_{i_0},x_{i_1},\dots,x_{i_p}\\y_{j_0},y_{j_1},\dots,y_{j_q}\end{matrix};f\right] \tag{11}$$

for any $(x_i,y_j)\in I\times J$, $i\in\{0,1,\dots,p\}$, $j\in\{0,1,\dots,q\}$ where $(i_0,i_1,\dots,i_p)$, $(j_0,j_1,\dots,j_q)$ are permutations of $(0,1,\dots,p)$, respectively $(0,1,\dots,q)$.

Remark 1.1. *In the above identities, we consider that the knots $x_0,x_1,\dots,x_p$ and respectively $y_0,y_1,\dots,y_q$ are distinct.*

In all what follow the brackets denote divided differences and for the results concerning the mentioned differences see ([6]) and the references from there.

2. Main results

We shall prove:

Theorem 2.1. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function, $(x, y) \in [0,1] \times [0,1]$ and $m, n \in \mathbb{N}$. The remainder therm of (8) can be represented under the form*

$$\begin{aligned}&(R_{mn}f)(x,y)\\&= \frac{xy(1-x)(1-y)}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} p_{m-1,k}(x)p_{n-1,j}(y)\left[\begin{array}{ll} x, \frac{k}{m}, \frac{k+1}{m} & \\ y, \frac{j}{n}, \frac{j+1}{n} & ;f\end{array}\right].\end{aligned} \tag{12}$$

Proof. Taking the definition (6) into account, yields

$$\begin{aligned}&(R_{m,n}f)(x,y) = f(x,y) - (U_{m,n}f)(x,y)\\&=\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)\left\{f(x,y) - f\left(\frac{k}{m}, y\right) - f\left(x, \frac{j}{n}\right) + f\left(\frac{k}{m}, \frac{j}{n}\right)\right\}.\end{aligned} \tag{13}$$

Using the (9) identity, we get:

$$\begin{aligned}&(R_{m,n}f)(x,y)\\&= \frac{1}{mn}\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)(mx-k)(ny-j)\left[\begin{array}{ll} x, \frac{k}{m} & \\ y, \frac{j}{n} & ;f\end{array}\right].\end{aligned} \tag{14}$$

Next, the elementary identity

$$(mx-k)(ny-j) = \{(m-k)x - k(1-x)\}\{(n-j)y - j(1-y)\}$$

leads to:

$$\begin{aligned}&(R_{mn}f)(x,y)\\
&= \frac{1}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} \frac{m!}{k!(m-k-1)!}\frac{n!}{j!(n-j-1)!} x^{k+1}y^{j+1}(1-x)^{m-k}(1-y)^{n-j}\left[\begin{array}{ll} x, \frac{k}{m} & \\ y, \frac{j}{n} & ;f\end{array}\right]\\
&- \frac{1}{mn}\sum_{k=0}^{m-1}\sum_{j=1}^{n} \frac{m!}{k!(m-k-1)!}\frac{n!}{(j-1)!(n-j)!} x^{k+1}y^{j}(1-x)^{m-k}(1-y)^{n+1-j}\left[\begin{array}{ll} x, \frac{k}{m} & \\ y, \frac{j}{n} & ;f\end{array}\right]\\
&- \frac{1}{mn}\sum_{k=1}^{m}\sum_{j=0}^{n-1} \frac{m!}{(k-1)!(m-k)!}\frac{n!}{j!(n-j-1)!} x^{k}y^{j+1}(1-x)^{m+1-k}(1-y)^{n-j}\left[\begin{array}{ll} x, \frac{k}{m} & \\ y, \frac{j}{n} & ;f\end{array}\right]\\
&+ \frac{1}{mn}\sum_{k=1}^{m}\sum_{j=1}^{n} \frac{m!}{(k-1)!(m-k)!}\frac{n!}{(j-1)!(n-j)!} x^{k}(1-x)^{m+1-k}y^{j}(1-y)^{n+1-j}\left[\begin{array}{ll} x, \frac{k}{m} & \\ y, \frac{j}{n} & ;f\end{array}\right].\end{aligned} \tag{15}$$

Let us to introduce the following notations:

$$T_1 = \frac{1}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1} \frac{m!}{k!(m-k-1)!}\frac{n!}{j!(n-j-1)!} x^{k+1}y^{j+1}(1-x)^{m-k}(1-y)^{n-j}\left[\begin{array}{ll} x, \frac{k}{m} & \\ y, \frac{j}{n} & ;f\end{array}\right], \tag{16}$$

$$T_2 = \frac{1}{mn}\sum_{k=0}^{m-1}\sum_{j=1}^{n} \frac{m!}{k!(m-k-1)!}\frac{n!}{(j-1)!(n-j)!} x^{k+1}y^{j}(1-x)^{m-k}(1-y)^{n+1-j}\left[\begin{array}{ll} x, \frac{k}{m} & \\ y, \frac{j}{n} & ;f\end{array}\right], \tag{17}$$

$$T_3 = \frac{1}{mn}\sum_{k=1}^{m}\sum_{j=0}^{n-1} \frac{m!}{(k-1)!(m-k)!}\frac{n!}{j!(n-j-1)!} x^{k}y^{j+1}(1-x)^{m+1-k}(1-y)^{n-j}\left[\begin{array}{ll} x, \frac{k}{m} & \\ y, \frac{j}{n} & ;f\end{array}\right], \tag{18}$$

$$T_4=\frac{1}{mn}\sum_{k=1}^{m}\sum_{j=1}^{n}\frac{m!}{(k-1)!(m-k)!}\frac{n!}{(j-1)!(n-j)!}x^k(1-x)^{m+1-k}(1-y)^{n+1-j}\begin{bmatrix} x,\frac{k}{m} & ;f\\ y,\frac{j}{n} & \end{bmatrix}. \quad (19)$$

After some elementary transformations, taking the above notations into account, we can write:

$$T_1=xy(1-x)(1-y)\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}p_{m-1,k}(x)\,p_{n-1,j}(y)\begin{bmatrix} x,\frac{k}{m} & ;f\\ y,\frac{j}{n} & \end{bmatrix},$$

$$T_2=xy(1-x)(1-y)\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}p_{m-1,k}(x)\,p_{n-1,j}(y)\begin{bmatrix} x,\frac{k}{m} & ;f\\ y,\frac{j+1}{n} & \end{bmatrix},$$

$$T_3=xy(1-x)(1-y)\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}p_{m-1,k}(x)\,p_{n-1,j}(y)\begin{bmatrix} x,\frac{k+1}{m} & ;f\\ y,\frac{j}{n} & \end{bmatrix},$$

$$T_4=xy(1-x)(1-y)\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}p_{m-1,k}(x)\,p_{n-1,j}(y)\begin{bmatrix} x,\frac{k+1}{m} & ;f\\ y,\frac{j+1}{n} & \end{bmatrix}.$$

From (15), taking the last expressions of (16), (17), (18) and (19) into account, one arrives to:

$$\begin{aligned}&(R_{mn}f)(x,y) \qquad (20)\\ &=xy(1-x)(1-y)\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}p_{m-1,k}(x)\,p_{n-1,j}(y)\left\{\begin{bmatrix} x,\frac{k}{m} & ;f\\ y,\frac{j}{n} & \end{bmatrix}\right.\\ &\left.-\begin{bmatrix} x,\frac{k}{m} & ;f\\ y,\frac{j+1}{n} & \end{bmatrix}-\begin{bmatrix} x,\frac{k+1}{m} & ;f\\ y,\frac{j}{n} & \end{bmatrix}+\begin{bmatrix} x,\frac{k+1}{m} & ;f\\ y,\frac{j+1}{n} & \end{bmatrix}\right\}.\end{aligned}$$

Next, taking the identities (10) and (11) into account, one arrives to

$$(R_{mn}f)(x,y)=\frac{xy(1-x)(1-y)}{mn}\sum_{k=0}^{m-1}\sum_{j=0}^{n-1}p_{m-1,k}(x)\,p_{n-1,j}(y)\begin{bmatrix} x,\frac{k}{m},\frac{k+1}{m} & ;f\\ y,\frac{j}{n},\frac{j+1}{n} & \end{bmatrix}$$

which is the desired identity (12). □

Theorem 2.2. *Let $p,q\in\mathbb{N}_0$, $a\le x_0<x_1<\cdots<x_p\le b$, $c\le y_0<y_1<\cdots<y_q\le d$ and function $f:[a,b]\times[c,d]\to\mathbb{R}$. If $f\in C^{(p-1,q-1)}([a,b]\times[c,d])$ and exists $\dfrac{\partial^{p+q}f}{\partial x^p\partial y^q}$ on $]a,b[\times]c,d[$ then, there exists $(\xi,\eta)\in]a,b[\times]c,d[$ such that*

$$\begin{bmatrix} x_0,x_1,\dots,x_p & ;f\\ y_0,y_1,\dots,y_q & \end{bmatrix}=\frac{1}{p!q!}\,\frac{\partial^{p+q}f}{\partial x^p\partial y^q}(\xi,\eta). \quad (21)$$

Proof. Applying the method of parametric extension (see [6]) and the mean-value theorem for one dimensional divided differences, there exist $\xi\in]a,b[$ and respectively

$\eta \in]c,d[$, such that

$$\begin{aligned}
&\left[\begin{matrix} x_0, x_1, \ldots, x_p \\ y_0, y_1, \ldots, y_q \end{matrix} ; f\right] = [y_0, y_1, \ldots, y_q; [x_0, x_1, \ldots, x_p; f]_x]_y \\
&= \left[y_0, y_1, \ldots, y_q; \frac{1}{p!} \frac{\partial^p f}{\partial x^p}(\xi, \cdot)\right]_y = \frac{1}{p!}\left[y_0, y_1, \ldots, y_q; \frac{\partial^p f}{\partial x^p}(\xi, \cdot)\right]_y \\
&= \frac{1}{p!q!} \frac{\partial^{p+q} f}{\partial x^p \partial y^q}(\xi, \eta),
\end{aligned}$$

so the equality (21) holds. $\square$

Theorem 2.3. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function with the properties that $f \in C^{(1,1)}([0,1] \times [0,1])$, there exists $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ on $]0,1[\times]0,1[$ and $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ is bounded on $[0,1] \times [0,1]$. Then the inequalities*

$$|(R_{m,n} f)(x,y)| \leq \frac{xy(1-x)(1-y)}{4mn} M(f) \leq \frac{1}{64mn} M(f) \tag{22}$$

hold for any $(x,y) \in [0,1] \times [0,1]$ and any $m, n \in \mathbb{N}$, where

$$M(f) = \sup_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \right|.$$

Proof. Because $x(1-x) \leq \dfrac{1}{4}$, $y(1-y) \leq \dfrac{1}{4}$ and $\sum\limits_{k=0}^{m-1} \sum\limits_{j=0}^{n-1} p_{m-1,k}(x), p_{n-1,j}(y) = 1$, for any $x, y \in [0,1]$, from (12) and Theorem 2.2, the inequalities from (22) follow. $\square$

Remark 2.1. *From (8) and (22) it results that in the conditions of Theorem 2.3, the sequence $(U_{m,n} f)_{m,n \geq 1}$ converges uniform to the function f.*

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