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Some results on Lorentzian β-Kenmotsu manifolds

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Abstract. The object of the present paper is to study Lorentzian β-Kenmotsu manifolds satisfying certain conditions.

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1. Introduction

In [14], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes:
(1) homogeneous normal contact Riemannian manifolds with $c > 0$,
(2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and
(3) a warped product space $\mathbf{R} \times_f \mathbf{C}$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [7] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [7]. In the Gray-Hervella classification of almost Hermitian manifolds [6], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [5]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [11] if the product manifold $M \times \mathbf{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$([8],[9]) coincides with the class of the trans-Sasakian structures of type (α, β). In fact, in [9], local nature of the two subclasses, namely, C_5 and C_6 structures, of trans-Sasakian structures are characterized completely.

We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [3], β-Kenmotsu [7] and α-Sasakian [7] respectively. In [15] it is proved that trans-Sasakian structures are generalized quasi-Sasakian [10]. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

An almost contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure [11] if $(M \times \mathbf{R}, J, G)$ belongs to the class W_4[6],where J is the almost complex structure on $M \times \mathbf{R}$ defined by

$$J(X, fd/dt) = (\phi X - f\xi, \eta(X)fd/dt)$$

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for all vector fields X on M and smooth functions f on $M \times \mathbf{R}$, and G is the product metric on $M \times \mathbf{R}$. This may be expressed by the condition [4]

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

Theorem 1.1. [1] *A trans-sasakian structure of type (α, β) with β a nonzero constant is always β-Kenmotsu*

In this case β becomes a constant. If $\beta = 1$, then β-Kenmotsu manifold is Kenmotsu.

The present paper dals with the study of Lorentzian β-Kenmotsu manifold satisfying certian conditions. After preliminaries, in section 3 we study Lorentzian β-Kenmotsu manifold satisfying the condition $R(X,Y).\widetilde{P} = 0$, where $\widetilde{P}$ is the pseudo projective curvature tensor and $R(X,Y)$ is considred as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y and it is shown that in a Lorentzian β-Kenmotsu manifold satisfying the condition $R(X,Y).P = 0$ is an η-Einstein manifold. Section 4 is devoted to the study of pseudo projectively recurrent Lorentzian β-Kenmotsu manifolds. In the last section we show that in a Lorentzian β-Kenmotsu manifold the transformation μ which leaves the curvature tensor invariant is an isometry and the infinitesimal paracontact transformationwhich leaves a Ricci tensor invariant is an infinitesimal strict paracontact transformation.

2. Preliminaries

A differentiable manifold M of dimension n is called Lorentzian Kenmotsu manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy

$$\eta\xi = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{1}$$

$$\phi^2 X = X + \eta(X)\xi, \quad g(X,\xi) = \eta(x), \tag{2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{3}$$

for all $X, Y \in TM$.
Also if Lorentzian Kenmotsu manifold M satisfies

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \tag{4}$$

$$(\nabla_X \eta)(Y) = \beta[g(X,Y) - \eta(X)\eta(Y)], \tag{5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g, then M is called Lorentzian β-Kenmotsu mnifold.
Further, on an Lorentzian β-Kenmotsu manifold M the following relations hold ([1], [2])

$$\eta(R(X,Y)Z) = \beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{6}$$

$$R(\xi, X)Y = \beta^2(\eta(Y)X - g(X,Y)\xi), \tag{7}$$

$$R(X,Y)\xi = \beta^2(\eta(X)Y - \eta(Y)X), \tag{8}$$

$$S(X,\xi) = -(n-1)\beta^2\eta(X), \tag{9}$$

$$Q\xi = -(n-1)\beta^2\xi, \tag{10}$$

$$S(\xi,\xi) = (n-1)\beta^2. \tag{11}$$

The pseudo projective curvature tensor on a Riemannian manifold is given by [12]

$$\begin{aligned}\widetilde{P}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ & \frac{r}{n}\left[\frac{a}{n-1}+b\right][g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{12}$$

. where a and b constants such that $a, b \neq 0$.

3. Lorentzian β-Kenmotsu Manifold satisfying $R(X,Y)\cdot\widetilde{P}=0$

From (6),(12) and (9) we have

$$\begin{aligned}\eta(\widetilde{P}(X,Y)Z) &= a\beta^2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\ &+b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] \\ &-\frac{r}{n}\left[\frac{a}{n-1}+b\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].\end{aligned} \tag{13}$$

Putting $Z=\xi$ in (13) we get

$$\eta(\widetilde{P}(X,Y)\xi) = 0. \tag{14}$$

Again taking $X=\xi$ in (13), we have

$$\begin{aligned}\eta(\widetilde{P}(\xi,Y)Z) &= \left[\frac{r}{n}\left(\frac{a}{n-1}+b\right)+a\beta^2\right][g(Y,Z)+\eta(Y)\eta(Z)] \\ &-bS(Y,Z) + (n-1)b\beta^2\eta(Y)\eta(Z).\end{aligned} \tag{15}$$

Now ,

$$\begin{aligned}(R(X,Y)\cdot\widetilde{P})(U,V)Z &= R(X,Y)\widetilde{P}(U,V)Z - \widetilde{P}(R(X,Y)U,V)Z \\ &-\widetilde{P}(U,R(X,Y)V)Z - \widetilde{P}(U,V)R(X,Y)Z.\end{aligned}$$

Let $R(X,Y)\cdot\widetilde{P}=0$, then we have

$$\begin{aligned}&R(X,Y)\cdot\widetilde{P}(U,V)Z - \widetilde{P}(R(X,Y)U,V)Z \\ &-\widetilde{P}(U,R(X,Y)V)Z - \widetilde{P}(U,V)R(X,Y)Z = 0.\end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned}&g[R(\xi,Y)\widetilde{P}(U,V)Z,\xi] - g[\widetilde{P}(R(\xi,Y)U,V)Z,\xi] \\ &-g[\widetilde{P}(U,R(\xi,Y)V)Z,\xi] - g[\widetilde{P}(U,V)R(\xi,Y)Z,\xi] = 0.\end{aligned} \tag{17}$$

From this it follows that

$$\begin{aligned}\widetilde{P}(U,V,Z,Y) &+ \eta(Y)\eta(\widetilde{P}(U,V)Z) - \eta(U)\eta(\widetilde{P}(Y,V)Z) \\ &+ g(U,Y)\eta(\widetilde{P}(\xi,V)Z) - \eta(V)\eta(\widetilde{P}(U,Y)Z) \\ &+ g(Y,V)\eta(\widetilde{P}(U,\xi)Z) - \eta(Z)\eta(\widetilde{P}(U,V)Y) = 0.\end{aligned} \tag{18}$$

where $\widetilde{P}(U,V,Z,Y) = g(\widetilde{P}(U,V)Z,Y)$.
Putting $Y=U$ in (18), we get

$$\begin{aligned}\widetilde{P}(U,V,Z,U) &+ g(U,U)\eta(\widetilde{P}(\xi,V)Z) + g(U,V)\eta(\widetilde{P}(U,\xi)Z) \\ &- \eta(V)\eta(\widetilde{P}(U,U)Z) - \eta(Z)\eta(\widetilde{P}(U,V)U) = 0.\end{aligned} \tag{19}$$

Let $\{e_i\}$, $i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (19) for $U = e_i$ yields

$$\eta(P(\xi, V)Z) = (a-b)\left[\frac{r}{n(n-1)} + \beta^2\right]\eta(V)\eta(Z). \tag{20}$$

From (15) and (20) we have

$$\begin{aligned} S(V,Z) &= \left[\frac{a}{b}\beta^2 + \frac{r}{n}\left(\frac{a}{b(n-1)} + 1\right)\right] g(V,Z) \\ &+ \left[\frac{a}{b}\beta^2 + \frac{r}{n}\left(\frac{a}{b(n-1)} + 1\right) + (n-1)\beta^2\right. \\ &\left.- \frac{(a-b)}{b}\left(\frac{r}{n(n-1)} + \beta^2\right)\right]\eta(V)\eta(Z). \end{aligned} \tag{21}$$

Taking $Z = \xi$ in (21), then using (1) and (9) we obtain

$$r = -n(n-1)\beta^2. \tag{22}$$

Now using (13), (14), (21) and (22) in (18) we get

$$\widetilde{P}(U, V, Z, Y) = 0. \tag{23}$$

From (23) it follows that

$$P(U, V)Z = 0. \tag{24}$$

Therefore the Lorentzian β-Kenmotsu manifold is pseudo projectively flat. Hence we can state

Theorem 3.1. *If in an Lorentzian β-Kenmotsu manifold M $n > 1$ the relation $R(X, Y) \cdot \widetilde{P} = 0$ holds, then the manifold is pseudo projectively flat.*

4. Pseudo projectively flat Lorentzian β-Kenmotsu Manifold

Suppose that $P(X, Y)Z = 0$. Then from (12), we have

$$R(X, Y)Z = \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]. \tag{25}$$

From (25), we have

$$\begin{aligned} R(X,Y,Z,W) &= -\frac{b}{a}[S(Y,Z)g(X,W) - S(X,Z)g(Y,Z)] \\ &+ \frac{r}{n}\left[\frac{1}{n-1} + \frac{b}{a}\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \quad (26) \end{aligned}$$

$$\tag{27}$$

where $R(X, Y, Z, W) = g(R(X, Y)Z, W)$.
Putting $W = \xi$ in (26), we get

$$\begin{aligned} \eta(R(X,Y)Z) &= -\frac{b}{a}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] \\ &+ \frac{r}{n}\left[\frac{1}{n-1} + \frac{b}{a}\right][g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \end{aligned} \tag{28}$$

Again taking $X=\xi$ in (28), and using (1),(6) and (9), we get

$$S(Y,Z) = \left[\frac{r}{n}\left(\frac{a}{b(n-1)}+1\right)+\frac{a}{b}\beta^2\right]g(Y,Z) \quad (29)$$

$$+\left[\frac{r}{n}\left(\frac{a}{b(n-1)}+1\right)+(n-1)\beta^2+\frac{a}{b}\beta^2\right]\eta(Y)\eta(Z) \quad (30)$$

Therefore, the manifold is η-Einstein.
From (29), it follows that

$$r=-n(n-1)\beta^2. \quad (31)$$

Hence we can state

Theorem 4.1. *A Pseudo projectively flat Lorentzian β-Kenmotsu manifold M $n>1$ is an η-Einstein manifold.*

Thus from Theorems 3.1 and 4.1, we conclude

Theorem 4.2. *A Lorentzian β-Kenmotsu manifold M satisfying $R(X,Y)\cdot\widetilde{P}=0$ is an η-Einstein manifold and also a manifold of negative curvature $-n(n-1)\beta^2$.*

5. Pseudo projectively recurrent Lorentzian β-Kenmotsu Manifold

A non-flat Riemannian manifold M is said to be pseudo projectively recurrent if the pseudo-projective curvature tensor $\widetilde{P}$ satisfies the condition $\nabla\widetilde{P}=A\otimes\widetilde{P}$, where A is an everywhere non-zero 1-form. We now define a function f on M by $f^2=g(\widetilde{P},\widetilde{P})$, where the metric g is extended to the inner product between the tensor fields in the standard fashion.

Then we know that $f(Yf)=f^2A(Y)$. So from this we have

$$Yf=fA(Y) \quad (because f\neq 0). \quad (32)$$

From (32) we have

$$X(Yf)=\frac{1}{f}(Xf)(Yf)+(XA(Y))f.$$

Hence

$$X(Yf)-Y(Xf)=\{XA(Y)-YA(X)\}f.$$

Therefore we get

$$(\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]})f=\{XA(Y)-YA(X)-A([X,Y])\}f.$$

Since the left hand side of the above equation is identically zero and $f\neq 0$ on M by our assumption. We obtain

$$dA(X,Y)=0. \quad (33)$$

that is the 1-form A is closed.

Now, from $(\nabla_X\widetilde{P})(U,V)Z=A(X)\widetilde{P}(U,V)Z$, we get

$$(\nabla_U\nabla_V\widetilde{P})(X,Y)Z=\{UA(V)+A(U)A(V)\}\widetilde{P}(X,Y)Z.$$

Hence from (33), we get

$$(R(X,Y).\widetilde{P})(U,V)Z=[2dA(X,Y)]\widetilde{P}(U,V)Z=0. \quad (34)$$

Therefore, for a pseudo projectively recurrent manifold, we have

$$R(X,Y)\widetilde{P}=0 \;\; for \;\; all \;\; X,Y. \quad (35)$$

Thus, we can state the following:

Theorem 5.1. *A pseudo projectively recurrent Lorentzian β-Kenmotsu manifold M is an η-Einstein manifold.*

Since for a pseudo projectively symmetric Lorentzian β-Kenmotsu manifold M, $(n > 1)$. we have $(\nabla_U \widetilde{P})(X, Y)Z = 0$ which implies $R(X, Y).\widetilde{P} = 0$. We can state the following:

Lemma 5.1. *A pseudo projectively symmetric Lorentzian β-Kenmotsu manifold M, $(n > 1)$ is an η-Einstein manifold.*

6. Some transformation in Lorentzian β-Kenmotsu Manifold

We now consider a transformation μ which transform a Lorentzian β-Kenmotsu structure (ϕ, ξ, η, g) into another Lorentzian β-Kenmotsu structure $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$. We denote by the notation 'bar' the geometric objects which are transformed by the transformation μ.

We first suppose that in a Lorentzian β-Kenmotsu manifold the Riemannian curvature tensor remains invariant with respect to the transformation μ.

Thus we have

$$\overline{R}(X, Y)Z = R(X, Y)Z \tag{36}$$

for all X, Y, Z.

This gives, $\eta(\overline{R}(X, Y)Z) = \eta(R(X, Y)Z)$, and hence by virtue of (6) we get

$$\eta(\overline{R}(X, Y)Z) = \beta^2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X). \tag{37}$$

Putting $Y = \overline{\xi}$ in (37) and then using (7) we obtain

$$\eta(\overline{\xi})g(X, Z) - \eta(X)g(\overline{\xi}, Z) = \eta(\overline{\xi})\overline{g}(X, Z) - \overline{\eta}(Z)\eta(X). \tag{38}$$

Interchanging X and Z

$$\eta(\overline{\xi})g(X, Z) - \eta(X)g(\overline{\xi}, X) = \eta(\overline{\xi})\overline{g}(X, Z) - \overline{\eta}(X)\eta(Z). \tag{39}$$

Substracting (39) from (38) we obtain

$$\eta(Z)g(\overline{\xi}, X) - \eta(X)g(\overline{\xi}, Z) = \overline{\eta}(X)\eta(Z) - \overline{\eta}(Z)\eta(X). \tag{40}$$

Putting $Z = \xi$ in (40) we obtain by using (7)

$$-g(\overline{\xi}, X) - g(\overline{\xi}, \xi)\eta(X) = -\overline{\eta}(X) - \overline{\eta}(\xi)\eta(X). \tag{41}$$

Also from (36) we have

$$\overline{S}(X, Y) = S(X, Y)$$

and hence

$$\overline{S}(\xi, \overline{\xi}) = S(\xi, \xi)$$

This gives by virtue of (9) that

$$\overline{\eta}(\xi) = \eta(\overline{\xi}) \tag{42}$$

Using (42) in (41) and since $\eta(\overline{\xi}) = g(\overline{\xi}, \xi)$ we get

$$\overline{\eta}(X) = g(\overline{\xi}, X) \tag{43}$$

By virtue of (43) we get from (39) that

$$[g(X, Z) - \overline{g}(X, Z)]\eta(\overline{\xi}).$$

This implies

$$g(X,Z) = \overline{g}(X,Z).$$

Hence we can state the following:

Theorem 6.1. *In a Lorentzian β-Kenmotsu manifold the transformation μ which leaves the curvature tensor invariant and $\eta(\overline{\xi}) \neq 0$.*

Definition 6.1. *A vector field V on a contact manifold with contact form η is said to be an infinitesimal contact transformation* [13] *if V satisfies*

$$(\pounds_V \eta)X = \sigma\eta(X) \tag{44}$$

for a scalar function σ, where $\pounds_v$ denotes the Lie differentiation with respect to V. Especially , if σ vanishes identically, then it is called an infinitesimal strict contact transformation [13]

Let us now suppose that in a Lorentzian β-Kenmotsu manifold, the infinitesimal contact transformation leaves Ricci tensor invariant. Then we have

$$(\pounds_V S)(X,Y) = 0$$

which gives

$$(\pounds_V S)(X,\xi) = 0 \tag{45}$$

Now,

$$(\pounds_V S)(X,\xi) = \pounds_V S(X,\xi) - S(\pounds_V X,\xi) - S(X,\pounds_V \xi). \tag{46}$$

By virtue of (9) and (45) we get from (46) that

$$-(n-1)\beta^2(\pounds_V \eta)(X) - S(X,\pounds_V \xi) = 0. \tag{47}$$

Using (44) in (47) we obtain

$$S(X,\pounds_V \xi) = -(n-1)\beta^2\sigma\eta(X). \tag{48}$$

Putting $X = \xi$ in (48) and then using (9), we get

$$\eta(\pounds_V \xi) = -\sigma. \tag{49}$$

Again putting $X = \xi$ in (44), we have

$$(\pounds_V \eta)(\xi) = -\sigma.$$

that is,

$$\pounds_V(\eta(\xi)) - \eta(\pounds_V \xi) = \sigma. \tag{50}$$

By virtue of (49) and (50) we get]

$$\sigma = 0.$$

Thus we can state the following:

Theorem 6.2. *In a Lorentzian β-Kenmotsu manifold, the infinitesimal contact transformation which leaves the Ricci tensor invariant is an infinitesimal strict contact transformation.*

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