Annals of the University of Craiova, Math. Comp. Sci. Ser.
Volume 35, 2008, Pages 21–31
ISSN: 1223-6934

ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR SOME NONLOCAL DIFFUSION PROBLEMS

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Abstract. In this paper, we address the following initial-value problem

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy - a(x)f(u(x,t)) \text{ in } \overline{\Omega}\times(0,\infty),$$
$$u(x,0) = u_0(x) > 0 \text{ in } \overline{\Omega},$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, J: $\mathbb{R}^N \longrightarrow \mathbb{R}$ is a kernel which is nonnegative, symmetric, bounded and $\int_{\mathbb{R}^N} J(z)dz = 1$, $f : [0,\infty) \longrightarrow [0,\infty)$ is a C^1 convex, increasing function, $\int^\infty \frac{ds}{f(s)} < \infty$, $f(0) = 0$, $f'(0) = 0$, the initial data $u_0 \in C^0(\overline{\Omega})$, $u_0(x) > 0$ for $x \in \overline{\Omega}$, and the potential $a \in C^0(\overline{\Omega})$, $a(x) > 0$ for $x \in \overline{\Omega}$. We reveal that the solution of the above problem exists globally and tends to zero uniformly in $x \in \overline{\Omega}$ as t approaches infinity. The description of its asymptotic behavior is also given under some conditions.

2000 Mathematics Subject Classification. 35B40; 45A07; 35G10.
Key words and phrases. Nonlocal diffusion, asymptotic behavior.

1. Introduction

Consider the following initial-value problem

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy - a(x)f(u(x,t)) \text{ in } \overline{\Omega}\times(0,\infty), \quad (1)$$
$$u(x,0) = u_0(x) > 0 \text{ in } \overline{\Omega}, \quad (2)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $J : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a kernel which is nonnegative, symmetric, bounded and $\int_{\mathbb{R}^N} J(z)dz = 1$, $f : [0,\infty) \longrightarrow [0,\infty)$ is a C^1 convex, increasing function, $\int^\infty \frac{ds}{f(s)} < \infty$, $f(0) = 0$, $f'(0) = 0$, the initial data $u_0 \in C^0(\overline{\Omega})$, $u_0(x) > 0$ for $x \in \overline{\Omega}$, and the potential $a \in C^0(\overline{\Omega})$, $a(x) > 0$ for $x \in \overline{\Omega}$. Recently nonlocal diffusion problems have been the subject of investigations of many authors (see, [1], [2], [4]-[7], [13]-[18], [20], [21], [23], [26], [27], [29], [30], and the references cited therein). Nonlocal evolution equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy,$$

and variations of it, have been used by many authors to model diffusion processes (see, [4]-[6], [13], [20], [21]). The solution $u(x,t)$ can be interpreted as the density of a single population at the point x, at time t, and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then, the convolution $(J*u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving at position x from all other places, and $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(x,t)dy$ is the rate at which they

Received: 01 October 2008.

are leaving location x to travel to any other site (see, [20]). Let us notice that the absorption term $-a(x)f(u(x,t))$ can be rewritten as follows

$$-a(x)f(u(x,t)) = -\int_{\mathbb{R}^N} J(x-y)a(x)f(u(x,t))dy.$$

In view of this equality, the absorption term $-a(x)f(u(x,t))$ can be interpreted as a force that increases the rate at which individuals are leaving location x to travel to all other sites. It is the reason why we shall see later that, because of the absorption term, the density of the population $u(x,t)$ will tend to zero uniformly in $x \in \overline{\Omega}$ as t approaches infinity. The equation $u_t = J*u-u$ is called a nonlocal diffusion equation because the diffusion of the density u at the point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $J*u$. On the other hand, for our equation defined in (1) , the integration is taken over Ω. As we have mentioned, the integral $\int_\Omega J(x-y)(u(y,t)-u(x,t))dy$ considers the individuals arriving or leaving position x from other places. Since, we have imposed that the diffusion takes place only in Ω, no individuals may enter or leave the domain. This is the reason why in the title of the paper, we have added Neumann boundary conditions.

In this paper, we are interested in the global existence and asymptotic behavior of solutions of the problem $(1)-(2)$. For local diffusion problems, the asymptotic behavior of solutions has been the subject of investigations of several authors (see, [3], [9]-[12], [24], [25] and the references cited therein). Recently, in [29], Pazoto and Rossi considered the problem $(1)-(2)$ in the case where $\Omega = \mathbb{R}^N$, $a(x) = 1$, and $f(u) = u^p$ with $p > 0$. They showed that, if $p > 1$, then the solution u of $(1)-(2)$ exists globally and tends to zero uniformly in $x \in \mathbb{R}^N$ as t approaches infinity. They also described the asymptotic behavior of the solution u. Analogous results have been obtained by Nabongo and Boni in [26] for the problem described in $(1)-(2)$ in the case where $f(s) = s^p$ with $p > 1$. Our purpose in this paper is to extend the above results considering the problem described in $(1)-(2)$ where the nonlinearity is more general. The remainder of the paper is organized in the following manner. In the next section, we prove the local existence and uniqueness of solutions, and prove that the solution u of $(1)-(2)$ exists globally. In the last section, we show that the solution u of $(1)-(2)$ tends to zero as t approaches infinity uniformly in $x \in \overline{\Omega}$. A complete description of its asymptotic behavior is also given under some conditions.

2. Local existence

In this section, we shall establish the existence and uniqueness of solutions of $(1)-(2)$ in $\Omega \times (0,T)$ for small T.

Let t_0 be fixed and define the function space $Y_{t_0} = \{u;\ u \in C([0,t_0],C(\overline{\Omega}))\}$ equipped with the norm defined by $\|u\|_{Y_{t_0}} = \max_{0\le t\le t_0}\|u\|_\infty$ for $u \in Y_{t_0}$. It is easy to see that Y_{t_0} is a Banach space. Introduce the set $X_{t_0} = \{u;\ u \in Y_{t_0}, \|u\|_{Y_{t_0}} \le b_0\}$, where $b_0 = 2\|u_0\|_\infty + 1$. We observe that X_{t_0} is a nonempty bounded closed convex subset of Y_{t_0}. Define the map R as follows

$$R : X_{t_0} \longrightarrow X_{t_0},$$

$$R(v)(x,t) = u_0(x) + \int_0^t \int_\Omega J(x-y)(v(y,s)-v(x,s))dyds - a(x)\int_0^t f(v(x,s))ds.$$

Theorem 2.1. *Assume that $u_0 \in Y_{t_0}$. Then R maps X_{t_0} into X_{t_0} and R is strictly contractive if t_0 is appropriately small relative to $\|u_0\|_\infty$.*

Proof. Using the fact that $\int_\Omega J(x-y)dy \leq \int_{\mathbb{R}^N} J(x-y)dy = 1$, a straightforward computation reveals that

$$|R(v)(x,t) - u_0(x)| \leq 2\|v\|_{Y_{t_0}} t + \|a\|_\infty f(\|v\|_{Y_{t_0}})t,$$

which implies that $\|R(v)\|_{Y_{t_0}} \leq \|u_0\|_\infty + 2b_0 t_0 + \|a\|_\infty f(b_0)t_0$. If

$$t_0 \leq \frac{b_0 - \|u_0\|_\infty}{2b_0 + \|a\|_\infty f(b_0)}, \tag{3}$$

then

$$\|R(v)\|_{Y_{t_0}} \leq b_0.$$

Therefore if (3) holds, then R maps X_{t_0} into X_{t_0}. Now, we are going to prove that the map R is strictly contractive. Let $t_0 > 0$, and let $v, z \in X_{t_0}$. Setting $\alpha = v - z$, we discover that

$$|(R(v) - R(z))(x,t)| \leq |\int_0^t \int_\Omega J(x-y)(\alpha(y,s) - \alpha(x,s))dyds|$$

$$+|a(x)| \int_0^t (f(v(x,s)) - f(z(x,s)))ds|.$$

Use Taylor's expansion to obtain

$$|(R(v) - R(z))(x,t)| \leq 2\|\alpha\|_{Y_{t_0}} t + t\|a\|_\infty \|v - z\|_{Y_{t_0}} f'(\|\beta\|_{Y_{t_0}}),$$

where β is an intermediate value between v and z. We deduce that

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq 2\|\alpha\|_{Y_{t_0}} t_0 + t_0\|a\|_\infty \|v - z\|_{Y_{t_0}} f'(\|\beta\|_{Y_{t_0}}),$$

which implies that

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq (2t_0 + t_0\|a\|_\infty f'(b_0))\|v - z\|_{Y_{t_0}}.$$

If

$$t_0 \leq \frac{1}{4 + 2\|a\|_\infty f'(b_0)},$$

then

$$\|R(v) - R(z)\|_{Y_{t_0}} \leq \frac{1}{2}\|v - z\|_{Y_{t_0}}.$$

Hence, we see that $R(v)$ is a strict contraction in Y_{t_0} and the proof is complete. □

It follows from the contraction mapping principle that for appropriately chosen $t_0 \in (0,1)$, R has a unique fixed point $u(x,t) \in Y_{t_0}$ which is a solution of $(1) - (2)$. In order to prove that the solution is global, we need to show that

$$\|u(\cdot,t)\|_\infty \leq \|u_0\|_\infty \quad \text{for} \quad t > 0.$$

To demonstrate this estimate, we proceed in the following manner. Multiply both sides of (3) by $(u(x,t) - \|u_0\|_\infty)_+$ and integrate over Ω to obtain

$$\frac{d}{dt}\int_\Omega \frac{(u(x,t) - \|u_0\|_\infty)_+^2}{2} dx$$

$$= \int_\Omega \int_\Omega J(x-y)(u(y,t) - u(x,t))(u(x,t) - \|u_0\|_\infty)_+ dxdy$$

$$- \int_\Omega a(x) f(u(x,t))(u(x,t) - \|u_0\|_\infty)_+ dx,$$

where $(x)_+$ denotes $\max(x,0)$. Due to the fact that the kernel J is symmetric, we note that

$$\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))\psi(x)dxdy$$
$$=-\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(y)-\varphi(x))(\psi(y)-\psi(x))dxdy.$$

It is also easy to see that $(A-B)(A_+-B_+)\geq(A_+-B_+)^2$. Use the above relations to arrive at

$$\frac{d}{dt}\int_\Omega\frac{(u(x,t)-\|u_0\|_\infty)_+^2}{2}dx$$
$$\leq-\frac{1}{2}\int_\Omega\int_\Omega J(x-y)|(u(y,t)-\|u_0\|_\infty)_+-(u(x,t)-\|u_0\|_\infty)_+|^2dxdy$$
$$-\int_\Omega a(x)f(u(x,t))(u(x,t)-\|u_0\|_\infty)_+dx\leq 0.$$

We infer that $\int_\Omega\frac{(u(x,t)-\|u_0\|_\infty)_+^2}{2}dx=0$, which implies that

$$\|u(\cdot,t)\|_\infty\leq\|u_0\|_\infty\quad\text{for}\quad t>0,$$

and our estimate is proved.

3. Asymptotic behavior of solutions

In this section, we show that the solution u of $(1)-(2)$ tends to zero as t approaches infinity uniformly in $x\in\overline{\Omega}$. We also describe its asymptotic behavior as $t\to\infty$. Before starting, let us prove the following lemma which is a version of the maximum principle for nonlocal problems.

Lemma 3.1. *Let $b\in C^0(\overline{\Omega}\times[0,\infty))$, and let $u\in C^{0,1}(\overline{\Omega}\times[0,\infty))$ satisfying the following inequalities*

$$u_t-\int_\Omega J(x-y)(u(y,t)-u(x,t))dy+b(x,t)u(x,t)\geq 0\ \ in\ \ \overline{\Omega}\times(0,\infty),$$
$$u(x,0)\geq 0\quad in\quad\overline{\Omega}.$$

Then, we have $u(x,t)\geq 0$ in $\overline{\Omega}\times(0,\infty)$.

Proof. Let T_0 be any quantity satisfying $T_0\in(0,\infty)$, and let λ be such that $b(x,t)-\lambda>0$ in $\overline{\Omega}\times[0,T_0]$. Introduce the function $z(x,t)=e^{\lambda t}u(x,t)$, and suppose that $m=\min_{x\in\overline{\Omega},t\in[0,T_0]}z(x,t)$. Since the function $z(x,t)$ is continuous in the compact $\overline{\Omega}\times[0,T_0]$, then it attains its minimum at a point in $\overline{\Omega}\times[0,T_0]$, that is, there exists $(x_0,t_0)\in\overline{\Omega}\times[0,T_0]$ such that $m=z(x_0,t_0)$. We get $z(x_0,t_0)\leq z(x_0,t)$ for $t\leq t_0$ and $z(x_0,t_0)\leq z(y,t_0)$ for y$\in\Omega$, which implies that

$$z_t(x_0,t_0)\leq 0,\tag{4}$$

and

$$\int_\Omega J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy\geq 0.\tag{5}$$

Using the first inequality of the lemma, it is not hard to see that

$$z_t(x_0,t_0)-\int_\Omega J(x_0-y)(z(y,t_0)-z(x_0,t_0))dy+(b(x_0,t_0)-\lambda)z(x_0,t_0)\geq 0.\tag{6}$$

It follows from (4) – (6) that $(b(x_0, t_0) - \lambda)z(x_0, t_0) \geq 0$, which implies that $z(x_0, t_0) \geq 0$ because $b(x_0, t_0) - \lambda > 0$. We deduce that $u(x, t) \geq 0$ in $\overline{\Omega} \times [0, T_0]$, which leads us to the result. □

Another version of the maximum principle for nonlocal problems is the following comparison lemma.

Lemma 3.2. *Let $u, v \in C^{0,1}(\overline{\Omega} \times (0, \infty))$ satisfying the following inequalities*

$$u_t(x,t) - \int_\Omega J(x-y)(u(y,t) - u(x,t))dy + a(x)f(u(x,t)) >$$

$$v_t(x,t) - \int_\Omega J(x-y)(v(y,t) - v(x,t))dy + a(x)f(v(x,t)) \quad in \quad \overline{\Omega} \times (0, \infty),$$

$$u(x,0) > v(x,0) \ \ in \ \overline{\Omega}.$$

Then, we have $u(x,t) > v(x,t)$ in $\overline{\Omega} \times (0, \infty)$.

Proof. Let $w = u - v$ in $\overline{\Omega} \times (0, \infty)$, and let t_0 be the first $t > 0$ such that $w(x, t) > 0$ in $\overline{\Omega} \times [0, t_0)$, but $w(x_0, t_0) = 0$ for a certain $x_0 \in \overline{\Omega}$. Since $w(y, t_0) \geq w(x_0, t_0)$ for $y \in \Omega$, $w(\cdot, t_0) \not\equiv w(x_0, t_0)$ in Ω, we deduce that

$$\int_\Omega J(x_0 - y)(w(y, t_0) - w(x_0, t_0))dy > 0.$$

We observe that $w_t(x_0, t_0) \leq 0$ because $w(x_0, t) \geq w(x_0, t_0)$ for $t \in [0, t_0)$. It follows that

$$w_t(x_0, t_0) - \int_\Omega J(x_0 - y)(w(y, t_0) - w(x_0, t_0))dy$$
$$+ a(x_0)(f(u(x_0, t_0)) - f(v(x_0, t_0))) \leq 0.$$

But, this contradicts the first strict inequality of the lemma, and the proof is complete. □

Remark 3.1. *Since the initial data u_0 is positive in $\overline{\Omega}$, if we modify slightly the proof of Lemma 3.2, we easily see that the solution u is also positive in $\overline{\Omega} \times (0, \infty)$.*

Let $F(s) = \int_s^1 \frac{d\sigma}{f(\sigma)}$, and let $H(s)$ be the inverse of $F(s)$. In this notation, the initial-value problem

$$\beta'(t) = -\lambda f(\beta(t)), \ \ t > 0, \ \ \beta(0) = 1 \quad (\lambda > 0), \tag{7}$$

has the unique solution $\beta(t) = H(\lambda t)$. It follows from $f(0) = 0$, $f'(0) = 0$ that $0 < f(t) < t$ for $0 < t < \delta$ $(\delta > 0)$ and hence

$$F(0) = \infty, \quad F(1) = 0 \ \text{ and } \ H(0) = 1, \quad H(\infty) = 0,$$

which implies that $\beta(\infty) = 0$. The function $\beta(t)$ will be used later in the construction of supersolutions and subsolutions of (1) – (2) to obtain the asymptotic behavior of solutions. In what follows, we suppose that for given $\lambda > 0$, δ satisfies the following equality

$$\delta = -\lambda + \frac{1}{|\Omega|} \int_\Omega a(x)dx. \tag{8}$$

Consider the following problem

$$-\lambda - \int_\Omega J(x-y)(\psi(y) - \psi(x))dy + a(x) = \delta \ \text{ in } \ \Omega. \tag{9}$$

Since J is symmetric, we observe that

$$\int_\Omega \int_\Omega J(x-y)(\psi(y)-\psi(x))dydx = 0. \tag{10}$$

We have the following result.

Lemma 3.3.
(i) ψ *solution of* (9) *implies that* ψ *+const is also solution of* (9).
(ii) There exists a solution of (9) *if and only if* δ *satisfies the equality in* (8).

Proof. The proof of (i) is straightforward. In order to prove (ii), we proceed in the following manner. Introduce the operator L defined as follows

$$L\psi(x) = \int_\Omega J(x-y)\psi(y)dy \quad \text{in} \quad \overline{\Omega}.$$

Hence, the problem (9) can be rewritten in the following manner

$$L\psi(x) = b(x) \quad \text{in} \quad \overline{\Omega}, \tag{11}$$

where

$$b(x) = -\delta - \lambda + \psi(x)\int_\Omega J(x-y)dy + a(x).$$

It is not hard to see that the operator L is self-adjoint. It follows by virtue of Fredholm theory that a solution ψ of (11) exists if and only if

$$\int_\Omega L\psi(x)dx = \int_\Omega b(x)dx. \tag{12}$$

In view of (10), we note that (12) is satisfied if and only if δ obeys the equality in (8). This finishes the proof. □

The following result reveals that the solution u of the problem $(1)-(2)$ tends to zero uniformly in $x \in \overline{\Omega}$ as $t \to \infty$.

Theorem 3.1. *Let* u *be the solution of* $(1)-(2)$. *Then, we have*

$$\lim_{t\to\infty} u(x,t) = 0 \;\; \textit{uniformly in} \quad x \in \overline{\Omega}.$$

Proof. From Remark 3.1, we know that the solution u is positive in $\overline{\Omega} \times (0,\infty)$. Introduce the function $w(x,t)$ defined by

$$w(x,t) = \beta(t) + \psi(x)f(\beta(t)) \;\; \text{in} \;\; \overline{\Omega} \times [1,\infty),$$

where $\beta(t)$ and $\psi(x) > 0$ are solutions of (7) and (9), respectively for $0 < \lambda \le \frac{\int_\Omega a(x)dx}{2|\Omega|}$, which implies that $\delta > 0$. A straightforward computation reveals that

$$w_t - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + a(x)f(w)$$

$$= f(\beta(t))\left(-\lambda - \lambda\psi(x)f'(\beta(t)) - \int_\Omega J(x-y)(\psi(y)-\psi(x))dy\right)$$

$$+a(x)f\left(\beta(t) + \psi(x)f(\beta(t))\right) \quad \text{in} \quad \overline{\Omega} \times [1,\infty).$$

Applying the mean value theorem, we get

$$f(\beta(t) + \psi(x)f(\beta(t))) = f(\beta(t)) + \psi(x)f(\beta(t))f'(M(x,t)),$$

where $M(x,t)$ is an intermediate value between $\beta(t)$ and $\beta(t)+\psi(x)f(\beta(t))$. It follows from (9) that

$$w_t - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + a(x)f(w)$$
$$= f(\beta(t))(\delta - \lambda\psi(x)f'(\beta(t)) - a(x)\psi(x)f'(M(x,t))) \quad \text{in} \quad \overline{\Omega}\times[1,\infty).$$

Due to the fact that $f(0)=0$ and $f'(0)=0$, we deduce that there exists $T\geq 1$ such that

$$w_t - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + a(x)f(w) > 0 \;\text{ in }\; \Omega\times(T,\infty).$$

Let $k\geq 1$ be large enough that

$$u(x,0) < kw(x,T) \;\text{ in }\; \overline{\Omega}.$$

We observe that

$$(kw)_t - \int_\Omega J(x-y)(kw(y,t)-kw(x,t))dy + a(x)f(kw) > 0 \;\text{ in }\; \Omega\times(T,\infty).$$

It follows from Lemma 3.2 that

$$u(x,t) < kw(x,T+t) \;\text{ in }\; \Omega\times(0,\infty).$$

Use the fact that $u(x,t)>0$ in $\overline{\Omega}\times(0,\infty)$ and

$$\lim_{t\to\infty} w(x,t) = 0 \;\text{ uniformly } \quad \text{in } \; x\in\overline{\Omega},$$

to complete the rest of the proof. □

Up to now, we have seen that the solution u of $(1)-(2)$ exists globally and tends to zero as t approaches infinity in $x\in\overline{\Omega}$. In the sequel, we shall describe its asymptotic behavior under some conditions.

Theorem 3.2. *Assume that there exists a constant $C_2>0$ such that*

$$\lim_{s\to\infty}\frac{sf(H(s))}{H(s)} \leq C_2,$$

and let u be the solution of $(1)-(2)$. Then, we have

$$u(x,t) = H(C_0 t)(1+o(1)) \quad \textit{as} \quad t\to\infty,$$

where $C_0 = \frac{\int_\Omega a(x)dx}{|\Omega|}$.

The proof of the above theorem is based on the following lemmas.

Lemma 3.4. *Let u be the solution of $(1)-(2)$. Then, for any $\varepsilon>0$ small enough, there exist positive times τ and T_1 such that*

$$u(x,t+\tau) \leq \beta_1(t+T_1) + \psi_1(x)f(\beta_1((t+T_1))) \quad \textit{in} \quad \overline{\Omega}\times(0,\infty),$$

where $\beta_1(t)$ and $\psi_1(x)>0$ are solutions of (7) and (9), respectively for $\lambda = C_0-\varepsilon/2$.

Proof. Introduce the function $w(x,t)$ defined as follows

$$w(x,t) = \beta_1(t) + \psi_1(x)f(\beta_1(t)) \quad \text{in} \quad \overline{\Omega}\times[1,\infty).$$

Since $C_0 = \frac{\int_\Omega a(x)dx}{|\Omega|}$, then $\delta = \varepsilon/2 > 0$. As in the proof of Theorem 3.1, we get

$$w_t - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + a(x)f(w)$$
$$= f(\beta_1(t))(\delta - \lambda\psi_1(x)f'(\beta_1(t)) - a(x)\psi_1(x)f'(M_1(x,t))) \quad \text{in} \quad \overline{\Omega}\times[1,\infty),$$

where $M_1(x,t)$ is an intermediate value between $\beta_1(t)$ and $\beta_1(t)+\psi_1(x)f(\beta_1(t))$. It follows that there exists a time $T_1 \geq 1$ such that

$$w_t - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + a(x)f(w) > 0 \ \text{ in } \ \overline{\Omega}\times(T_1,\infty).$$

Since u tends to zero as t approaches infinity uniformly in $x \in \overline{\Omega}$, we deduce that there exists a time $\tau \geq T_1$ such that

$$u(x,\tau) < w(x,T_1) \ \text{ in } \ \overline{\Omega}.$$

Setting $z(x,t) = u(x,t+\tau-T_1)$, we observe that

$$z_t - \int_\Omega J(x-y)(z(y,t)-z(x,t))dy + a(x)f(z) = 0 \ \text{ in } \ \overline{\Omega}\times(T_1,\infty),$$

$$z(x,T_1) = u(x,\tau) < w(x,T_1) \ \text{ in } \ \overline{\Omega}.$$

It follows from Lemma 3.2 that

$$z(x,t) < w(x,t) \ \text{ in } \ \overline{\Omega}\times(T_1,\infty),$$

which implies that

$$u(x,t+\tau-T_1) < w(x,t) \ \text{ in } \ \overline{\Omega}\times(T_1,\infty).$$

Consequently

$$u(x,t+\tau) \leq w(x,t+T_1) \ \text{ in } \ \overline{\Omega}\times(0,\infty),$$

and the proof is complete. □

Lemma 3.5. *Let u be the solution of $(1)-(2)$. Then, there exists a time $T_2 \geq 1$ such that*

$$u(x,t+1) \geq \beta_2(t+T_2) + \psi_2(x)f(\beta_2(t+T_2)) \ \text{ in } \ \overline{\Omega}\times(0,\infty),$$

where $\beta_2(t)$ and $\psi_2(x) > 0$ are solutions of (7) and (9), respectively for $\lambda = C_0 + \varepsilon/2$.

Proof. From Remark 3.1, we know that u is positive in $\overline{\Omega}\times(0,\infty)$. Introduce the function $w(x,t)$ defined as follows

$$w(x,t) = \beta_2(t) + \psi_2(x)f(\beta_2(t)) \quad \text{in} \quad \overline{\Omega}\times[1,\infty).$$

Since $C_0 = \frac{\int_\Omega a(x)dx}{|\Omega|}$, then $\delta = -\varepsilon/2 < 0$. As in the proof of Theorem 3.1, we get

$$w_t - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + a(x)f(w)$$
$$= f(\beta_2(t))(\delta - \lambda\psi_2(x)f'(\beta_2(t)) - a(x)\psi_2(x)f'(M_2(x,t))) \quad \text{in} \quad \overline{\Omega}\times[1,\infty).$$

where $M_2(x,t)$ is an intermediate value between $\beta_2(t)$ and $\beta_2(t)+\psi_2(x)f(\beta_2(t))$. It follows that there exists a time $T_1 \geq 1$ such that

$$w_t - \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + a(x)f(w) < 0 \ \text{ in } \ \overline{\Omega}\times(T_1,\infty).$$

Since w tends to zero as t approaches infinity uniformly in $x \in \overline{\Omega}$, there exists a time $T_2 \geq T_1$ such that

$$w(x,T_2) < u(x,1) \ \text{ in } \ \overline{\Omega}.$$

Set $z(x,t) = w(x,t+T_2-1)$. We observe that

$$z_t - \int_\Omega J(x-y)(z(y,t)-z(x,t))dy + a(x)f(z) < 0 \ \text{ in } \ \overline{\Omega}\times(1,\infty),$$

$$z(x,1) = w(x,T_2) < u(x,1) \ \text{ in } \ \overline{\Omega}.$$

It follows from Lemma 3.2 that

$$z(x,t) < u(x,t) \text{ in } \overline{\Omega} \times (1,\infty),$$

which implies that

$$w(x,t+T_2-1) < u(x,t) \text{ in } \overline{\Omega} \times (1,\infty).$$

Consequently, we get

$$u(x,t+1) \geq w(x,t+T_2) \text{ in } \overline{\Omega} \times (0,\infty).$$

Thus, we have

$$u(x,t+1) \geq \beta_2(t+T_2) + \psi_2(x) f(\beta_2(t+T_2)) \text{ in } \overline{\Omega} \times (0,\infty),$$

and the proof is complete. □

Lemma 3.6. *Let* $\beta(t,\lambda)$ *be a solution of* (7). *Then*
(i) for $\gamma > 0$,

$$\lim_{t\to 0} \frac{\beta(t+\gamma,\lambda)}{\beta(t,\lambda)} = 1.$$

(ii) if $\lim_{s\to\infty} \frac{sf(H(s))}{H(s)} \leq C_2$ *and* $\alpha > 0$, *then*

$$1 \geq \lim_{t\to\infty} \sup \frac{\beta(t,\lambda+\alpha)}{\beta(t,\lambda)} \geq \lim_{t\to\infty} \inf \frac{\beta(t,\lambda+\alpha)}{\beta(t,\lambda)} \geq 1 - \frac{C_2\alpha}{\lambda}, \tag{13}$$

$$1 \leq \lim_{t\to\infty} \inf \frac{\beta(t,\lambda-\alpha)}{\beta(t,\lambda)} \leq \lim_{t\to\infty} \sup \frac{\beta(t,\lambda-\alpha)}{\beta(t,\lambda)} \leq 1 + \frac{2C_2\alpha}{\lambda}, \tag{14}$$

for α *small enough.*

Proof. (i) Since $\beta_\lambda(t) = \beta(t,\lambda)$ is decreasing and convex, then we get

$$\beta(t,\lambda) - \gamma\lambda f(\beta(t,\lambda)) \leq \beta(t+\lambda,\lambda) \leq \beta(t,\lambda),$$

which implies $\lim_{t\to\infty} \frac{\beta(\gamma+t,\lambda)}{\beta(t,\lambda)} = 1$ because $\lim_{s\to 0} \frac{f(s)}{s} = 0$.

(ii) We have

$$1 \geq \frac{\beta(t,\lambda+\alpha)}{\beta(t,\lambda)} = \frac{H(\lambda t + \alpha t)}{H(\lambda t)} \geq \frac{H(\lambda t) - \alpha t f(H(\lambda t))}{H(\lambda t)}.$$

Since $\lim_{s\to\infty} \frac{sf(H(s))}{H(s)} \leq C_2$, we obtain (13). We also get by means of (13) the following inequalities:

$$1 \leq \lim_{t\to\infty} \inf \frac{\beta(t,\lambda-\alpha)}{\beta(t,\lambda)} \leq \lim_{t\to\infty} \sup \frac{\beta(t,\lambda-\alpha)}{\beta(t,\lambda)} \leq \frac{1}{1-\frac{C_2\alpha}{\lambda-\alpha}} \leq 1 + \frac{2C_2\alpha}{\lambda}$$

which yields (13). This ends the proof. □

Now, we are in a position to prove the main result of this paper.

Proof of Theorem 3.2. It follows from Lemmas 3.4, 3.5 and 3.6 that, for any $\varepsilon > 0$ small enough, we have

$$1 - k_1\varepsilon \leq \lim_{t\to\infty} \inf \frac{u(x,t)}{\beta(t)} \leq \lim_{t\to\infty} \sup \frac{u(x,t)}{\beta(t)} \leq 1 + k_2\varepsilon,$$

where k_1 and k_2 are two positive constants. Consequently

$$u(x,t) = \beta(t)(1+o(1)) \text{ as } t \to \infty,$$

which gives the desired result.□

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