A completeness theorem for three-valued temporal predicate logic

CARMEN CHIRIȚĂ

ABSTRACT. The main result in this paper is a completeness theorem for the three-valued temporal predicate calculus, obtained by providing a semantical interpretation for this logic and by using the Henkin models to define a canonical model used to prove the completeness.

2000 Mathematics Subject Classification. 03G20; 03B44; 03B52.

Key words and phrases. Three-valued temporal predicate logic, Henkin models.

1. Introduction

The classical temporal logic is obtained from bivalent logic by adding the tense operators $G$ ("it is always going to be the case that") and $H$ ("it has always been the case that"). By starting from other logical systems and adding appropriate tense operators we can produce new temporal logics. In [5] we have studied a complete three-valued temporal propositional calculus based on the Łukasiewicz three-valued logic.

The goal of this paper is to construct a temporal logical system for the predicate calculus based on the three-valued Łukasiewicz logic. This logical system is obtained from the Łukasiewicz logic described in [5] by adding the quantifiers. The main result is a completeness theorem for this logical system, whose proof uses a Henkin-style method (see [8]).

The paper is organized as follows:

In Section 2 we recall from [1] and [11] some basic definitions and results on the three-valued Łukasiewicz logic: the syntax, the semantic, the completeness theorem and a list of provable sentences.

Section 3 contains a short presentation of the three-valued temporal propositional logic $TL_3$ and the completeness theorem proved in [5].

In Section 4 we define the language and the logical structure of the three-valued temporal predicate logic $PTL_3$. We study the consistent sets of formulas and we prove that any consistent theory of $PTL_3$ can be embedded in a Henkin theory.

Section 5 deals with the semantic of $PTL_3$. We define the structures of $PTL_3$ and we construct the canonical model associated with a maximal consistent Henkin theory. The satisfiability of formulas in canonical model is characterized in terms of maximal consistent Henkin theories.

Section 6 contains the proof of completeness theorem for $PTL_3$. This proof is based on the properties of canonical model (cf. Theorem 5.1).

Received: 10 November 2008.
2. Three-valued Lukasiewicz propositional logic

The first system of three-valued logic was constructed by Lukasiewicz in 1920 in connection with the investigation of modalities (see [14]). His main idea was to consider a third truth-value \( \frac{1}{2} \) between 0 (false) and 1 (truth). The interpretation for the sentences of the three-valued logic is defined in \( L_3 = \{0, \frac{1}{2}, 1\} \). The algebraic structures for the three-valued Lukasiewicz logic were introduced by Gr.C. Moisil in [15] under the name of three-valued Lukasiewicz algebras (see also [16], [1]). Today these structures are known as Lukasiewicz-Moisil algebras (see [1]). We shall use the Wajsberg axiomatization of the three-valued Lukasiewicz logic ([1]). The sentences of the three-valued Lukasiewicz propositional calculus \( L_3 \) are obtained from a countable set \( V \) of propositional variables and the logical connectives \( \neg \) and \( \rightarrow \), according to the following rules:

(i) the propositional variables are sentences;
(ii) if \( p \) and \( q \) are sentences then \( \neg p \) and \( p \rightarrow q \) are sentences;
(iii) every sentence is obtained by applying a finite number of times the above rules (i) and (ii).

In what follows, we will denote the set of sentences of \( L_3 \) by \( E \).

We are going to use the following abbreviations:

\[
\begin{align*}
\varphi \lor \psi & := (\varphi \rightarrow \psi) \rightarrow \psi \\
\varphi \land \psi & := \neg (\neg \varphi \lor \neg \psi) \\
\varphi \rightarrow \psi & := (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi) \\
\varphi \oplus \psi & := \neg \varphi \rightarrow \psi \\
\varphi \odot \psi & := \neg (\neg \varphi \oplus \neg \psi) \\
\sim \varphi & := \varphi \rightarrow \neg \varphi
\end{align*}
\]

The axioms of three-valued Lukasiewicz propositional calculus are sentences of one of the following forms:

(A1) \( p \rightarrow (q \rightarrow p) \)
(A2) \( (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) \)
(A3) \( ((p \rightarrow \neg p) \rightarrow p) \rightarrow p \)
(A4) \( (\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p) \)

Three-valued Lukasiewicz propositional logic uses modus ponens (m.p) as rule of inference:

\[
\frac{p, p \rightarrow q}{q}
\]

A proof of a sentence \( p \) is a finite sequence \( p_1, \ldots, p_n = p \) of sentences such that for any \( i \leq n \) we have one of the following:

(a) \( p_i \) is an axiom;
(b) there exists \( j, k < i \) such that \( p_k \) is the sentence \( p_j \rightarrow p_i \).

A sentence \( p \) is provable (\( \vdash p \)) if there is at least one proof of it.

The following proposition collects the main provable sentences of \( L_3 \).

**Proposition 2.1.** ([1]) The following sentences are provable in the three-valued Lukasiewicz logic:

(t1) \( p \rightarrow (q \rightarrow p) \),
(t2) \( (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)) \),
(t3) \( p \rightarrow p \),
(t4) \( p \rightarrow q \leftrightarrow (\neg q \rightarrow \neg p) \),
(15) $p \leftrightarrow \neg
p$,
(16) $\neg p \rightarrow (p \rightarrow q)$,
(17) $(p \rightarrow (p \rightarrow (q \rightarrow r))) \rightarrow ((p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow (p \rightarrow r)))$,
(18) $\sim \sim p \rightarrow p$,
(19) $(p \rightarrow \sim p) \rightarrow \sim p$,
(20) $(p \wedge (q \wedge r)) \leftrightarrow ((p \wedge q) \wedge r)$,
(21) $(p \wedge q) \leftrightarrow (q \wedge p)$,
(22) $(p \wedge q) \rightarrow p$,
(23) $p \rightarrow (q \rightarrow (p \wedge q))$,
(24) $(p \rightarrow (q \rightarrow ((p \wedge q) \rightarrow r)))$,
(25) $(p \rightarrow (p \rightarrow q))$,
(26) $(p \rightarrow (p \rightarrow q))$.

3. Three-valued temporal propositional logic

In this section we present a three-valued temporal logic $L_3$ based on the three-valued Łukasiewicz propositional calculus [5]. Our axiomatization is inspired from the axioms of the three-valued Łukasiewicz logic in [1] and from the Ostermann system from [17] (see also Leuştean [13]) and introduces two temporal operators $G$ and $H$.

The symbols of the three-valued propositional temporal logic are:

(i) a countable set $AF$ of atomic sentences, denoted by $v_0, v_1, ...$,
(ii) the propositional connectives $\neg, \rightarrow$,
(iii) the temporal operators $G$ and $H$.

The set $E$ of sentences of $L_3$ is defined by the canonical induction.

We shall use the $\vee, \wedge, \leftrightarrow, \oplus, \odot$ and $\sim$ defined in the previous section.
We also define:

\[ Fp \ := \ \neg G \neg p \]

\[ Pp \ := \ \neg H \neg p \]

\( \mathcal{T} \mathcal{L}_3 \) has the following axioms:

(T1) the axioms of the three-valued Lukasiewicz logic (the axioms (A1)-(A4) in section 2)

(T2) \( G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq) \),

\( H(p \rightarrow q) \rightarrow (Hp \rightarrow Hq) \),

(T3) \( G(p \oplus p) \leftrightarrow (Gp \oplus Gp) \),

\( H(p \oplus p) \leftrightarrow (Hp \oplus Hp) \),

(T4) \( p \rightarrow Gp \),

\( p \rightarrow Hp \).

The notion of formal proof in the three-valued temporal logic is defined in terms of the above axioms and the following inference rules:

\[ p, p \rightarrow q \vdash q; \]

\[ p \vdash Gp; \]

\[ p \vdash Hp. \]

We will denote by \( \vdash_{\mathcal{T} \mathcal{L}_3} p \) the fact that \( p \) is provable in \( \mathcal{T} \mathcal{L}_3 \).

A frame is a pair \( \mathcal{F} = \langle W, R \rangle \), where \( W \) is a not-empty set and \( R \) is a binary relation on \( W \).

An evaluation of \( \mathcal{T} \mathcal{L}_3 \) in \( \mathcal{F} \) is a function \( V : E \times W \rightarrow L_3 = \{0, \frac{1}{2}, 1\} \) such that, for all \( p, q \in E \) and \( s \in W \), the following equalities hold:

(i) \( V(\neg p, s) = 1 - V(p, s) \),

(ii) \( V(p \rightarrow q, s) = \min\{1, 1 - V(p, s) + V(q, s)\} \),

(iii) \( V(Gp, s) = \min\{V(p, t) | sRt\} \), for all \( p, q \in E \), \( s \in W \)

\( V(Hp, s) = \min\{V(p, t) | tRs\} \), for all \( p, q \in E \), \( s \in W \).

A sentence \( p \in E \) is universally valid in \( \mathcal{T} \mathcal{L}_3 \) if for every frame \( \langle W, R \rangle \) and for any evaluation \( V : E \times W \rightarrow L_3 \) we have \( V(p, s) = 1 \), for all \( s \in W \).

We recall from [5] the following completeness result.

Theorem 3.1. (Completeness Theorem) For any sentence \( \varphi \) of \( \mathcal{T} \mathcal{L}_3 \),

\[ \vdash_{\mathcal{T} \mathcal{L}_3} \varphi \iff \vdash_{\mathcal{T} \mathcal{L}_3} \varphi \]

4. Syntax of three-valued temporal predicate logic

In this section we shall define the three-valued temporal predicate logic \( \mathcal{P} \mathcal{T} \mathcal{L}_3 \) by adding to \( \mathcal{T} \mathcal{L}_3 \) the universal quantifier \( \forall \). The logical structure of \( \mathcal{P} \mathcal{T} \mathcal{L}_3 \) is obtained by enriching the axiomatization of \( \mathcal{T} \mathcal{L}_3 \) with the new axioms (A6)-(A9) and the generalization rule of inference. We study the consistent sets of formulas and the Henkin theories of \( \mathcal{P} \mathcal{T} \mathcal{L}_3 \).

A lot of properties of consistent sets follows as in the case of classical temporal logic and we omit their proofs. We prove that any consistent theory of \( \mathcal{P} \mathcal{T} \mathcal{L}_3 \) can be embedded in a Henkin theory of an extended language \( \mathcal{P} \mathcal{T} \mathcal{L}_3 \) obtained by adding to \( \mathcal{P} \mathcal{T} \mathcal{L}_3 \) the new constants of \( C \).

The alphabet of \( \mathcal{P} \mathcal{T} \mathcal{L}_3 \) consists of the following primitive symbols:

- a countable set \( V \) of variable symbols, denoted by \( x, y, z, \ldots \)
- an arbitrary set of constant symbols.
• an arbitrary set of predicate symbols; each predicate symbol $P$ has associated a
natural number $n > 0$ (the order or arity of $P$).
• the propositional connectives $\neg, \rightarrow$.
• the temporal operators $G$ and $H$.
• the universal quantifier $\forall$.
• the parantheses $(, ), [ , ]$.

A term of $\mathcal{PTL}_3$ is a variable symbol or a constant symbol. An atomic formula of $\mathcal{PTL}_3$ has the form $\varphi(t_1, t_2, \ldots, t_n)$ where $\varphi$ is an $n$-ary $P$ symbol and $t_1, t_2, \ldots, t_n$ are terms.

We will inductively define the set $\text{Form}$ of formulas:
(i) the atomic formulas are formulas.
(ii) if $\varphi \in \text{Form}$ and $\psi \in \text{Form}$ then $\varphi \rightarrow \psi$ and $\neg \varphi \in \text{Form}$.
(iii) if $\varphi \in \text{Form}$ then $G\varphi \in \text{Form}, H\varphi \in \text{Form}$.
(iv) if $\varphi \in \text{Form}$ and $x$ is a variable symbol then $\forall x \varphi$ is a formula.

We also define:

$$F \varphi := \neg G \neg \varphi$$
$$P \varphi := \neg H \neg \varphi$$
$$\exists x \varphi := \neg \forall x \neg \varphi$$

The notion of subformula is defined by induction:
• $\varphi$ is a subformula of $\varphi$.
• any subformula of $\varphi$ is a subformula of $\neg \varphi$.
• any subformula of $\varphi$ or $\psi$ is a subformula of $\varphi \rightarrow \psi$.
• any subformula of $\varphi$ is a subformula of $\forall \varphi$.

An occurrence of a variable $x$ in a formula $\varphi$ is free if $x$ does not belong to any occurrence of a subformula of $\varphi$ having the form $\forall x \psi$. Otherwise, an occurrence of $x$ in $\varphi$ is bound.

We say that $x$ is free in $\varphi$ if any occurrence of $x$ is free in $\varphi$. A sentence is a formula with no free variables. We will write $\varphi(x_1, \ldots, x_n)$ if all the free variables of $\varphi$ are among $\{x_1, \ldots, x_n\}$. We’ll denote by $FV(\varphi)$ the set of free variables of $\varphi$.

A theory is a set of formulas.

The axioms of $\mathcal{PTL}_3$ are:
(A0) the axioms of the three-valued logic.
(A1) $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$
$H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$
(A2) $G\varphi \oplus G\psi \rightarrow G(\varphi \oplus \psi)$
$H\varphi \oplus H\psi \rightarrow H(\varphi \oplus \psi)$
(A3) $G(\varphi \oplus \varphi) \rightarrow G\varphi \oplus G\varphi$
$H(\varphi \oplus \varphi) \rightarrow H\varphi \oplus H\varphi$
(A4) $F\varphi \oplus F\varphi \rightarrow F(\varphi \oplus \varphi)$
$P\varphi \oplus P\varphi \rightarrow P(\varphi \oplus \varphi)$
(A5) $\neg \forall x \varphi \rightarrow (\forall x \neg \varphi)$,
$\varphi \rightarrow \neg \forall x \neg \varphi$
(A6) $\forall x \varphi(x) \rightarrow \varphi(t)$, where $t$ is a term
(A7) $\forall x(\varphi \rightarrow \psi(x)) \rightarrow (\varphi \rightarrow \forall x \psi(x))$, where $x$ is not free in $\varphi$
(A8) $\forall x(\varphi \oplus \varphi) \leftrightarrow \forall x \varphi \oplus \forall x \varphi$
(A9) $\forall x(\varphi \oplus \varphi) \leftrightarrow \forall x \varphi \oplus \forall x \varphi$

$\mathcal{PTL}_3$ has the following rules of inference:
The formal theorems of $\mathcal{PTL}_3$ are obtained from axioms by applying a finite number of times the rules of inference. We denote by $\vdash \varphi$ the fact that $\varphi$ is a formal theorem. The syntactic deduction is defined by:

$$\Gamma \vdash \varphi \iff \text{there exists } \gamma_1, \ldots, \gamma_n \in \Gamma \text{ with } \vdash \bigwedge_{i=1}^{n} \gamma_i \to \varphi$$

We say that a set $\Gamma$ of formulas is consistent if there is no formula $\varphi$ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$; otherwise we say that $\Gamma$ is inconsistent.

A consistent set $\Gamma$ is said to be maximal consistent if $\varphi \in \Gamma$ for any formula $\varphi$ such that $\Gamma \cup \{\varphi\}$ is consistent.

We are going to present some formal theorems and properties for three-valued temporal predicate calculus. The proofs are similar to the corresponding results for Łukasiewicz predicate logic[10].

**Proposition 4.1.** Let $\Sigma \subseteq \text{Form}$ and $p \in \text{Form}$.

(i) $\Sigma$ is inconsistent iff $\Sigma \vdash r$ for any formula $r$.
(ii) $\Sigma \cup \{p\}$ is inconsistent iff $\Sigma \vdash \neg p$.
(iii) $\Sigma \cup \{\neg p\}$ is inconsistent iff $\Sigma \vdash p$.
(iv) $\Sigma$ is consistent iff every finite subset of $\Sigma$ is consistent.
(v) If $\Sigma$ is consistent, then for any formula $p$, at least one of $\Sigma \cup \{p\}$ and $\Sigma \cup \{\neg p\}$ is consistent.

**Proposition 4.2.** Let $\Sigma$ be a maximal consistent set and $p, q \in \text{Form}$.

(i) $\Sigma \vdash p$ implies $p \in \Sigma$.
(ii) If $\Sigma \subseteq \Gamma$ and $\Gamma$ is consistent, then $\Sigma = \Gamma$.
(iii) $p \in \Sigma$ iff $\neg p \notin \Sigma$.
(iv) $p \lor q \in \Sigma$ iff $(p \in \Sigma$ or $q \in \Sigma)$.
(v) $p \land q \in \Sigma$ iff $(p \in \Sigma$ and $q \in \Sigma)$.
(vi) $p \circ q \in \Sigma$ iff $(p \in \Sigma$ and $q \in \Sigma)$.
(vii) If $p \in \Sigma$ or $q \in \Sigma$, then $p \vartriangle q \in \Sigma$.
(viii) If $(p \to q) \in \Sigma$, then $p \in \Sigma$ implies $q \in \Sigma$.
(ix) If $(p \leftrightarrow q) \in \Sigma$, then $p \in \Sigma$ iff $q \in \Sigma$.

**Lemma 4.1.** (Lindenbaum’s Lemma) Every consistent set of formulas is contained in a maximal consistent set.

**Lemma 4.2.** Let $\Delta$ and $\Gamma$ be maximal consistent sets of formulas. The following are equivalent:

(a) $\varphi \in \Gamma \Rightarrow P\varphi \in \Delta$, for all formula $\varphi$.
(b) $\psi \in \Delta \Rightarrow F\psi \in \Gamma$, for all formula $\psi$.
(c) $G\gamma \in \Gamma \Rightarrow \gamma \in \Delta$, for all formula $\gamma$. 

(d) $H\delta \in \Delta \Rightarrow \delta \in \Gamma$, for all formula $\delta$.

**Lemma 4.3.** If $\Sigma$ is a maximal consistent set of formulas and $\gamma$ a formula of $\mathcal{PTL}_3$, then:

(a) If $F\gamma \in \Sigma$ then there exists a maximal consistent set $\Delta$ with $\Sigma \prec \Delta$ and $\gamma \in \Delta$.

(b) If $P\gamma \in \Sigma$ then there exists a maximal consistent set $\Gamma$ with $\Gamma \prec \Sigma$ and $\gamma \in \Gamma$.

**Proposition 4.3.** If $x$ is a variable, $\varphi$ and $\psi$ are formulas and $x$ is not free in $\psi$ then

1. $\vdash \forall x(\varphi \rightarrow \psi) \iff (\exists x\varphi \rightarrow \psi)$
2. $\vdash \exists x(\psi \rightarrow \varphi) \iff (\psi \rightarrow \exists x\varphi)$

**Lemma 4.4.** If $\vdash \varphi \rightarrow \psi$ then $\vdash \varphi^2 \rightarrow \psi^2$, where we denote $\varphi^2 = \varphi \circ \varphi$.

**Proposition 4.4.** If $\varphi$ is a formula then

$\vdash \exists \varphi^2 \rightarrow (\exists \varphi)^2$

**Lemma 4.5.** Let $T$ a theory and $\varphi$ a formula. The following are equivalent:

- $T \cup \{\varphi\}$ is inconsistent.
- $T \vdash \neg \varphi^2$.

**Proposition 4.5.** Any consistent theory can be embedded in a maximal consistent theory.

Let $C$ a set of new constants having the same cardinality as $\mathcal{PTL}_3$ and $\mathcal{PTL}_3(C)$ the language obtained from $\mathcal{PTL}_3$ by adding the constants of $C$.

**Lemma 4.6.** Let $T$ a theory of $\mathcal{PTL}_3$, $\varphi(x)$ a formula of $\mathcal{PTL}_3$ and $c \in C$. We have:

$$T \vdash \forall x \varphi(x) \text{ in } \mathcal{PTL}_3 \text{ iff } T \vdash \varphi(c) \text{ in } \mathcal{PTL}_3(C)$$

**Definition 4.1.** A consistent theory $T$ of $\mathcal{PTL}_3(C)$ is said to be a Henkin theory if for any formula $\varphi(x)$ in $\mathcal{PTL}_3(C)$ there exists $c \in C$ such that $T \vdash \exists x \varphi(x) \rightarrow \varphi(c)$.

The following lemma will be the main tool in proving the properties of the canonical model (see the proof of Theorem 5.1).

**Lemma 4.7.** Let $T$ be a consistent theory in $\mathcal{PTL}_3$. Then, there is a set $C$ of new constants and a Henkin theory $\bar{T}$ in $\mathcal{PTL}_3(C)$ such that $T \subseteq \bar{T}$.

**Proof.** Let $\alpha$ be the cardinal of the language $\mathcal{PTL}_3$. Let $C$ a set of new constants such that $|C| = \alpha$. Then $|\mathcal{PTL}_3(C)| = \alpha$. Let us consider an enumeration $\{c_\xi\}_{\xi < \alpha}$ of $C$ such that $c_\beta \neq c_\gamma$ for all $\gamma < \beta < \alpha$. We can take an enumeration $\{\varphi_\xi(x_\xi)\}_{\xi < \alpha}$ of the formulas of $\mathcal{PTL}_3(C)$ with at most one free variable. We will construct by transfinite induction an increasing sequence of theories in $\mathcal{PTL}_3(C)$: $T = T_0 \subseteq T_1 \subseteq \ldots \subseteq T_\xi \subseteq \ldots$ with $\xi < \alpha$ and a sequence $\{d_\xi\}_{\xi < \alpha}$ of constants in $C$ such that the following conditions hold:

- $T_\xi$ is consistent in $\mathcal{PTL}_3(C)$.
- If $\xi = \mu + 1$ is an successor ordinal then $T_\xi = T_\mu \cup \{\exists x_\mu \varphi_\mu(x_\mu) \rightarrow \varphi_\mu(d_\mu)\}$ where $d_\mu$ is the first constant in $C$ which does not appear in $T_\mu$.
- If $\xi$ is a non-zero limit ordinal then $T_\xi = \bigcup_{\mu < \xi} T_\mu$.

Let’s assume that $T_\mu$ is consistent and $T_{\mu + 1} = T_\mu \cup \{\exists x_\mu \varphi_\mu(x_\mu) \rightarrow \varphi_\mu(d_\mu)\}$ is inconsistent in $\mathcal{PTL}_3(C)$. By the Lemma 4.5 we obtain: $T_\mu \vdash \neg(\exists x_\mu \varphi_\mu(x_\mu) \rightarrow \varphi_\mu(d_\mu))^2$. 
Since $d_y$ does not appear in $T^\mu$, using Lemma 4.6 we get:

$T^\mu \vdash \forall y(\exists x\varphi(x, y)) \rightarrow \varphi(y, y)^2$,

where $y$ is a variable does not appear in $\varphi(x, y)$. Thus $T^\mu \vdash \neg\exists y(\exists x\varphi(x, y)) \rightarrow \varphi(y, y)^2$ and by proposition 4.4 we get $T^\mu \vdash \neg(\exists y(\exists x\varphi(x, y)) \rightarrow \varphi(y, y)^2)$. Using proposition 4.3 (2) we obtain $T^\mu \vdash \neg(\exists y\varphi_n(y))^2$. From $T^\mu \vdash (\exists x\varphi_n(x) \rightarrow \exists y\varphi_n(y))^2$ we obtain a contradiction because $T^\mu$ is assumed consistent. Thus $T^\mu+1$ is consistent. If $\xi$ is a non-zero limit ordinal and the theories $T^\mu, \mu < \xi$ are consistent then $T_\xi = \bigcup T^\mu$ is consistent. We denote $\bar{T} = \bigcup_{\mu < \xi} T_\mu$.

Using the fact that $T^\mu, \mu < \alpha$ is consistent and $T^\mu \subset \bar{T}$, for all $\mu < \alpha$ we obtain that $\bar{T}$ is consistent.

Let’s show that $\bar{T}$ is a Henkin theory. Let $\varphi(x) \in PT_L(C)$ with at most one free variable, hence there exists $n$ with $\varphi(x) = \varphi_n(x_n)$.

Hence $\exists x\varphi(x) \rightarrow \varphi(e_n) = \exists x\varphi_n(x_n) \rightarrow \varphi_n(e_n) \in T_{n+1} \subset \bar{T}$, where $e_n$ is the first constant in $C$ does not appear in $T_n$.

We obtain that $\bar{T} \vdash \exists x\varphi(x) \rightarrow \varphi(e_n)$ and $\bar{T}$ is a Henkin theory. \hfill $\square$

5. The semantic of $PT_L$ and the canonical model

This section concerns with the semantic of $PT_L$. We define the structures corresponding to $PT_L$ and the interpretation of formulas in these structures. This definition combines the Kripke semantics and three-valued semantics.

The contribution of this section is the construction of the canonical model associated with a maximal consistent Henkin theory. The idea of this construction is inspired from [8]. The main result of this section (Theorem 5.1) expresses the satisfiability of formulas in the canonical model by their position w.r.t. the maximal consistent Henkin theories.

A structure of the three-valued temporal predicate calculus has the form: $A = \langle \langle K, R \rangle, \{A_k, k \in K \} \rangle$ where $K$ is a nonempty set, $R$ is a binary relation on $K$ and $A_k$ is a three-valued structure of the form $A_k = \langle A_k, \{P(A_k) \}_{P \text{ predicate}}, \{c^{A_k} \}_{c \text{ constant}} \rangle$ where:

- $A_k$ is a nonempty set called the universe of structure;
- $P(A_k) : A_k^n \rightarrow L_3$, where $n$ is the arity of $P$, is the interpretation of the predicate $P$ in $A_k$;
- $c^{A_k} \in A_k$ is the interpretation of $c$ in $A_k$.

Let $A_k$ be a three-valued structure, $\varphi(x_1, \ldots, x_n)$ be a formula and $a_1, \ldots, a_n \in A_k$, $k \in K$. We will define inductively $\lbrack \varphi(a_1, \ldots, a_n) \rbrack_k \in L_3$.

(a) If $\varphi(x_1, \ldots, x_n) = P(x_1, \ldots, x_n)$ where $P$ is a $n$-ary predicate, $\lbrack \varphi(a_1, \ldots, a_n) \rbrack_k = \lbrack P(a_1, \ldots, a_n) \rbrack_k = P(A_k)(a_1, \ldots, a_n)$.

(b) If $\varphi(x_1, \ldots, x_n) = \neg \psi(x_1, \ldots, x_n)$ then $\lbrack \varphi(a_1, \ldots, a_n) \rbrack_k = \neg \lbrack \psi(a_1, \ldots, a_n) \rbrack_k = 1 - \lbrack \psi(a_1, \ldots, a_n) \rbrack_k$.

(c) If $\varphi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n) \rightarrow \theta(x_1, \ldots, x_n)$ then $\lbrack \varphi(a_1, \ldots, a_n) \rbrack_k = \neg \lbrack \psi(a_1, \ldots, a_n) \rbrack_k \rightarrow \lbrack \theta(a_1, \ldots, a_n) \rbrack_k = \min \{1, 1 - \lbrack \psi(a_1, \ldots, a_n) \rbrack_k + \lbrack \theta(a_1, \ldots, a_n) \rbrack_k \}$.

(d) If $\varphi(x_1, \ldots, x_n) = \forall x \psi(x, x_1, \ldots, x_n)$ then $\lbrack \varphi(a_1, \ldots, a_n) \rbrack_k = \min \{1, 1 - \lbrack \psi(a_1, \ldots, a_n) \rbrack_k + \lbrack \theta(a_1, \ldots, a_n) \rbrack_k \}$.

(e) If $\varphi(x_1, \ldots, x_n) = G \psi(x_1, \ldots, x_n)$ then $\lbrack \varphi(a_1, \ldots, a_n) \rbrack_k = \lbrack \lbrack \psi(a_1, \ldots, a_n) \rbrack_k \rbrack_{k+1} = \lbrack \lbrack \psi(a_1, \ldots, a_n) \rbrack_k \rbrack_{k+1} = \lbrack \lbrack \psi(a_1, \ldots, a_n) \rbrack_k \rbrack_{k+1}$.

(f) If $\varphi(x_1, \ldots, x_n) = H \psi(x_1, \ldots, x_n)$ then $\lbrack \varphi(a_1, \ldots, a_n) \rbrack_k = \lbrack \lbrack \psi(a_1, \ldots, a_n) \rbrack_k \rbrack_{k+1} = \lbrack \lbrack \psi(a_1, \ldots, a_n) \rbrack_k \rbrack_{k+1}$.
Definition 5.1. If $A = \langle (K, R), \{A_k, k \in K \} \rangle$ is a structure, $k \in K$ and $a_1, ..., a_n \in A_k$ we will denote:

$$A \models_k \varphi(a_1, ..., a_n) \iff \|\varphi(a_1, ..., a_n)\|_k = 1$$

Let $C$ be a set of new constants with the same cardinal number as the language $\mathcal{PTL}_3$ and $\Sigma$ be a maximal consistent Henkin theory of the language $\mathcal{PTL}_3(C)$. In what follows we shall define a structure named the canonical model of $\Sigma$.

Let $C_1, C_2, ...$ a denumerable sequence of sets of new constants such that
- $C \cap C_i = \emptyset$, for all $i$;
- $C_i \cap C_j = \emptyset$, for all $i \neq j$.

For any natural number $n \geq 1$, $\mathcal{PTL}_3(C \cup C_1 \cup ... \cup C_n)$ is the language obtained from $\mathcal{PTL}_3$ by adding the constants of $C \cup C_1 \cup ... \cup C_n$.

Let us denote by $K$ the family of the sets $\Delta$ having the following properties:

(i) there exists a natural number $n \geq 1$ such that $\Delta$ is a maximal consistent Henkin theory of $\mathcal{PTL}_3(C \cup C_1 \cup ... \cup C_n)$.

(ii) $\Sigma \subseteq \Delta$.

We consider $A_\Delta = C \cup C_1 \cup ... \cup C_n$ where $n$ is the smallest natural number with $\Delta \subseteq \mathcal{PTL}_3(C \cup C_1 \cup ... \cup C_n)$ and $\Delta \in K$. We will organize each $A_\Delta$, $\Delta \in K$ like a three-valued structure for the language $\mathcal{PTL}_3$ with the following properties:

- If $R$ is a three-valued predicate then the $n$-ary relation $R^{A_\Delta}$ on $A_\Delta$ is defined:

$$R^{A_\Delta}_n : A_\Delta^n \rightarrow L_3.$$  

$$R^{A_\Delta}_n = \begin{cases} 1, & \text{if } R(c_1, ..., c_n) \in \Delta \\ 0, & \text{if } \neg R(c_1, ..., c_n) \in \Delta \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

- If $c$ is a constant symbol then its interpretation in $A_\Delta$ is $c^{A_\Delta} = c$.

We will define the binary relation $\prec$ on $K$: if $\Delta \subseteq \Gamma \subseteq K$ then we say that $\Gamma \prec \Delta$ if the conditions of the Lemma 4.2 hold.

We have defined a structure, $A = \langle (K, \prec), \{A_\Delta, \Delta \in K \} \rangle$ for the language $\mathcal{PTL}_3$.

Theorem 5.1. For every formula $\varphi(x_1, ..., x_n)$ of $\mathcal{PTL}_3$, for every $\Delta \in K$ and for all $c_1, ..., c_n \in A_\Delta$ we have the equivalence:

$$A \models \varphi(c_1, ..., c_n) \iff \varphi(c_1, ..., c_n) \in \Delta$$

Proof. We will prove by induction of $\varphi(x_1, ..., x_n)$.

(a) $\varphi$ is an atomic formula, i.e $\varphi(x_1, ..., x_n) = P(x_1, ..., x_n)$ where $P$ is a three-valued predicate. We have the equivalence:

$$A \models \varphi(c_1, ..., c_n) \iff \|P(c_1, ..., c_n)\|_\Delta = 1 \text{ (from Definition 5.1)} \iff P^{A_\Delta}(c_1, ..., c_n) = 1 \iff P(c_1, ..., c_n) \in \Delta \iff \varphi(c_1, ..., c_n) \in \Delta.$$

(b) $\varphi(x_1, ..., x_n) = \neg \psi(x_1, ..., x_n)$, where for $\psi$ the hypothesis is satisfied: $A \models \psi(c_1, ..., c_n) \iff \psi(c_1, ..., c_n) \in \Delta$. We have:

$$A \models \varphi(c_1, ..., c_n) \iff A \models \neg \psi(c_1, ..., c_n) \iff \|\neg \psi(c_1, ..., c_n)\|_\Delta = 1 \iff \|\psi(c_1, ..., c_n)\|_\Delta = 0 \iff A \not\models \psi(c_1, ..., c_n) \iff \psi(c_1, ..., c_n) \notin \Delta \iff \neg \psi(c_1, ..., c_n) \in \Delta \text{ (from the fact that } \Delta \text{ is maximal consistent, i.e } \varphi(c_1, ..., c_n) \in \Delta).$$

(c) $\varphi(x_1, ..., x_n) = \psi(x_1, ..., x_n) \rightarrow \theta(x_1, ..., x_n)$.

(\Rightarrow) Assume that $A \models \varphi(c_1, ..., c_n)$. We have:

$$A \models \varphi(c_1, ..., c_n) \rightarrow \theta(c_1, ..., c_n) \iff \|\varphi(c_1, ..., c_n)\|_\Delta \rightarrow \|\theta(c_1, ..., c_n)\|_\Delta = 1 \iff \min\{1, 1 - \|\psi(c_1, ..., c_n)\|_\Delta, \|\theta(c_1, ..., c_n)\|_\Delta\} = 1 \iff \|\psi(c_1, ..., c_n)\|_\Delta \leq \|\theta(c_1, ..., c_n)\|_\Delta \leq 1.$$
\[ \| \theta(c_1, ..., c_n) \|_\Delta. \]

We consider the following cases:

1. Let \( \| \psi(c_1, ..., c_n) \|_\Delta = 0. \) This implies that \( A \not \models_\Delta \psi(c_1, ..., c_n) \iff \psi(c_1, ..., c_n) \not \in \Delta \) (by inductive hypothesis) \( \iff \neg \psi(c_1, ..., c_n) \in \Delta \) (from \( \Delta \)-maximal consistent). Using (t6) and Proposition 4.2 ((i) and (viii)) we obtain \( \psi(c_1, ..., c_n) \rightarrow \theta(c_1, ..., c_n) \in \Delta \) hence \( \varphi(c_1, ..., c_n) \in \Delta. \)

2. Let \( \| \psi(c_1, ..., c_n) \|_\Delta = 1. \)

Then it is necessary that \( \| \theta(c_1, ..., c_n) \|_\Delta = 1. \) From this we have: \( A \models_\Delta \psi(c_1, ..., c_n) \) and \( A \models_\Delta \theta(c_1, ..., c_n) \) and by inductive hypothesis we obtain \( \psi(c_1, ..., c_n) \in \Delta \) and \( \theta(c_1, ..., c_n) \in \Delta. \) Using (t1) and Proposition 4.2 ((i) and (viii)) we obtain that \( \psi(c_1, ..., c_n) \rightarrow \theta(c_1, ..., c_n) \in \Delta, \) hence \( \varphi(c_1, ..., c_n) \in \Delta. \)

3. Let \( \| \psi(c_1, ..., c_n) \|_\Delta = \frac{1}{2}. \) Hence \( \| \theta(c_1, ..., c_n) \|_\Delta \in \{ \frac{1}{2}, 1 \}. \)

(a) If \( \| \theta(c_1, ..., c_n) \|_\Delta = \frac{1}{2} \) then we have: \( A \not \models_\Delta \psi(c_1, ..., c_n) \) and \( A \not \models_\Delta \theta(c_1, ..., c_n) \) and by inductive hypothesis \( \psi(c_1, ..., c_n) \not \in \Delta \) and \( \theta(c_1, ..., c_n) \not \in \Delta, \) so from the fact that \( \Delta \) is maximal consistent we obtain \( \neg \psi(c_1, ..., c_n) \in \Delta \) and \( \neg \theta(c_1, ..., c_n) \in \Delta. \) Using (t1),(t4) and Proposition 4.2 ((i) and (viii)) it follows \( \psi(c_1, ..., c_n) \rightarrow \theta(c_1, ..., c_n) \in \Delta, \) so \( \varphi(c_1, ..., c_n) \in \Delta. \)

(b) If \( \| \theta(c_1, ..., c_n) \|_\Delta = 1 \) then the Definition 5.1 we obtain \( A \models_\Delta \theta(c_1, ..., c_n) \) and by inductive hypothesis \( \theta(c_1, ..., c_n) \in \Delta. \) Using (t1) and Proposition 4.2 ((i) and (viii)) we have \( \psi(c_1, ..., c_n) \rightarrow \theta(c_1, ..., c_n) \in \Delta, \) hence \( \varphi(c_1, ..., c_n) \in \Delta. \)

(\( \Leftarrow \)) Assume that \( \varphi(c_1, ..., c_n) \in \Delta. \)

This is equivalent to \( \psi(c_1, ..., c_n) \rightarrow \theta(c_1, ..., c_n) \in \Delta. \) We consider the following cases:

(a) Assume that \( \psi(c_1, ..., c_n) \in \Delta. \) Using Proposition 4.2(viii) we obtain \( \theta(c_1, ..., c_n) \in \Delta. \) By the inductive hypothesis about \( \psi \) and \( \theta \) it follows that \( A \models_\Delta \psi(c_1, ..., c_n) \) and \( A \models_\Delta \theta(c_1, ..., c_n) \) and using Definition 5.1 we have \( \| \psi(c_1, ..., c_n) \|_\Delta = 1 \) and \( \| \theta(c_1, ..., c_n) \|_\Delta = 1. \)

Thus \( \| \varphi(c_1, ..., c_n) \|_\Delta = 1, \) so \( A \models_\Delta \varphi(c_1, ..., c_n). \)

(b) Assume that \( \psi(c_1, ..., c_n) \not \in \Delta. \) Because \( \Delta \) is maximal consistent, it results that \( \neg \psi(c_1, ..., c_n) \in \Delta \) and by inductive hypothesis \( A \models_\Delta \neg \psi(c_1, ..., c_n). \) By Definition 5.1 we have \( \| \psi(c_1, ..., c_n) \|_\Delta = 0. \)

It follows that \( \| \psi(c_1, ..., c_n) \|_\Delta \rightarrow \| \theta(c_1, ..., c_n) \|_\Delta = 1, \) so \( \| \psi(c_1, ..., c_n) \|_\Delta = 1, \) hence \( A \models_\Delta \varphi(c_1, ..., c_n). \)

(d) \( \psi(x_1, ..., x_n) = G \psi(x_1, ..., x_n), \) where for \( \psi \) the hypothesis of induction is satisfied, i.e.

\[ A \models_\Delta \psi(c_1, ..., c_n) \iff \psi(c_1, ..., c_n) \in \Delta. \]

(\( \Leftarrow \)) Assume \( \varphi(c_1, ..., c_n) \in \Delta, \) i.e \( G \psi(c_1, ..., c_n) \in \Delta. \)

Let \( \Delta \prec \Delta', \) and by definition of \( \prec \) it results that \( \psi(c_1, ..., c_n) \in \Delta'. \) By inductive hypothesis we obtain \( A \models_\Delta' \psi(c_1, ..., c_n). \) So, \( A \models_\Delta' \psi(c_1, ..., c_n) \) for all \( \Delta' \) with \( \Delta \prec \Delta' \) and we get

\[ A \models_\Delta G \psi(c_1, ..., c_n). \]

(\( \Rightarrow \)) Assume \( G \psi(c_1, ..., c_n) \not \in \Delta, \) so \( \neg G \psi(c_1, ..., c_n) \in \Delta. \)

Hence \( F \neg \psi(c_1, ..., c_n) \in \Delta. \)

By Lemma 4.3 and from \( F \neg \psi(c_1, ..., c_n) \in \Delta \) it follows that there exists \( \Delta' \in K, \Delta \prec \Delta' \) and \( \neg \psi(c_1, ..., c_n) \in \Delta'. \)

Because \( \Delta' \) is maximal consistent we have \( \psi(c_1, ..., c_n) \not \in \Delta', \) so \( A \not \models_\Delta' \psi(c_1, ..., c_n) \).
ψ(c₁, ..., cₙ). We proved that there exists Δ × Δ' with $A \not\models _Δ ψ(c₁, ..., cₙ)$, hence $A \not\models_Δ Gψ(c₁, ..., cₙ)$.

(c) $ϕ(x₁, ..., xₙ) = \forall xψ(x, x₁, ..., xₙ)$.

($⇒$) Suppose that $A \models _Δ ϕ(c₁, ..., cₙ)$, i.e. $A \models _Δ \forall xψ(x, c₁, ..., cₙ)$ and by Definition 5.1 we get $∥\forall xψ(x, c₁, ..., cₙ)∥_Δ = 1$ $⇒$ $A \models _Δ [ϕ(a, c₁, ..., cₙ)]_a = 1$. It follows that for all $a \in A_Δ$\(\exists aA_Δ\) we have the following equivalence:

$$\parallel ψ(a, c₁, ..., cₙ) \parallel_Δ = 1 \iff \forall a \in A_Δ, A \models _Δ ψ(a, c₁, ..., cₙ).$$

By the inductive hypothesis we obtain: for all $a \in A_Δ, ψ(a, c₁, ..., cₙ) \in Δ$ and using Lemma 4.6 it follows that $\forall xψ(x, c₁, ..., cₙ) \in Δ$, so $ψ(c₁, ..., cₙ) \in Δ$.

($⇐$) Suppose that $ϕ(c₁, ..., cₙ) \in Δ$. We have: $\forall xψ(x, c₁, ..., cₙ) \in Δ$ $⇒$ for all $a \in A_Δ, A \models _Δ ψ(a, c₁, ..., cₙ) \in Δ$. By the inductive hypothesis we get for all $a \in A_Δ, A \models _Δ ψ(a, c₁, ..., cₙ)$ $⇒$ $A \models _Δ \forall xψ(x, c₁, ..., cₙ)$, hence $A \models _Δ ϕ(c₁, ..., cₙ)$.

\[\square\]

Remark 5.1. Since $Σ \subseteq Δ$, for each $Σ \subseteq Δ$, it follows that $A \models _Δ ϕ(c₁, ..., cₙ)$ for all $Δ \in Κ$, $ϕ(x₁, ..., xₙ)$ in $PTL₃$ and $c₁, ..., cₙ \in A_Δ$.

6. Completeness theorem

This section contains the main result of this paper: the strong completeness theorem for $PTL₃$. The proof of the completeness theorem is based on Theorem 5.1.

Theorem 6.1. If $Γ$ is a set of formulas of $PTL₃$ and $ϕ$ is a formula of $PTL₃$ then we have the following equivalence:

$$Γ ⊢ ϕ \iff Γ \models ϕ$$

Proof. ($⇒$) Assume that $Γ \not\models ϕ$. We get $Γ \cup \{¬ϕ\}$ is consistent and by Theorem 5.1 it follows that there exists a structure $A$ such that $A \models Γ \cup \{¬ϕ\}$. It follows that $A \models Γ$ and $A \not\models ϕ$, hence $Γ \not\models ϕ$.

($⇒$) By the induction on the concept $Γ \vdash ϕ$.

(1) Suppose that $ϕ \in Γ$. Let $A$ be a model for $Γ$. Then we have $A \models ϕ$, for all $ψ \in Γ$, hence $A \models ϕ$. Because $A$ is an arbitrary model for $Γ$, we obtain that $Γ \models ϕ$.

(2) Suppose that $ϕ$ is an axiom. Let $A$ be a model for $Γ$.

(a) Let $ϕ = G(θ → ψ) → (Gθ → Gψ)$. We must prove that $A \models ϕ$, i.e $A \models _k ϕ$, for all moments $k$. Suppose that $A \models _k G(θ → ψ)$, and $A \models _k Gθ$. By the definition of the concept $A \models _k ϕ$ we obtain: $A \models _{k '} θ → ψ$, $A \models _{k '} θ$ for all $kRk'$, so $A \models _{k '} ψ$ for all $kRk'$. It follows that $A \models _k Gψ$.

(b) Let $ϕ = ψ → Gψ$. We must prove that $A \models ψ$ implies $A \models Gψ$ i.e for all moments $k$, $A \models _k ψ$ implies $A \models _k Gψ$.

Proving $A \models _k Gψ$ is equivalent to showing that for all $kRk'$, $A \models _{k '} Pψ$ or to the following: for all $kRk'$ there exists $k''Rk'$ with $A \models _{k ''} ψ$. We assumed that $A \models _k ψ$, hence, for $k'' = k$ we obtain $A \models _k Gψ$ i.e $A \models _k ψ → Gψ$.

In a similar way we can prove the remaining axioms.
(a) Suppose that \( \varphi \) was obtained by the rule \( \Gamma \vdash \psi, \Gamma \vdash \psi \rightarrow \varphi \) and \( \Gamma \models \psi, \Gamma \models \psi \rightarrow \varphi \).

Let \( \mathcal{A} \) be a model for \( \Gamma \). We have: \( \mathcal{A} \models \psi \) and \( \mathcal{A} \models \psi \rightarrow \varphi \), i.e. \( \mathcal{A} \models_k \psi, \mathcal{A} \models_k \psi \rightarrow \varphi \) for all \( k \), so \( \mathcal{A} \models_k \varphi \), for all \( k \). It follows that \( \mathcal{A} \models_k \varphi \).

(b) Suppose that \( \varphi(x_1, \ldots, x_n) = \forall x \psi(x, x_1, \ldots, x_n) \), obtained by the rule \( \Gamma \vdash \psi \) and \( \Gamma \models \psi \). We want to prove that \( \Gamma \models \varphi \). Let \( \mathcal{A} \) be a model for \( \Gamma \). It follows that \( \mathcal{A} \models \psi \), i.e. \( \mathcal{A} \models_k \psi \), for all moments \( k \). This is equivalent with \( \mathcal{A} \models_k \varphi(a, a_1, \ldots, a_n) \) for all \( a \in A_k \) and we have \( \mathcal{A} \models_k \varphi \).

(c) Assume that \( \varphi = G \psi \), obtained by the rule \( \Gamma \vdash \psi \) and \( \Gamma \models \psi \). We must prove that \( \Gamma \models \varphi \), i.e. \( \Gamma \models G \psi \). Let \( \mathcal{A} \) be a model for \( \Gamma \). Then \( \mathcal{A} \models \psi \iff \mathcal{A} \models_k \psi \), for all \( k \), so, we have \( \mathcal{A} \models_k \psi \) for all \( k \in kR^k \), i.e. \( \mathcal{A} \models_k G \psi \).

Theorem 6.2 (The completeness theorem). For any formula \( \varphi \) of \( \mathcal{PTL}_3 \) the following equivalence holds:

\[ \vdash \varphi \iff \models \varphi \]

Proof. By Theorem 6.1, with \( \Gamma = \emptyset \). \( \square \)

References


(Carmen Chiriță) University of Bucharest
Faculty of Mathematics and Computer Science
Academiei 14, RO 010014, Bucharest, Romania
E-mail address: stama@funinf.cs.unibuc.ro