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A completeness theorem for three-valued temporal predicate logic

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Abstract. The main result in this paper is a completeness theorem for the three-valued temporal predicate calculus, obtained by providing a semantical interpretation for this logic and by using the Henkin models to define a canonical model used to prove the completeness.

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1. Introduction

The classical temporal logic is obtained from bivalent logic by adding the tense operators G ("it is always going to be the case that") and H (" it has always been the case that"). By starting from other logical systems and adding appropiate tense operators we can produce new temporal logics. In [5] we have studied a complete three-valued temporal propositional calculus based on the Łukasiewicz three-valued logic.

The goal of this paper is to construct a temporal logical system for the predicate calculus based on the three-valued logic . This logical system is obtained from the Łukasiewicz logic described in [5] by adding the quantifiers. The main result is a completeness theorem for this logical system, whose proof uses a Henkin-style method (see [8]).

The paper is organized as follows:

In Section 2 we recall from [1] and [11] some basic definitions and results on the three-valued Łukasiewicz logic: the syntax, the semantic, the completeness theorem and a list of provable sentences.

Section 3 contains a short presentation of the three-valued temporal propositional logic $\mathcal{TL}_3$ and the completeness theorem proved in [5].

In Section 4 we define the language and the logical structure of the three-valued temporal predicate logic $\mathcal{PTL}_3$. We study the consistent sets of formulas and we prove that any consistent theory of $\mathcal{PTL}_3$ can be embedded in a Henkin theory.

Section 5 deals with the semantic of $\mathcal{PTL}_3$. We define the structures of $\mathcal{PTL}_3$ and we construct the canonical model associated with a maximal consistent Henkin theory. The satisfiability of formulas in canonical model is characterized in terms of maximal consistent Henkin theories.

Section 6 contains the proof of completeness theorem for $\mathcal{PTL}_3$. This proof is based on the properties of canonical model (cf. Theorem 5.1).

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2. Three-valued Lukasiewicz propositional logic

The first system of three-valued logic was constructed by Lukasiewicz in 1920 in connection with the investigation of modalities (see [14]). His main idea was to consider a third truth-value $\frac{1}{2}$ between 0 (false) and 1(truth). The interpretation for the sentences of the three-valued logic is defined in $L_3 = \{0, \frac{1}{2}, 1\}$. The algebraic structures for the three-valued Łukasiewicz logic were introduced by Gr.C.Moisil in [15] under the name of three-valued Łukasiewicz algebras (see also [16], [1]). Today these structures are known as Łukasiewicz-Moisil algebras (see [1]). We shall use the Wajsberg axiomatization of the three-valued Lukasiewicz logic ([1]). The sentences of the three-valued Lukasiewicz propositional calculus $\mathcal{L}_3$ are obtained from a countable set V of propositional variables and the logical conectives $\neg$ and $\rightarrow$, according to the following rules:

(i) the propositional variables are sentences;

(ii) if p şi q are sentences then $\neg p$ and $p \rightarrow q$ are sentences;

(iii) every sentence is obtained by applying a finite number of times the above rules (i) and (ii).

In what follows, we will denote the set of sentences of $\mathcal{L}_3$ by E.
We are going to use the following abbreviations:

$$\begin{aligned}
\varphi \vee \psi &:= ((\varphi \rightarrow \psi) \rightarrow \psi) \\
\varphi \wedge \psi &:= \neg(\neg\varphi \vee \neg\psi) \\
\varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \\
\varphi \oplus \psi &:= \neg\varphi \rightarrow \psi \\
\varphi \odot \psi &:= \neg(\neg\varphi \oplus \neg\psi) \\
\sim \varphi &:= \varphi \rightarrow \neg\varphi
\end{aligned}$$

The axioms of three-valued Lukasiewicz propositional calculus are sentences of one of the following forms:
(A1) $p \rightarrow (q \rightarrow p)$
(A2) $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$
(A3) $((p \rightarrow \neg p) \rightarrow p) \rightarrow p$
(A4) $(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)$
Three-valued Lukasiewicz propositional logic uses modus ponens (m.p) as rule of inference:

$$\frac{p, p \rightarrow q}{q}$$

A proof of a sentence p is a finite sequence $p_1, ..., p_n = p$ of sentences such that for any $i \leq n$ we have one of the following:
(a) p_i is an axiom;
(b) there exists $j, k < i$ such that p_k is the sentence $p_j \rightarrow p_i$.
A sentence p is *provable* ($\vdash p$) if there is at least one proof of it.
The following proposition collects the main provable sentences of $\mathcal{L}_3$.

Proposition 2.1. ([1]) *The following sentences are provable in the three-valued Luckasiewicz logic :*
(t1) $p \rightarrow (q \rightarrow p)$,
(t2) $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$,
(t3) $p \rightarrow p$,
(t4) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$,

(t5) $p \leftrightarrow \neg\neg p$,
(t6) $\neg p \to (p \to q)$,
(t7) $(p \to (p \to (q \to r))) \to ((p \to (p \to q)) \to (p \to (p \to r)))$,
(t8) $\sim\sim p \to p$,
(t9) $(p \to \sim p) \to \sim p$,
(t10) $(p \wedge (q \wedge r)) \leftrightarrow ((p \wedge q) \wedge r)$,
(t11) $(p \wedge q) \leftrightarrow (q \wedge p)$,
(t12) $(p \wedge q) \to p$,
(t13) $p \to (q \to (p \wedge q))$,
(t14) $(p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r)$,
(t15) $(p \vee q) \leftrightarrow (q \vee p)$,
(t16) $p \to (p \vee q)$,
(t17) $(p \odot (q \odot r)) \leftrightarrow ((p \odot q) \odot r)$,
(t18) $(p \odot q) \leftrightarrow (q \odot p)$,
(t19) $p \odot q \to p$,
(t20) $p \to (q \to p \odot q)$,
(t21) $(p \to (q \to r)) \leftrightarrow (p \odot q \to r)$,
(t22) $(p \to q) \to (p \odot r \to q \odot r)$,
(t23) $(p \oplus (q \oplus r)) \leftrightarrow ((p \oplus q) \oplus r)$,
(t24) $(p \oplus q) \leftrightarrow (q \oplus p)$,
(t25) $p \to p \oplus q$,
(t26) $(p \vee q) \to (p \oplus q)$.
(t27) $\neg(p \to q) \to p$.
(t28) $\neg(p \to q) \to \neg q$.
(t29) $p \to (\neg q \to \neg(p \to q))$.
(t30) $\sim p \to (\sim \neg q \to (p \to q))$.

Definition 2.1. An interpretation of $\mathcal{L}_3$ is an arbitrary function $v : E \to L_3$ such that:

- $v(p \to q) = v(p) \to v(q)$
- $v(\neg p) = \neg v(p)$

for all $p, q \in E$.
We say that a sentence p is valid ($\models p$) if $v(p) = 1$ for any interpretation v.

Theorem 2.1. *(Completeness Theorem) For any sentence p of $\mathcal{L}_3$,*

$$\vdash p \text{ iff } \models p$$

3. Three-valued temporal propositional logic

In this section we present a three-valued temporal logic $\mathcal{TL}_3$ based on the three-valued Lukasiewicz propositional calculus [5]. Our axiomatization is inspired from the axioms of the three-valued Lukasiewicz logic in [1] and from the Ostermann system from [17] (see also Leuştean [13]) and introduces two temporal operators G and H.
The symbols of the three-valued propositional temporal logic are:
(i) a countable set AF of atomic sentences, denoted by $v_0, v_1, ...,$
(ii) the propositional connectives $\neg, \to$,
(iii) the temporal operators G and H.
The set E of sentences of $\mathcal{TL}_3$ is defined by the canonical induction.
We shall use the $\vee, \wedge, \leftrightarrow, \oplus, \odot$ and $\sim$ defined in the previous section.

We also define:

$$\begin{array}{rcl} Fp & := & \neg G\neg p \\ Pp & := & \neg H\neg p \end{array}$$

$\mathcal{TL}_3$ has the following axioms:

(T1) the axioms of the three-valued Lukasiewicz logic (the axioms (A1)-(A4) in section 2)

(T2) $G(p \to q) \to (Gp \to Gq)$,
$H(p \to q) \to (Hp \to Hq)$,

(T3) $G(p \oplus p) \leftrightarrow (Gp \oplus Gp)$,
$H(p \oplus p) \leftrightarrow (Hp \oplus Hp)$,

(T4) $p \to GPp$,
$p \to HFp$,

The notion of formal proof in the three-valued temporal logic is defined in terms of the above axioms and the following inference rules:

$$\frac{p, p \to q}{q}; \frac{p}{Gp}; \frac{p}{Hp}$$

We will denote by $\vdash_{\mathcal{TL}_3} p$ the fact that p is provable in $\mathcal{TL}_3$.

A *frame* is a pair $\mathcal{F} = \langle W, R\rangle$,where W is a not-empty set and R is a binary relation on W.

An *evaluation* of $\mathcal{TL}_3$ in $\mathcal{F}$ is a function $V : E \times W \to L_3 = \{0, \frac{1}{2}, 1\}$ such that, for all $p, q \in E$ and $s \in W$, the following equalities hold:

(i) $V(\neg p, s) = 1 - V(p, s)$,

(ii) $V(p \to q, s) = min\{1, 1 - V(p, s) + V(q, s)\}$,

(iii) $V(Gp, s) = min\{V(p, t)|sRt\}$, for all $p, q \in E, s \in W$
$V(Hp, s) = min\{V(p, t)|tRs\}$, for all $p, q \in E, s \in W$

A sentence $p \in E$ is *universally valid* in $\mathcal{TL}_3$ ($\models_{\mathcal{TL}_3} p$) if for every frame (W, R) and for any evaluation $V : E \times W \to L_3$ we have
$V(p, s) = 1$, for all $s \in W$.

We recall from [5] the following completeness result.

Theorem 3.1. *(Completeness Theorem) For any sentence φ of $\mathcal{TL}_3$,*

$$\vdash_{\mathcal{TL}_3} \varphi \text{ iff } \models_{\mathcal{TL}_3} \varphi$$

4. Syntax of three-valued temporal predicate logic

In this section we shall define the three-valued temporal predicate logic $\mathcal{PTL}_3$ by adding to $\mathcal{TL}_3$ the universal quantifier $\forall$. The logical structure of $\mathcal{PTL}_3$ is obtained by enriching the axiomatization of $\mathcal{TL}_3$ with the new axioms (A6)-(A9) and the generalization rule of inference. We study the consistent sets of formulas and the Henkin theories of $\mathcal{PTL}_3$.

A lot of properties of consistent sets follows as in the case of classical temporal logic and we omit their proofs. We prove that any consistent theory of $\mathcal{PTL}_3$ can be embedded in a Henkin theory of an extended language $\mathcal{PTL}_3$ obtained by adding to $\mathcal{PTL}_3$ the new constants of C.

The alphabet of $\mathcal{PTL}_3$ consists of the following primitive symbols:

- a countable set V of variable symbols, denoted by x,y,z,..,.
- an arbitrary set of constant symbols.

- an arbitrary set of predicate symbols; each predicate symbol P has associated a natural number $n > 0$ (the order or arity of P).
- the propositional connectives $\neg, \rightarrow$.
- the temporal operators G and H.
- the universal quantifier $\forall$.
- the parantheses : (,), [,].

A *term* of $\mathcal{PTL}_3$ is a variable symbol or a constant symbol. An *atomic formula* of $\mathcal{PTL}_3$ has the form $\varphi(t_1, t_2, ..t_n)$ where φ is a n-ary P symbol and $t_1, t_2, ...t_n$ are terms.

We will inductively define the set $Form$ of $formulas$:

(i) the atomic formulas are formulas.
(ii) if $\varphi \in Form$ and $\psi \in Form$ then $\varphi \rightarrow \psi$ and $\neg\varphi \in Form$.
(iii) if $\varphi \in Form$ then $G\varphi \in Form, H\varphi \in Form$.
iv) if $\varphi \in Form$ and x is a variable symbol then $\forall x\varphi$ is a formula.

We also define:

$$\begin{aligned} F\varphi &:= \neg G \neg \varphi \\ P\varphi &:= \neg H \neg \varphi \\ \exists x \varphi &:= \neg \forall x \neg \varphi \end{aligned}$$

The notion of *subformula* is defined by induction:

- φ is a subformula of φ.
- any subformula of φ is a subformula of $\neg\varphi$
- any subformula of φ or ψ is a subformula of $\varphi \rightarrow \psi$.
- any subformula of φ is a subformula of $\forall x\varphi$.

An occurence of a variable x in a formula φ is *free* if x does not belongs to any occurence of a subformula of φ having the form $\forall x\psi$. Otherwise, an occurence of x in φ is *bound*.

We say that x is *free* in φ if any occurence of x is free in φ. A *sentence* is a formula with no free variables. We will write $\varphi(x_1, ..., x_n)$ if all the free variables of φ are among $\{x_1, ...x_n\}$. We'll denote by $FV(\varphi)$ the set of free variables of φ.

A *theory* is a set of formulas.

The axioms of $\mathcal{PTL}_3$ are:

(A0) the axioms of the three-valued logic.
(A1) $G(\varphi \rightarrow \psi) \rightarrow (G\varphi \rightarrow G\psi)$
$H(\varphi \rightarrow \psi) \rightarrow (H\varphi \rightarrow H\psi)$
(A2) $G\varphi \oplus G\psi \rightarrow G(\varphi \oplus \psi)$
$H\varphi \oplus H\psi \rightarrow H(\varphi \oplus \psi)$
(A3) $G(\varphi \oplus \varphi) \rightarrow G\varphi \oplus G\varphi$
$H(\varphi \oplus \varphi) \rightarrow H\varphi \oplus H\varphi$
(A4) $F\varphi \oplus F\varphi \rightarrow F(\varphi \oplus \varphi)$
$P\varphi \oplus P\varphi \rightarrow P(\varphi \oplus \varphi)$
(A5) $\varphi \rightarrow GP\varphi$
$\varphi \rightarrow HF\varphi$
(A6) $\forall x\varphi(x) \rightarrow \varphi(t)$, where t is a term
(A7) $\forall x(\varphi \rightarrow \psi(x)) \rightarrow (\varphi \rightarrow \forall x\psi(x))$, where x is not free in φ
(A8) $\forall x(\varphi \oplus \varphi) \leftrightarrow \forall x\varphi \oplus \forall x\varphi$
(A9) $\forall x(\varphi \odot \varphi) \leftrightarrow \forall x\varphi \odot \forall x\varphi$

$\mathcal{PTL}_3$ has the following rules of inference:

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi} \qquad \text{(Modus Ponens)}$$

$$\frac{\varphi}{\forall x \varphi} \qquad \text{(Generalization)}$$

$$\frac{\varphi}{G\varphi} \qquad \text{(Temporal Generalization)}$$

$$\frac{\varphi}{H\varphi} \qquad \text{(Temporal Generalization)}$$

The formal theorems of $\mathcal{PTL}_3$ are obtained from axioms by applying a finite number of times the rules of inference. We denote by $\vdash \varphi$ the fact that φ is a formal theorem. *The sintactic deduction* is defined by:

$$\Gamma \vdash \varphi \iff \text{there exists } \gamma_1, ...\gamma_n \in \Gamma \text{ with } \vdash \bigwedge_{i=1}^{n} \gamma_i \rightarrow \varphi$$

We say that a set Γ of formulas is *consistent* if there is no formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$; otherwise we say that Γ is *inconsistent*.
A *consistent* set Γ is said to be *maximal consistent* if $\varphi \in \Gamma$ for any formula φ such that $\Gamma \cup \{\varphi\}$ is consistent.

We are going to present some formal theorems and properties for three-valued temporal predicate calculus. The proofs are similar to the corresponding results for Łukasiewicz predicate logic[10].

Proposition 4.1. *Let* $\Sigma \subseteq Form$ *and* $p \in Form$.
(i) Σ *is inconsistent iff* $\Sigma \vdash r$ *for any formula* r.
(ii) $\Sigma \cup \{p\}$ *is inconsistent iff* $\Sigma \vdash \sim p$.
(iii) $\Sigma \cup \{\sim p\}$ *is inconsistent iff* $\Sigma \vdash p$.
(iv) Σ *is consistent iff every finite subset of* Σ *is consistent.*
(v) *If* Σ *is consistent, then for any formula* p*, at least one of* $\Sigma \cup \{p\}$ *and* $\Sigma \cup \{\sim p\}$ *is consistent.*

Proposition 4.2. *Let* Σ *be a maximal consistent set and* $p, q \in Form$.
(i) $\Sigma \vdash p$ *implies* $p \in \Sigma$.
(ii) *If* $\Sigma \subseteq \Gamma$ *and* Γ *is consistent, then* $\Sigma = \Gamma$.
(iii) $p \in \Sigma$ *iff* $\sim p \notin \Sigma$.
(iv) $p \vee q \in \Sigma$ *iff* ($p \in \Sigma$ *or* $q \in \Sigma$).
(v) $p \wedge q \in \Sigma$ *iff* ($p \in \Sigma$ *and* $q \in \Sigma$).
(vi) $p \odot q \in \Sigma$ *iff* ($p \in \Sigma$ *and* $q \in \Sigma$).
(vii) *If* $p \in \Sigma$ *or* $q \in \Sigma$*, then* $p \oplus q \in \Sigma$.
(viii) *If* $(p \rightarrow q) \in \Sigma$*, then* $p \in \Sigma$ *implies* $q \in \Sigma$.
(ix) *If* $(p \leftrightarrow q) \in \Sigma$*, then* $p \in \Sigma$ *iff* $q \in \Sigma$.

Lemma 4.1. *(Lindenbaum's Lemma) Every consistent set of formulas is contained in a maximal consistent set.*

Lemma 4.2. *Let* Δ *and* Γ *be maximal consistent sets of formulas. The following are equivalent:*
(a) $\varphi \in \Gamma \Rightarrow P\varphi \in \Delta$*, for all formula* φ.
(b) $\psi \in \Delta \Rightarrow F\psi \in \Gamma$*, for all formula* ψ.
(c) $G\gamma \in \Gamma \Rightarrow \gamma \in \Delta$*, for all fromula* γ.

(d) $H\delta \in \Delta \Rightarrow \delta \in \Gamma$, *for all fromula* δ.

Lemma 4.3. *If* Σ *is a maximal consistent set of formulas and* γ *a formula of* $\mathcal{PTL}_3$, *then:*
(a) *If* $F\gamma \in \Sigma$ *then there exists a maximal consistent set* Δ *with* $\Sigma \prec \Delta$ *and* $\gamma \in \Delta$.
(b) *If* $P\gamma \in \Sigma$ *then there exists a maximal consistent set* Γ *with* $\Gamma \prec \Sigma$ *and* $\gamma \in \Gamma$.

Proposition 4.3. *If* x *is a variable,* φ *and* ψ *are formulas and* x *is not free in* ψ *then*
(1) $\vdash \forall x(\varphi \to \psi) \leftrightarrow (\exists x\varphi \to \psi)$
(2) $\vdash \exists x(\psi \to \varphi) \leftrightarrow (\psi \to \exists x\varphi)$

Lemma 4.4. *If* $\vdash \varphi \to \psi$ *then* $\vdash \varphi^2 \to \psi^2$, *where we denote* $\varphi^2 = \varphi \odot \varphi$.

Proposition 4.4. *If* φ *is a formula then*

$$\vdash \exists\varphi^2 \leftrightarrow (\exists\varphi)^2$$

Lemma 4.5. *Let* T *a theory and* φ *a formula. The following are equivalent:*
- $T \cup \{\varphi\}$ *is inconsistent.*
- $T \vdash \neg\varphi^2$.

Proposition 4.5. *Any consistent theory can be embedded in a maximal consistent theory.*

Let C a set of new constants having the same cardinality as $\mathcal{PTL}_3$ and $\mathcal{PTL}_3(C)$ the language obtained from $\mathcal{PTL}_3$ by adding the constants of C.

Lemma 4.6. *Let* T *a theory of* $\mathcal{PTL}_3$, $\varphi(x)$ *a formula of* $\mathcal{PTL}_3$ *and* $c \in C$. *We have:*

$$T \vdash \forall x\varphi(x) \text{ in } \mathcal{PTL}_3 \text{ iff } T \vdash \varphi(c) \text{ in } \mathcal{PTL}_3(C)$$

Definition 4.1. A consistent theory T of $\mathcal{PTL}_3(C)$ is said to be a *Henkin theory* if for any formula $\varphi(x)$ in $\mathcal{PTL}_3(C)$ there exists $c \in C$ such that $T \vdash \exists x\varphi(x) \to \varphi(c)$.

The following lemma will be the main tool in proving the properties of the canonical model (see the proof of Theorem 5.1).

Lemma 4.7. *Let* T *be a consistent theory in* $\mathcal{PTL}_3$. *Then, there is a set* C *of new constants and a Henkin theory* $\overline{T}$ *in* $\mathcal{PTL}_3(C)$ *such that* $T \subseteq \overline{T}$.

Proof. Let α be the cardinal of the language $\mathcal{PTL}_3$. Let C a set of new constants such that $|C| = \alpha$. Then $|\mathcal{PTL}_3(C)| = \alpha$. Let us consider an enumeration $\{c_\xi\}_{\xi<\alpha}$ of C such that $c_\beta \neq c_\gamma$ for all $\gamma < \beta < \alpha$. We can take an enumeration $\{\varphi_\xi(x_\xi)\}_{\xi<\alpha}$ of the formulas of $\mathcal{PTL}_3(C)$ with at most one free variable. We will construct by transfinite induction an increasing sequence of theories in $\mathcal{PTL}_3(C)$: $T = T_0 \subseteq T_1 \subseteq ... \subseteq T_\xi \subseteq ...$ with $\xi < \alpha$ and a sequence $\{d_\xi\}_{\xi<\alpha}$ of constants in C such that the following conditions hold:
- T_ξ is consistent in $\mathcal{PTL}_3(C)$.
- If $\xi = \mu + 1$ is an succesor ordinal then $T_\xi = T_\mu \cup \{\exists x_\mu\varphi_\mu(x_\mu) \to \varphi_\mu(d_\mu)\}$ where d_μ is the first constant in C which does not appear in T_μ.
- If ξ is a non-zero limit ordinal then $T_\xi = \bigcup_{\mu<\xi} T_\mu$.

Let's assume that T_μ is consistent and $T_{\mu+1} = T_\mu \cup \{\exists x_\mu\varphi_\mu(x_\mu) \to \varphi_\mu(d_\mu)\}$ is inconsistent in $\mathcal{PTL}_3(C)$.
By the Lemma 4.5 we obtain: $T_\mu \vdash \neg(\exists x_\mu\varphi_\mu(x_\mu) \to \varphi_\mu(d_\mu))^2$.

Since d_μ does not apear in T_μ, using Lemma 4.6 we get:
$T_\mu \vdash \forall y \neg(\exists x_\mu \varphi_\mu(x_\mu) \to \varphi_\mu(y))^2$, where y is a variable does not apear in $\varphi_\mu(x_\mu)$. Thus $T_\mu \vdash \neg\exists y(\exists x_\mu \varphi_\mu(x_\mu) \to \varphi_\mu(y))^2$ and by proposition 4.4 we get $T_\mu \vdash \neg(\exists y(\exists x_\mu \varphi_\mu(x_\mu) \to \varphi_\mu(y)))^2$. Using proposition 4.3 (2) we obtain $T_\mu \vdash \neg(\exists x_\mu \varphi_\mu(x_\mu) \to \exists y \varphi_\mu(y))^2$. From $T_\mu \vdash (\exists x_\mu \varphi_\mu(x_\mu) \to \exists y \varphi_\mu(y))^2$ we obtain a contradiction because T_μ is assumed consistent. Thus $T_{\mu+1}$ is consistent. If ξ is a non-zero limit ordinal and the theories $T_{\mu,\mu<\xi}$ are consistent then $T_\xi = \bigcup_{\mu<\xi} T_\mu$ is consistent. We denote $\overline{T} = \bigcup_{n<\alpha} T_n$.
Using the fact that $T_{\mu,\mu<\alpha}$ is consistent and $T_\mu \subset \overline{T}$, for all $\mu < \alpha$ we obtain that $\overline{T}$ is consistent.
Let's show that $\overline{T}$ is a Henkin theory. Let $\varphi(x) \in \mathcal{PTL}_3(C)$ with at most one free variable, hence there exists n with $\varphi(x) = \varphi_n(x_n)$.
Hence $\exists x \varphi(x) \to \varphi(e_n) = \exists x_n \varphi_n(x_n) \to \varphi_n(e_n) \in T_{n+1} \subseteq \overline{T}$, where e_n is the first constant in C does not appear in T_n.
We obtain that $\overline{T} \vdash \exists x \varphi(x) \to \varphi(e_n)$ and $\overline{T}$ is a Henkin theory. □

5. The semantic of $\mathcal{PTL}_3$ and the canonical model

This section concerns with the semantic of $\mathcal{PTL}_3$. We define the structures corresponding to $\mathcal{PTL}_3$ and the interpretation of formulas in these structures. This definition combines the Kripke semantics and three-valued semantics.

The contribution of this section is the construction of the canonical model associated with a maximal consistent Henkin theory. The idea of this construction is inspired from [8]. The main result of this section (Theorem 5.1) expresses the satisfiability of formulas in the canonical model by their position w.r.t. the maximal consistent Henkin theories.

A structure of the three-valued temporal predicate calculus has the form: $\mathcal{A} = \langle (K,R), \{\mathcal{A}_k, k \in K\}\rangle$ where K is a nonempty set, R is a binary relation on K and $\mathcal{A}_k$ is a three-valued structure of the form $\mathcal{A}_k = \langle A_k, \{P^{\mathcal{A}_k}\}_{P:predicate}, \{c^{\mathcal{A}_k}\}_{c:constant}\rangle$ where :

- A_k is a nonempty set called the universe of structure;
- $P^{\mathcal{A}_k} : A_k^n \to L_3$, where n is the arity of P, is the interpretation of the predicate P in $\mathcal{A}_k$.
- $c^{\mathcal{A}_k} \in A_k$ is the interpretation of c in $\mathcal{A}_k$.

Let $\mathcal{A}_k$ be a three-valued structure, $\varphi(x_1,..,x_n)$ be a formula and $a_1,..,a_n \in A_k$, $k \in K$. We will define inductively $\|\varphi(a_1,..,a_n)\|_k \in L_3$.

(a) If $\varphi(x_1,..,x_n) = P(x_1,..,x_n)$ where P is a n-ary predicat, $\|\varphi(a_1,..,a_n)\|_k = \|P(a_1,..,a_n)\|_k = P^{\mathcal{A}_k}(a_1,..,a_n)$.
(b) If $\varphi(x_1,..,x_n) = \neg\psi(x_1,..,x_n)$ then $\|\varphi(a_1,..,a_n)\|_k = \neg\|\psi(a_1,..,a_n)\|_k = 1 - \|\psi(a_1,..,a_n)\|_k$.
(c) If $\varphi(x_1,..,x_n) = \psi(x_1,..,x_n) \to \theta(x_1,..,x_n)$ then $\|\varphi(a_1,..,a_n)\|_k =$
$= \|\psi(a_1,..,a_n)\|_k \to \|\theta(a_1,..,a_n)\|_k =$
$= min\{1, 1 - \|\psi(a_1,..,a_n)\|_k + \|\theta(a_1,..,a_n)\|_k\}$
(d) If $\varphi(x_1,..,x_n) = \forall x \psi(x, x_1,..,x_n)$ then $\|\varphi(a_1,..,a_n)\|_k =$
$= \|\forall x \psi(x, a_1,..,a_n)\|_k = \bigwedge_{a \in A_k} \|\psi(a, a_1,..,a_n)\|_k$
(e) If $\varphi(x_1,..,x_n) = G\psi(x_1,..,x_n)$ then $\|\varphi(a_1,..,a_n)\|_k =$
$= \bigwedge\{\|\psi(a_1,..,a_n)\|_{k'} | kRk'\}$
(f) If $\varphi(x_1,..,x_n) = H\psi(x_1,..,x_n)$ then $\|\varphi(a_1,..,a_n)\|_k =$
$= \bigwedge\{\|\psi(a_1,..,a_n)\|_{k'} | k'Rk\}$

Definition 5.1. If $\mathcal{A} = \langle (K, R), \{\mathcal{A}_k, k \in K\}\rangle$ is a stucture, $k \in K$ and $a_1, .., a_n \in A_k$ we will denote:

$$\mathcal{A} \models_k \varphi(a_1, ..a_n) \iff \|\varphi(a_1, .., a_n)\|_k = 1$$

Let C be a set of new constants with the same cardinal number as the language $\mathcal{PTL}_3$ and Σ be a maximal consistent Henkin theory in the language $\mathcal{PTL}_3(C)$.
In what follows we shall define a structure nammed the *canonical model* of Σ.
Let $C_1, C_2, ..$, a denumerable sequence of sets of new constants such that

- $C \cap C_i = \emptyset$, for all i;
- $C_i \cap C_j = \emptyset$, for all $i \neq j$.

For any natural number $n \geq 1$, $\mathcal{PTL}_3(C \cup C_1 \cup ... \cup C_n)$ is the language obtained from $\mathcal{PTL}_3$ by adding the constants of $C \cup C_1 \cup ... \cup C_n$.

Let us denote by $\mathcal{K}$ the family of the sets Δ having the following properties:

(i) there exists a natural number $n \geq 1$ such that Δ is a maximal consistent Henkin theory of $\mathcal{PTL}_3(C \cup C_1 \cup ... \cup C_n)$.
(ii) $\Sigma \subseteq \Delta$.

We consider $A_\Delta = C \cup C_1 \cup ... \cup C_n$ where n is the smallest natural number with $\Delta \subseteq \mathcal{PTL}_3(C \cup C_1 \cup ... \cup C_n)$ and $\Delta \in \mathcal{K}$. We will organize each A_Δ, $\Delta \in \mathcal{K}$ like a three-valued structure for the language $\mathcal{PTL}_3$ with the following properties:

- If R is a three-valued predicate then the n-ary relation $R^{\mathcal{A}_\Delta}$ on A_Δ is defined: $R^{\mathcal{A}_\Delta} : A_\Delta^n \longrightarrow L_3$,

$$R_\Delta^{\mathcal{A}} = \begin{cases} 1, & \text{if } R(c_1, .., c_n) \in \Delta \\ 0, & \text{if } \neg R(c_1, .., c_n) \in \Delta \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

- If c is a constant simbol then its interpretation in $\mathcal{A}_\Delta$ is $c^{\mathcal{A}_\Delta} = c$.

We will define the binary relation $\prec$ on $\mathcal{K}$: if Δ , $\Gamma \in \mathcal{K}$ then we say that $\Gamma \prec \Delta$ if the conditions of the Lemma 4.2 hold.

We have defined a structure, $\mathcal{A} = \langle (\mathcal{K}, \prec), \{\mathcal{A}_\Delta, \Delta \in \mathcal{K}\}\rangle$ for the language $\mathcal{PTL}_3$.

Theorem 5.1. *For every formula $\varphi(x_1, .., x_n)$ of $\mathcal{PTL}_3$, for every $\Delta \in \mathcal{K}$ and for all $c_1, ..., c_n \in A_\Delta$ we have the equivalence:*

$$\mathcal{A} \models_\Delta \varphi(c_1, .., c_n) \iff \varphi(c_1, .., c_n) \in \Delta$$

Proof. We will prove by induction of $\varphi(x_1, ..., x_n)$.

(a) φ is an atomic formula, i.e $\varphi(x_1, .., x_n) = P(x_1, .., x_n)$ where P is a three-valued predicate. We have the equivalence:
$\mathcal{A} \models_\Delta P(c_1, ..., c_n) \iff \|P(c_1, .., c_n)\|_\Delta = 1$ (from Definition 5.1) $\iff P^{\mathcal{A}_\Delta}(c_1, .., c_n) = 1 \iff P(c_1, .., c_n) \in \Delta \iff \varphi(c_1, .., c_n) \in \Delta$.

(b) $\varphi(x_1, .., x_n) = \neg\psi(x_1, .., x_n)$, where for ψ the hypothesis is satisfied: $\mathcal{A} \models_\Delta \psi(c_1, .., c_n) \iff \psi(c_1, .., c_n) \in \Delta$. We have:
$\mathcal{A} \models_\Delta \varphi(c_1, .., c_n) \iff \mathcal{A} \models_\Delta \neg\psi(c_1, .., c_n) \iff \|\neg\psi(c_1, .., c_n)\|_\Delta = 1 \iff \|\psi(c_1, .., c_n)\|_\Delta = 0 \iff \mathcal{A} \not\models_\Delta \psi(c_1, .., c_n) \iff \psi(c_1, .., c_n) \notin \Delta \iff \neg\psi(c_1, .., c_n) \in \Delta$ (from the fact that Δ is maximal consistent), i.e $\varphi(c_1, .., c_n) \in \Delta$.

(c) $\varphi(x_1, .., x_n) = \psi(x_1, .., x_n) \rightarrow \theta(x_1, .., x_n)$.
($\Rightarrow$) Assume that $\mathcal{A} \models_\Delta \varphi(c_1, .., c_n)$. We have :
$\mathcal{A} \models_\Delta (\psi(c_1, ..., c_n) \rightarrow \theta(c_1, .., c_n)) \iff \|\psi(c_1, .., c_n)\|_\Delta \rightarrow \|\theta(c_1, .., c_n)\|_\Delta = 1 \iff min\{1, 1 - \|\psi(c_1, .., c_n)\|_\Delta + \|\theta(c_1, .., c_n)\|_\Delta\} = 1 \iff \|\psi(c_1, .., c_n)\|_\Delta \leq$

$\|\theta(c_1,..,c_n)\|_\Delta$.

We consider the following cases:

(1) Let $\|\psi(c_1,..,c_n)\|_\Delta = 0$. This implies that $\mathcal{A} \not\models_\Delta \psi(c_1,..,c_n) \iff \psi(c_1,..,c_n) \notin \Delta$(by inductive hypothesis) $\iff \neg\psi(c_1,..,c_n) \in \Delta$ (from Δ -maximal consistent). Using (t6) and Proposition 4.2 ((i) and (viii)) we obtain $\psi(c_1,..,c_n) \to \theta(c_1,..,c_n) \in \Delta$ hence $\varphi(c_1,..,c_n) \in \Delta$.

(2) Let $\|\psi(c_1,..,c_n)\|_\Delta = 1$.
Then it is necessary that $\|\theta(c_1,..,c_n)\|_\Delta = 1$. From this we have: $\mathcal{A} \models_\Delta \psi(c_1,..,c_n)$ and $\mathcal{A} \models_\Delta \theta(c_1,..,c_n)$ and by inductive hypothesis we obtain $\psi(c_1,..,c_n) \in \Delta$ and $\theta(c_1,..,c_n) \in \Delta$. Using (t1) and Proposition 4.2 ((i) and (viii)) we obtain that $\psi(c_1,..,c_n) \to \theta(c_1,..,c_n) \in \Delta$, hence $\varphi(c_1,..,c_n) \in \Delta$.

(3) Let $\|\psi(c_1,..,c_n)\|_\Delta = \frac{1}{2}$. Hence $\|\theta(c_1,..,c_n)\|_\Delta \in \{\frac{1}{2}, 1\}$.

(a) If $\|\theta(c_1,..,c_n)\|_\Delta = \frac{1}{2}$ then we have: $\mathcal{A} \not\models_\Delta \psi(c_1,..,c_n)$ and $\mathcal{A} \not\models_\Delta \theta(c_1,..,c_n)$ and by inductive hypothesis
$\psi(c_1,..,c_n) \notin \Delta$ and $\theta(c_1,..,c_n) \notin \Delta$, so from the fact that Δ is maximal consistent we obtain $\neg\psi(c_1,..,c_n) \in \Delta$ and $\neg\theta(c_1,..,c_n) \in \Delta$. Using (t1),(t4) and Proposition 4.2 ((i) and (viii)) it follows $\psi(c_1,..,c_n) \to \theta(c_1,..,c_n) \in \Delta$, so $\varphi(c_1,..,c_n) \in \Delta$.

(b) If $\|\theta(c_1,..,c_n)\|_\Delta = 1$ then the Definition 5.1 we obtain $\mathcal{A} \models_\Delta \theta(c_1,..,c_n)$ and by inductive hypothesis $\theta(c_1,..,c_n) \in \Delta$. Using (t1) and Proposition 4.2 ((i) and (viii)) we have $\psi(c_1,..,c_n) \to \theta(c_1,..,c_n) \in \Delta$, hence $\varphi(c_1,..,c_n) \in \Delta$.

($\Leftarrow$) Assume that $\varphi(c_1,..,c_n) \in \Delta$.
This is equivalent to $\psi(c_1,...,c_n) \to \theta(c_1,..,c_n) \in \Delta$. We consider the following cases:

(a) Assume that $\psi(c_1,..,c_n) \in \Delta$. Using Proposition 4.2(viii) we obtain $\theta(c_1,..,c_n) \in \Delta$. By the inductive hypothesis about ψ and θ it follows that $\mathcal{A} \models_\Delta \psi(c_1,..,c_n)$ and $\mathcal{A} \models_\Delta \theta(c_1,..,c_n)$, and using Definition 5.1 we have $\|\psi(c_1,..c_n)\|_\Delta = 1$ and $\|\theta(c_1,..,c_n)\|_\Delta = 1$.
Thus $\|\varphi(c_1,..,c_n)\|_\Delta = 1$, so $\mathcal{A} \models_\Delta \varphi(c_1,..,c_n)$.

(b) Assume that $\psi(c_1,..,c_n) \notin \Delta$. Because Δ is maximal consistent, it results that $\neg\psi(c_1,..,c_n) \in \Delta$ and by inductive hypothesis $\mathcal{A} \models_\Delta \neg\psi(c_1,...,c_n)$. By Definition 5.1 we have $\|\psi(c_1,..,c_n)\|_\Delta = 0$.
It follows that $\|\psi(c_1,..,c_n)\|_\Delta \to \|\theta(c_1,..,c_n)\|_\Delta = 1$, so
$\|\varphi(c_1,..,c_n)\|_\Delta = 1$, hence $\mathcal{A} \models_\Delta \varphi(c_1,..,c_n)$.

(d) $\varphi(x_1,..,x_n) = G\psi(x_1,..,x_n)$, where for ψ the hypothesis of induction is satisfied , i.e
$\mathcal{A} \models_\Delta \psi(c_1,..,c_n) \iff \psi(c_1,..,c_n) \in \Delta$.

($\Leftarrow$) Assume $\varphi(c_1,..,c_n) \in \Delta$, i.e $G\psi(c_1,..,c_n) \in \Delta$.
Let $\Delta \prec \Delta'$, and by definition of $\prec$ it results that $\psi(c_1,...,c_n) \in \Delta'$. By inductive hypothesis we obtain $\mathcal{A} \models_{\Delta'} \psi(c_1,..,c_n)$. So, $\mathcal{A} \models_{\Delta'} \psi(c_1,..,c_n)$ for all Δ' with $\Delta \prec \Delta'$ and we get
$\mathcal{A} \models_\Delta G\psi(c_1,..,c_n)$.

($\Rightarrow$) Assume $G\psi(c_1,..,c_n) \notin \Delta$, so $\neg G\psi(c_1,..,c_n) \in \Delta$.
Hence $F\neg\psi(c_1,..,c_n) = \neg G\psi(c_1,..,c_n) \in \Delta$.
By Lemma 4.3 and from $F\neg\psi(c_1,..,c_n) \in \Delta$ it follows that there exists $\Delta' \in K$, $\Delta \prec \Delta'$ and $\neg\psi(c_1,..,c_n) \in \Delta'$.
Because Δ' is maximal consistent we have $\psi(c_1,..,c_n) \notin \Delta'$, so $\mathcal{A} \not\models_{\Delta'}$

$\psi(c_1,..,c_n)$. We proved that there exists $\Delta \prec \Delta'$ with $\mathcal{A} \not\models_{\Delta'} \psi(c_1,..,c_n)$, hence $\mathcal{A} \not\models_\Delta G\psi(c_1,..,c_n)$.

(e) $\varphi(x_1,..,x_n) = \forall x\psi(x, x_1,..,x_n)$.

($\Rightarrow$) Suppose that $\mathcal{A} \models_\Delta \varphi(c_1,..,c_n)$, i.e $\mathcal{A} \models_\Delta \forall x\psi(x, c_1,..,c_n)$ and by Definition 5.1 we get $\|\forall x\psi(x, c_1,..,c_n)\|_\Delta = 1 \iff$ $\bigwedge_{a\in A_\Delta} \|\psi(a, c_1,..,c_n)\|_\Delta = 1$. It follows that for all $a \in A_\Delta$ $\|\psi(a, c_1,..,c_n)\|_\Delta = 1 \iff$ for all $a \in A_\Delta, \mathcal{A} \models_\Delta \psi(a, c_1,..,c_n)$. By the inductive hypothesis we obtain: for all $a \in A_\Delta, \psi(a, c_1,..,c_n) \in \Delta$ and using Lemma 4.6 it follows that $\forall x\psi(x, c_1,..,c_n) \in \Delta$, so $\varphi(c_1,..,c_n) \in \Delta$.

($\Leftarrow$) Suppose that $\varphi(c_1,..,c_n) \in \Delta$. We have : $\forall x\psi(x, c_1,..,c_n) \in \Delta \iff$ for all $a \in A_\Delta, \psi(a, c_1,..,c_n) \in \Delta$. By the inductive hypothesis we get for all $a \in A_\Delta$, $\mathcal{A} \models_\Delta \psi(a, c_1,..,c_n) \iff$ $\mathcal{A} \models_\Delta \forall x\psi(x, c_1,..,c_n)$, hence $\mathcal{A} \models_\Delta \varphi(c_1,..,c_n)$.

□

Remark 5.1. *Since* $\Sigma \subseteq \Delta$, *for each* $\Sigma \subseteq \Delta$, *it follows that* $\mathcal{A} \models_\Delta \varphi(c_1,..,c_n)$ *for all* $\Delta \in \mathcal{K}$, $\varphi(x_1,..,x_n)$ *in* $\mathcal{PTL}_3$ *and* $c_1,..,c_n \in A_\Delta$.

6. Completeness theorem

This section contains the main result of this paper: the strong completeness theorem for $\mathcal{PTL}_3$. The proof of the completeness theorem is based on Theorem 5.1.

Theorem 6.1. *If* Γ *is a set of formulas of* $\mathcal{PTL}_3$ *and* φ *is a formula of* $\mathcal{PTL}_3$ *then we have the following equivalence:*

$$\Gamma \vdash \varphi \iff \Gamma \models \varphi$$

Proof. ($\Leftarrow$) Assume that $\Gamma \nvdash \varphi$. We get $\Gamma \cup \{\neg\varphi\}$ is consistent and by Theorem 5.1 it follows that there exists a structure $\mathcal{A}$ such that $\mathcal{A} \models \Gamma \cup \{\neg\varphi\}$. It follows that $\mathcal{A} \models \Gamma$ and $\mathcal{A} \not\models \varphi$, hence $\Gamma \not\models \varphi$.

($\Rightarrow$) By the induction on the concept $\Gamma \vdash \varphi$.

(1) Suppose that $\varphi \in \Gamma$. Let $\mathcal{A}$ be a model for Γ. Then we have $\mathcal{A} \models \psi$, for all $\psi \in \Gamma$, hence $\mathcal{A} \models \varphi$. Because $\mathcal{A}$ is an arbitrary model for Γ, we obtain that $\Gamma \models \varphi$.

(2) Suppose that φ is an axiom. Let $\mathcal{A}$ be a model for Γ.

(a) Let $\varphi = G(\theta \to \psi) \to (G\theta \to G\psi)$. We must prove that $\mathcal{A} \models \varphi$, i.e $\mathcal{A} \models_k \varphi$, for all moments k. Suppose that $\mathcal{A} \models_k G(\theta \to \psi)$, and $\mathcal{A} \models_k G\theta$. By the definition of the concept $\mathcal{A} \models_k \varphi$ we obtain: $\mathcal{A} \models_{k'} \theta \to \psi$, $\mathcal{A} \models_{k'} \theta$ for all kRk', so $\mathcal{A} \models_{k'} \psi$ for all kRk'. It follows that $\mathcal{A} \models_k G\psi$.

(b) Let $\varphi = \psi \to GP\psi$. We must prove that $\mathcal{A} \models \psi$ implies $\mathcal{A} \models GP\psi$ i.e for all moments k, $\mathcal{A} \models_k \psi$ implies $\mathcal{A} \models_k GP\psi$.

Proving $\mathcal{A} \models_k GP\psi$ is equivalent to showing that for all kRk' $\mathcal{A} \models_{k'} P\psi$ or to the following : for all kRk' there exists $k''Rk'$ with $\mathcal{A} \models_{k''} \psi$. We assumed that $\mathcal{A} \models_k \psi$, hence, for $k'' = k$ we obtain $\mathcal{A} \models_k GP\psi$ i.e $\mathcal{A} \models_k \psi \to GP\psi$.

In a similar way we can prove the remaining axioms.

(3) (a) Suppose that φ was obtained by the rule $\dfrac{\Gamma \vdash \psi, \Gamma \vdash \psi \to \varphi}{\Gamma \vdash \varphi}$ and $\Gamma \models \psi$, $\Gamma \models \psi \to \varphi$.
Let $\mathcal{A}$ be a model for Γ. We have: $\mathcal{A} \models \psi$ and $\mathcal{A} \models \psi \to \varphi$, i.e. $\mathcal{A} \models_k \psi$, $\mathcal{A} \models_k \psi \to \varphi$ for all k, so $\mathcal{A} \models_k \varphi$, for all k. It follows that $\mathcal{A} \models \varphi$, i.e. $\Gamma \models \varphi$.
(b) Suppose that $\varphi(x_1, .., x_n) = \forall x \psi(x, x_1, .., x_n)$, obtained by the rule $\dfrac{\Gamma \vdash \psi}{\Gamma \vdash \forall x \psi}$ and $\Gamma \models \psi$. We want to prove that $\Gamma \models \varphi$. Let $\mathcal{A}$ be a model for Γ. It follows that $\mathcal{A} \models \psi$, i.e. $\mathcal{A} \models_k \psi$, for all moments k. This is equivalent with $\mathcal{A} \models_k \psi(a, a_1, .., a_n)$ for all $a \in \mathcal{A}_k$ and we have $\mathcal{A} \models_k \varphi$.
(c) Assume that $\varphi = G\psi$, obtained by the rule $\dfrac{\Gamma \vdash \psi}{\Gamma \vdash G\psi}$, and $\Gamma \models \psi$. We must prove that $\Gamma \models \varphi$, i.e. $\Gamma \models G\psi$. Let $\mathcal{A}$ be a model for Γ. Then $\mathcal{A} \models \psi \iff \mathcal{A} \models_k \psi$, for all k, so, we have $\mathcal{A} \models_{k'} \psi$ for all kRk', i.e. $\mathcal{A} \models_k G\psi$.

□

Theorem 6.2 (The completeness theorem)**.** *For any formula φ of $\mathcal{PTL}_3$ the following equivalence holds:*

$$\vdash \varphi \iff \models \varphi$$

Proof. By Theorem 6.1, with $\Gamma = \emptyset$. □

References

[1] V. Boicescu, A. Filipoiu, G. Georgescu and S. Rudeanu, *Lukasiewicz-Moisil algebras*, North-Holland, 1991.
[2] J. P. Burgess, *Basic tense logic*, In: Dov Gabbay and F. Guenthner, Eds., Handbook of philosophical logic, chapter II.2, 89-134. Reidel, 1984.
[3] D. Buşneag, *Categories of algebraic logic*, Ed. Academiei Române, 2006.
[4] B. F. Chellas, *Modal logic: An Introduction*, Cambridge University Press, 1980.
[5] C. Chiriţă, *Three-valued Temporal Logic*, Annals of the Bucharest University, **LIV**, 2005, 75-90.
[6] R. Cignoli, I. M .L. D'Ottaviano and D. Mundici, *Algebraic Foundations of Many-valued Reasoning*, Kluwer, 2000.
[7] D. Diaconescu, G. Georgescu, *Tense operators on MV-algebras and Lukasiewicz-Moisil algebras*, Fundamenta Informaticae XX (2007), 1-30.
[8] H. Goldberg, H. Leblanc and G. Weaver, *A strong completeness theorem for 3-valued logic*, Notre Dame Journal of Formal Logic, **15**, 325-332, 1974.
[9] R. Goldblatt, *Logics of Time and Computation*, CSLI Lecture Notes No. 7, 1992.
[10] P.Hajek, *Metamathematics of fuzzy logic*, Kluwer Acad.Publ., Dordrecht, 1998
[11] G. E. Hughes and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London, 2 edition, 1968.
[12] S. Kripke, *Semantical analysis of modal logic I: normal modal propositional calculi*, Zeitschrift fur math. Logik und Grundlagen der Math, **9**, 67-96, 1963.
[13] L. Leuştean, *Canonical Models and Filtrations in Three-valued Propositional Modal Logic*, Multi. Val. Logic, **8**(5-6), 577-590, 2002.
[14] J. Łukasiewicz, *On three-valued logic (Polish)*, Ruch Filozoficzny, 5, 160-171.
[15] Gr.C.Moisil, *Recherches sur les logiques non-chrysippiennes*, Ann.Sci.Univ.Jassy, 26(1940) 431-466,
[16] Gr.C.Moisil, *Essais sur les logiques non-chrysippiennes*, Ed.Academiei, Bucarest, 1972
[17] P. Ostermann, *Many-valued propositional calculi*, Zeitschrift fur math. Logik und Grundlagen der Math., **34**, 345-354, 1988.

[18] H. Rasiowa, *An algebraic approach to non-classical logics*, North-Holland Publ., Amsterdam, Polish Scientific Publ., Warszawa, 1974.

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