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# Similarity Lukasiewicz-Moisil algebras

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Abstract. The aim of this paper is to introduce the notion of similarity $n$-valued Lukasiewicz-Moisil algebra and to present a general theory of similarity which generalize the case of $LM_n$-algebras and also of $MV$-algebras.

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## 1. Preliminaries

Gr. C. Moisil introduced in 1940 the 3-valued Łukasiewicz algebras as algebraic models for the 3-valued Łukasiewicz logics. In 1941 he defined the $n$-valued Łukasiewicz algebras. A. Rose showed that Łukasiewicz implications cannot be defined in $n$-valued Łukasiewicz algebras for $n \geq 5$, so the latter do not correspond to the $n$-valued Łukasiewicz logics. Thus the structures created by Moisil led to other logical systems, so called Moisil logics. Now these algebras are known under the name of Łukasiewicz-Moisil algebras ([3]).

This paper represents a part of my P.h. Thesis, *Contribution to the study of $LM_n$-algebras*, sustained in January 2007 at the Faculty of Mathematic and Computer Science, Bucharest (see [7]).

Basic definition and results useful for understanding the subsequent sections are recalled in **Section 2** of this paper, following especially the monograph [3].

In **Section 3** I propose a generalization of the variety of $LM_n$-algebras by adding a binary operator playing the role of similarity. I mention here that I was inspired by $MV$-algebras (see [11]) to study the notion of *similarity* on an $LM_n$-algebra, so that my results are very close to those of $MV$-algebras.

In **Section 4** I present the same theory as in **Section 3** but for another binary operation called "*strong similarity*", starting from another implication that makes an $LM_n$-algebra to be a Heyting algebra.

Heyting algebras constitute one of the fundamental structures generated by mathematical logic. Therefore the problem of investigating the relationships between Łukasiewicz-Moisil algebras and Heyting algebras is a natural one. The fact that every Moisil algebra is a Heyting algebra was first proved by Moisil [1942], [1963] for three-valued algebras, then generalized to the $n$-valued case (Moisil [1965]).

In **Section 5** I present a generalization of the theory of similarity and strong similarity $LM_n$-algebras from the previous sections and also, of similarity $MV$-algebra from [11].

## 2. Definitions. Examples. Basic results

Let $n$ be an integer, $n \geq 2$.

**Definition 2.1.** *(*[3]*)* An *n-valued Lukasiewicz-Moisil algebra*(shortly, $LM_n$-algebra) is an algebra $\mathcal{L} = (L, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ of type (2,2,1,0,0,$\{1\}_{1\leq i\leq n-1}$) satisfying the following conditions:
$(a_1)$ $(L, \wedge, \vee, N, 0, 1)$ is a De Morgan algebra,
$(a_2)$ $\varphi_1, ..., \varphi_{n-1} : L \to L$ are bounded lattice morphisms such that for every $x, y \in L$:
$(a_3)$ $\varphi_i(x) \vee N\varphi_i(x) = 1$ for every $i = 1, ..., n-1$,
$(a_4)$ $\varphi_i(x) \wedge N\varphi_i(x) = 0$ for every $i = 1, ..., n-1$,
$(a_5)$ $\varphi_i\varphi_j(x) = \varphi_j(x)$ for every $i, j = 1, ..., n-1$,
$(a_6)$ $\varphi_i(Nx) = N\varphi_j(x)$ for every $i, j = 1, ..., n-1$ with $i + j = n$,
$(a_7)$ $\varphi_1(x) \leq \varphi_2(x) \leq ... \leq \varphi_{n-1}(x)$,
$(a_8)$ If $\varphi_i(x) = \varphi_i(y)$ for every $i = 1, ..., n-1$, then $x = y$.

The relation $(a_8)$ is called *the determination principle.* The following relations are consequences of the determination principle:
$(c_1)$ If $x, y \in L$, then $x \leq y$ iff $\varphi_i(x) \leq \varphi_i(y)$ for all $i = 1, ..., n-1$,
$(c_2)$ $\varphi_1(x) \leq x \leq \varphi_{n-1}(x)$ for all $x \in L$.

An $LM_n$-algebra $\mathcal{L} = (L, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ will be denoted in the rest of this paper by its universe $L$.

**Remark 2.1.** The endomorphisms $\{\varphi_i\}_{1\leq i\leq n-1}$ are called *chrysippian endomor − phisms*.

**Examples:**

E$_1$. Let $L_n = \{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\}$.We define $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $Nx = 1 - x$ $(N(\frac{j}{n-1}) = \frac{n-1-j}{n-1})$ and $\varphi_i : L_n \to L_n, \varphi_i(\frac{j}{n-1}) = 0$ if $i + j < n$ and 1 if $i + j \geq n$, for $i, j = 1, ..., n-1$.
Then $(L_n, \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ is an $LM_n$-algebra.

E$_2$. If $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra, then $(B, \wedge, \vee, ', 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ is an $LM_n$-algebra, where $\varphi_i = 1_B$ for every $i \in \{1, ..., n-1\}$.

E$_3$. Let $(B, \vee, \wedge, ', 0, 1)$ a Boolean algebra and $D(B) = \{(x_1, ..., x_{n-1}) \in B^{n-1} : x_1 \leq ... \leq x_{n-1}\}$.
We define pointwise the infimum and the supremum, $N(x_1, ..., x_{n-1}) = (x'_{n-1},...,x'_1)$ and $\varphi_i(x_1, ..., x_{n-1}) = (x_i, ..., x_i)$ for all $i = 1, ..., n-1$.
Then $(D(B), \wedge, \vee, N, 0, 1, \{\varphi_i\}_{1\leq i\leq n-1})$ is an $LM_n$-algebra.

The set of all *complemented elements* of the bounded lattice $(L, \wedge, \vee, 0, 1)$ is denoted by $C(L)$ and it is called the *center of* $L$; it is easy to see that $(C(L), \vee, \wedge, N, 0, 1)$ is a Boolean algebra.

**Lemma 2.1.** *(*[3]*) Let* $L$ *be an* $LM_n$*-algebra.The following are equivalent:*
*(i)* $e \in C(L)$,
*(ii) there are* $i \in \{1, ..., n-1\}$ *and* $x \in L$ *such that* $e = \varphi_i(x)$,
*(iii) there is* $i \in \{1, ..., n-1\}$ *such that* $e = \varphi_i(e)$,
*(iv)* $e = \varphi_i(e)$ *for every* $i = 1, ..., n-1$,
*(v)* $\varphi_i(e) = \varphi_j(e)$ *for every* $i, j = 1, ..., n-1$.

**Remark 2.2.** If $x \in L$, then $\varphi_i(x) \in C(L)$ for every $i = 1, ..., n-1$.

**Lemma 2.2.** *(*[3]*) Let* $L$ *be an* $LM_n$*-algebra. The following are equivalent:*
*(i)* $e \in C(L)$,
*(ii)* $N\ e \in C(L)$,
*(iii)* $e \wedge Ne = 0$,
*(iv)* $e \vee Ne = 1$.

**Lemma 2.3.** *If $L$ is an $LM_n$-algebra, then for every $x \in L$:*
$(c_3)$ $x \wedge \varphi_1(Nx) = x \wedge N\varphi_{n-1}(x) = 0,$
$(c_4)$ $x \vee N\varphi_1(x) = x \vee \varphi_{n-1}(Nx) = 1.$

**Proof.** $(c_3)$. For every $x \in L$ we have $x \leq \varphi_{n-1}(x)$, so

$$x \wedge \varphi_1(Nx) = x \wedge N\varphi_{n-1}(x) \leq \varphi_{n-1}(x) \wedge N\varphi_{n-1}(x) = 0(\text{by } a_4),$$

hence $x \wedge \varphi_1(Nx) = 0$.

$(c_4)$. We have $x \geq \varphi_1(x)$(by $a_7$), so $x \vee N\varphi_1(x) \geq \varphi_1(x) \vee N\varphi_1(x) = 1$, hence $x \vee N\varphi_1(x) = 1$. ■

**Theorem 2.1.** *([1]) For an $LM_n$-algebra $L$ (with $0 \neq 1$), the following are equivalent:*
$(i)$ *$C(L) = \{0, 1\}$,*
$(ii)$ *$L$ is a chain,*
$(iii)$ *$L$ is subdirectly irreducible.*

**Corollary 2.1.** *([3]) Every chain which is an $LM_n$-algebra is finite.*

**Definition 2.2.** *([3])* Let $L$ and $L'$ be $LM_n$-algebras. A function $f : L \to L'$ is a *morphism* of $LM_n$-algebras iff it satisfies the following conditions, for every $x, y \in L$ :
$(i)$ $f(x \vee y) = f(x) \vee f(y)$,
$(ii)$ $f(x \wedge y) = f(x) \wedge f(y)$,
$(iii)$ $f(0) = 0, f(1) = 1$,
$(iv)$ $f(\varphi_i(x)) = \varphi_i(f(x))$ for every $i = 1, ..., n-1$.

**Remark 2.3.** It follows from $(a_6)$ and $(a_8)$ that

$$f(Nx) = Nf(x)$$

for every $x \in L$ (see [3], Remark 3.1.29 and the subsequent remark (3.1.52)).

**Definition 2.3.** *([3])* A nonempty subset $F \subseteq L$ is called an *n-filter* if $F$ is a lattice filter of $L$ and if $x \in F$ implies $\varphi_1(x) \in F$.

**Remark 2.4.** (i). From $(a_7)$ it follows that if $F \subseteq L$ is an $n$-filter and $x \in F$, then $\varphi_i(x) \in F$ for every $i \in \{1, ..., n-1\}$.

(ii). It is obvious that $x \in F$ iff $\varphi_1(x) \in F$.

**Definition 2.4.** A proper $n$-filter $F$ of $L$ is said to be *prime* if $F$ is prime as a lattice filter, i.e. for any $x, y \in L$, the condition $x \vee y \in F$ implies $x \in F$ or $y \in F$(see [3], p.33).

**Definition 2.5.** By a *maximal (minimal) n-filter* is meant a maximal (minimal) element in the family of proper $n$-filters ordered by set inclusion.

By a *maximal (minimal) prime n-filter* is meant a maximal (minimal) element in the family of prime $n$-filters ordered by set inclusion.

**Definition 2.6.** *([3])* A *congruence* of an $LM_n$-algebra $L$ is an equivalence relation of $L$ compatible with the operations $\wedge, \vee, N, \varphi_i$, for every $i = 1, ..., n-1$.

**Proposition 2.1.** *([3]) For an equivalence relation $\theta$ of an $LM_n$-algebra $L$, the following conditions are equivalent:*
$(i)$ *$\theta$ is a congruence of $L$,*
$(ii)$ *$\theta$ is compatible with $\wedge, \vee, \varphi_i$, for every $i = 1, ..., n-1$.*

**Theorem 2.2.** *([3],p.126) The category of $LM_n$-algebras is an equational class.*

If $L$ is an $LM_n$-algebra and $F$ is an $n$-filter, I consider the relation:

$$x \ mod\ F\ y \text{ iff there exists } f \in F \text{ such that } x \wedge f = y \wedge f.$$

**Remark 2.5.** Theorem 5.1.13(p. 251) from [3] proves that *mod F* is a congruence on $L$ and Proposition 5.1.31(p. 259), from the same book, shows that

$$x \ mod\ F\ y \text{ iff } \bigwedge_{i=1}^{n-1} [(N\varphi_i(x) \vee \varphi_i(y)) \wedge (\varphi_i(x) \vee N\varphi_i(y))] \in F.$$

In the following I denote by $L/F$ the quotient $LM_n$-algebra $L/mod\ F$.

**Theorem 2.3.** *([3])(Representation theorem of Moisil) Every $LM_n$-algebra can be embedded in a direct product of copies of the canonical $LM_n$-algebra $L_n$.*

**Corollary 2.2.** *([3]) Every $LM_n$-algebra is a subdirect product of subalgebras of the canonical $LM_n$-algebra $L_n$:*

*If $Spec_n(L)$ is the set of all prime $n$-filters of $L$, then $L$ is a subdirect product (as an $LM_n$-algebra) of the family $\{L/F : F \in Spec_n(L)\}$, where $i : L \to \prod_{F\in Spec_n(L)} L/F$ is the canonical representation (see [3], Proposition 6.1.5 and Theorem 5.2.3).*

**Corollary 2.3.** *Any identity valid in $LM_n$ chains holds in every $LM_n$-algebras.*

## 3. Similarity LM$_n$-algebra

Let $F$ be an $n$-filter of $L$ and $\sim_F$ the relation defined on $L$ as follows:

$x \sim_F y$ iff there exists $f \in F$ such that $\varphi_i(x) \wedge f = \varphi_i(y) \wedge f$, for every $i = 1, ..., n-1$.

**Definition 3.1.** For $x, y \in L$, I consider the implication

$$x \longrightarrow y = \bigwedge_{i=1}^{n-1} (N\varphi_i(x) \vee \varphi_i(y)).$$

Also, I denote

$$x \longleftrightarrow y = (x \longrightarrow y) \wedge (y \longrightarrow x).$$

So,

$$x \longleftrightarrow y = \bigwedge_{i=1}^{n-1} [(N\varphi_i(x) \vee \varphi_i(y)) \wedge (\varphi_i(x) \vee N\varphi_i(y))],$$

and it is obvious that $x \longrightarrow y, x \longleftrightarrow y \in C(L)$.

**Proposition 3.1.** *([3], Proposition 5.1.31, p.259) The relation $\sim_F$ is a congruence of $L$ and coincides with mod $F$.*

**Remark 3.1.** According to Remark 2.5 we have that:

$$x \sim_F y \text{ iff } x \ mod\ F\ y \text{ iff } x \longleftrightarrow y \in F.$$

**Proposition 3.2.** *For every $x, y \in L$ we have that $x \leq y$ iff $x \longrightarrow y = 1$.*

**Proof.** We have:

$$x \leq y \text{ iff } \varphi_i(x) \leq \varphi_i(y) \text{ iff } N\varphi_i(y) \leq N\varphi_i(x),\ i = 1, ..., n-1$$

iff

$$\bigwedge_{i=1}^{n-1} (N\varphi_i(x) \vee \varphi_i(y)) = 1 \text{ iff } x \longrightarrow y = 1.$$

$\blacksquare$

**Corollary 3.1.** *In every $LM_n$-algebra $L$, $x = y$ iff $x \longleftrightarrow y = 1$.*

**Remark 3.2.**

(*i*) If $x \leq y$ then $x \longleftrightarrow y = \bigwedge_{i=1}^{n-1} (\varphi_i(x) \vee N\varphi_i(y))$,

(*ii*) If $x, y \in C(L)$, then $x \longleftrightarrow y = (Nx \vee y) \wedge (x \vee Ny)$ and if $x \leq y$ then $x \longleftrightarrow y = x \vee Ny$,

(*iii*) If $x, y \in \{0, 1\}$ then $x \longleftrightarrow y = \{ \begin{array}{l} 1, x = y \\ 0, x \neq y. \end{array}$

**Lemma 3.1.** *In an $LM_n$-algebra $L$ we have:*

(1) $1 \longrightarrow x = x \longleftrightarrow 1 = \varphi_1(x)$,

(2) $x \longleftrightarrow y \leq (x \longleftrightarrow z) \longleftrightarrow (y \longleftrightarrow z)$,

(3) $x \longleftrightarrow y \leq (x \wedge z) \longleftrightarrow (y \wedge z)$,

(4) $(x \longrightarrow y) \vee (y \longrightarrow x) = 1$.

**Proof.** (1) We have that

$$1 \longrightarrow x = \bigwedge_{i=1}^{n-1} (N\varphi_i(1) \vee \varphi_i(x)) = \bigwedge_{i=1}^{n-1} (0 \vee \varphi_i(x)) = \bigwedge_{i=1}^{n-1} \varphi_i(x) = \varphi_1(x).$$

By Proposition 3.2 we have that $x \longrightarrow 1 = 1$, therefore $x \longleftrightarrow 1 = \varphi_1(x)$.

(2) According to Remark 3.2,(ii), the relation is equivalent with

$$x \longleftrightarrow y \leq [(x \longleftrightarrow z) \vee N(y \longleftrightarrow z)] \wedge [N(x \longleftrightarrow z) \vee (y \longleftrightarrow z)]$$

so it is equivalent with the system of two inequalities:

$$x \longleftrightarrow y \leq (x \longleftrightarrow z) \vee N(y \longleftrightarrow z)$$
$$x \longleftrightarrow y \leq N(x \longleftrightarrow z) \vee (y \longleftrightarrow z).$$

Since the operations $\longleftrightarrow$ and $\vee$ are commutative, it suffices to prove the first inequality. We will use two well-known relations from boolean calculus:
$(a \wedge Nb) \vee (Na \wedge b) = (a \vee b) \wedge (Na \vee Nb)$ and $(x \wedge a) \vee (Nx \wedge b) \geq a \wedge b$.

$$\begin{aligned} \text{So, } (x \longleftrightarrow z) \vee N(y \longleftrightarrow z) &= \bigwedge_{i=1}^{n-1} [(N\varphi_i(x) \vee \varphi_i(z)) \wedge (\varphi_i(x) \vee N\varphi_i(z))] \vee \\ &\vee \bigwedge_{j=1}^{n-1} [(\varphi_j(y) \wedge N\varphi_j(z)) \vee (N\varphi_j(y) \wedge \varphi_j(z))] \\ &\geq \bigwedge_{i=1}^{n-1} [(N\varphi_i(x) \wedge N\varphi_i(z)) \vee (\varphi_i(x) \wedge \varphi_i(z)) \vee (\varphi_i(y) \wedge N\varphi_i(z)) \vee \\ &\vee (N\varphi_i(y) \wedge \varphi_i(z))] \\ &= \bigwedge_{i=1}^{n-1} [((N\varphi_i(x) \vee \varphi_i(y)) \wedge N\varphi_i(z)) \vee (\varphi_i(z) \wedge (\varphi_i(x) \vee N\varphi_i(y)))] \\ &\geq \bigwedge_{i=1}^{n-1} [(N\varphi_i(x) \vee \varphi_i(y)) \wedge (\varphi_i(x) \vee N\varphi_i(y))] = x \longleftrightarrow y. \end{aligned}$$

To prove (3) it suffices to consider that $L$ is an $LM_n$-chain (according to Corollary 2.3 ).

(3) Without loss of generality, we can suppose $x \leq y$. In this case $x \wedge z \leq y \wedge z$, hence:

$$(x \wedge z) \longleftrightarrow (y \wedge z) = \bigwedge_{i=1}^{n-1} [\varphi_i(x \wedge z) \vee N\varphi_i(y \wedge z)] = \bigwedge_{i=1}^{n-1} [(\varphi_i(x) \wedge \varphi_i(z)) \vee$$

$$N(\varphi_i(y) \wedge \varphi_i(z))] = \bigwedge_{i=1}^{n-1} [(\varphi_i(x) \wedge \varphi_i(z)) \vee (N\varphi_i(y) \vee N\varphi_i(z))] =$$

$$\begin{aligned} &= \bigwedge_{i=1}^{n-1} [(\varphi_i(x) \vee N\varphi_i(y) \vee N\varphi_i(z)) \wedge (\varphi_i(z) \vee N\varphi_i(y) \vee N\varphi_i(z))] \\ &= \bigwedge_{i=1}^{n-1} [(\varphi_i(x) \vee N\varphi_i(y) \vee N\varphi_i(z)) \wedge (1 \vee N\varphi_i(y))] \\ &= \bigwedge_{i=1}^{n-1} (\varphi_i(x) \vee N\varphi_i(y) \vee N\varphi_i(z)) \geq \bigwedge_{i=1}^{n-1} (\varphi_i(x) \vee N\varphi_i(y)) \\ &= x \longleftrightarrow y. \end{aligned}$$

(4) According to Corollary 2.3, it suffices to consider that $L$ is a chain. So $x \leq y$ or $y \leq x$, hence $x \longrightarrow y = 1$ or $y \longrightarrow x = 1$. Therefore $(x \longrightarrow y) \vee (y \longrightarrow x) = 1$. ■

**Definition 3.2.** A *similarity $LM_n$-algebra* is a pair $(L, S)$ where $L$ is an $LM_n$-algebra and $S : L \times L \to L$ is a binary operation on $L$ such that the following properties hold for every $x, y, z \in L$ :

$(S_1)$ $S(x, x) = 1$,
$(S_2)$ $S(x, y) = S(y, x)$,
$(S_3)$ $S(x, y) \wedge S(y, z) \leq S(x, z)$,
$(S_4)$ $x \wedge S(x, y) \leq \varphi_{n-1}(y)$,
$(S_5)$ $S(x \longleftrightarrow y, 1) \leq S(x, z) \longleftrightarrow S(y, z)$.

An operator $S$ which satisfies $S_1$-$S_5$ will be called a *similarity operation* on $L$ (or, simply, a *similarity* on $L$).

If $S$ and $T$ are two similarities on $L$, I define

$$S \leq T \text{ iff } S(x, y) \leq T(x, y) \text{ for every } x, y \in L.$$

The notions of subalgebra and homomorphism are defined as usual.

**Remark 3.3.** From $(S_5)$ and Lemma 3.1,(1) it follows that

$$S(x \longleftrightarrow y, 1) \leq S(x, y) \longleftrightarrow S(y, y) = S(x, y) \longleftrightarrow 1 = \varphi_1(S(x, y)) \leq S(x, y)$$

for every $x, y \in L$.

**Examples:**

**1.** On every $LM_n$-algebra $L$, the operation $E(x, y) = x \longleftrightarrow y$ is a similarity.
Indeed, $E(x, x) = 1$ and $E(x, y) = E(y, x)$.
For $(S_3$ ) we will use the well-known boolean equality

$$(x \vee Ny) \wedge (Nx \vee y) = (x \wedge y) \vee (Nx \wedge Ny).$$

So,

$$\begin{aligned}(x \longleftrightarrow y)\wedge(y\longleftrightarrow z) &= \bigwedge_{i=1}^{n-1}[(N\varphi_i(x)\vee\varphi_i(y))\wedge(\varphi_i(x)\vee N\varphi_i(y))\wedge \\ &\quad\wedge(N\varphi_i(y)\vee\varphi_i(z))\wedge(\varphi_i(y)\vee N\varphi_i(z))] = \\ &= \bigwedge_{i=1}^{n-1}[(N\varphi_i(y)\vee(\varphi_i(x)\wedge\varphi_i(z)))\wedge(\varphi_i(y)\vee(N\varphi_i(x)\wedge N\varphi_i(z))] \\ &= \bigwedge_{i=1}^{n-1}[(N\varphi_i(x)\wedge N\varphi_i(y)\wedge N\varphi_i(z)))\vee(\varphi_i(x)\wedge\varphi_i(y)\wedge\varphi_i(z))] \\ &\leq \bigwedge_{i=1}^{n-1}[(N\varphi_i(x)\wedge N\varphi_i(z))\vee(\varphi_i(x)\wedge\varphi_i(z))] = x\longleftrightarrow z.\end{aligned}$$

Since $E(x,y)\leq N\varphi_{n-1}(x)\vee\varphi_{n-1}(y)$, it follows that

$$\begin{aligned}x\wedge E(x,y) &\leq x\wedge[N\varphi_{n-1}(x)\vee\varphi_{n-1}(y)] = [x\wedge N\varphi_{n-1}(x)]\vee[x\wedge\varphi_{n-1}(y)] \\ &= 0\vee[x\wedge\varphi_{n-1}(y)]\leq\varphi_{n-1}(y).\end{aligned}$$

From Lemma 3.1, (1), it follows that

$$E(x\longleftrightarrow y,1) = (x\longleftrightarrow y)\longleftrightarrow 1 = \varphi_1(x\longleftrightarrow y) = x\longleftrightarrow y,$$

hence $E(x\longleftrightarrow y,1)\leq E(x,z)\longleftrightarrow E(y,z)$ (by Lemma 3.1,(2)).

**2.** On every $LM_n$-algebra $L$, the operation $\Delta : L\times L\to L$ defined by

$$\Delta(x,y) = \{\begin{array}{c}1, x=y\\ 0, x\neq y\end{array}, \text{ for any } x,y\in L,$$

is also a similarity on $L$.

It is obvious that $\Delta(x,x)=1$ and $\Delta(x,y)=\Delta(y,x)$.

Also, $\Delta(x,y)\wedge\Delta(y,z) = \{\begin{array}{c}1, x=y=z\\ 0, \text{otherwise}\end{array} \leq \Delta(x,z)$.

It is easy to see that $x\wedge\Delta(x,y) = \{\begin{array}{c}x, x=y\\ 0, x\neq y\end{array} \leq \varphi_{n-1}(y)$.

For $(S_5)$ we have that $\Delta(x\longleftrightarrow y,1) = \{\begin{array}{c}1, x\longleftrightarrow y=1\\ 0, x\longleftrightarrow y\neq 1\end{array}$.

Therefore, $\Delta(x\longleftrightarrow y,1) = \{\begin{array}{c}1, x=y\\ 0, x\neq y\end{array}$ (according to Corollary 3.1). Also,

$$\Delta(x,z)\longleftrightarrow\Delta(y,z) = \{\begin{array}{cc}1, & \Delta(x,z)=\Delta(y,z)\\ 0, & \text{otherwise}\end{array}$$

(by Remark 3.2,(*iii*)).

Therefore $\Delta(x\longleftrightarrow y,1)\leq\Delta(x,z)\longleftrightarrow\Delta(y,z)$.

**Proposition 3.3.** *For any similarity $S$ on $L$ we have that:*

(1) $\Delta\leq S$,

(2) $E\leq S$ *iff* $\varphi_1(x)\leq S(x,1)$, *for every* $x\in L$.

**Proof.** (1). Obvious.

(2). " $\Rightarrow$ ". We have that $\varphi_1(x) = x\longleftrightarrow 1 = E(x,1)\leq S(x,1)$.

" $\Leftarrow$ ". $E(x,y) = x\longleftrightarrow y = \varphi_1(x\longleftrightarrow y)\leq S(x\longleftrightarrow y,1)\leq S(x,y)$(by Remark 3.3). Therefore $E\leq S$. ■

**Definition 3.3.** If $(L,S)$ is a similarity $LM_n$-algebra, then $F \subseteq L$ is an $S-filter$ if $F$ is an $n$-filter of $L$ and $S(x,y) \in F$ for every $x,y \in F$.

**Proposition 3.4.** *Let $(L,S)$ be a similarity $LM_n$-algebra and $F$ an n-filter. Then $F$ is an S-filter iff $S(x,1) \in F$ for every $x \in F$.*

**Proof.** " $\Rightarrow$ ". Because $1 \in F$ it follows that $S(x,1) \in F$ for every $x \in F$.

" $\Leftarrow$ ". If $x,y \in F$, then $x \longleftrightarrow y \in F$(by Remarks 2.4 and 3.1), hence $S(x \longleftrightarrow y, 1) \in F$.

But, $S(x \longleftrightarrow y,1) \leq S(x,y)$ (by Remark 3.3), hence $S(x,y) \in F$.

Therefore, $F$ is an $S$-filter. ■

**Proposition 3.5.** *If $(L,S)$ is a similarity $LM_n$-algebra and $F \subseteq L$ is an $S$-filter, then $\sim_F$ is a congruence with respect to the similarity $LM_n$-algebra $(L,S)$.*

**Proof.** We only have to prove that $\sim_F$ is compatible with $S$. Suppose that $x_1, x_2, y_1, y_2 \in L$ such that $x_1 \sim_F x_2$ and $y_1 \sim_F y_2$.

It follows that $x_1 \longleftrightarrow x_2 \in F$ and $y_1 \longleftrightarrow y_2 \in F$. Hence $S(x_1 \longleftrightarrow x_2, 1) \in F$ and $S(y_1 \longleftrightarrow y_2, 1) \in F$.

But

$$\begin{aligned} S(x_1 \longleftrightarrow x_2, 1) &\leq S(x_1,y_1) \longleftrightarrow S(y_1,x_2) \text{ and} \\ S(y_1 \longleftrightarrow y_2, 1) &\leq S(y_1,x_2) \longleftrightarrow S(x_2,y_2), \end{aligned}$$

hence $S(x_1,y_1) \longleftrightarrow S(y_1,x_2) \in F$ and $S(y_1,x_2) \longleftrightarrow S(x_2,y_2) \in F$. We have that, $S(x_1,y_1) \sim_F S(y_1,x_2)$ and $S(y_1,x_2) \sim_F S(x_2,y_2)$, hence $S(x_1,y_1) \sim_F S(x_2,y_2)$. ■

Because the class of $LM_n$-algebras is equational and any similarity is an algebraic function, it follows that the similarity $LM_n$-algebras form an equational class, hence:

**Remark 3.4.** If $(L,S)$ is a similarity $LM_n$-algebra and $F \subseteq L$ is an $S$-filter of $L$, if we denote the quotient $LM_n$-algebra $L/\sim_F$ by $L/F$, then $L/F$ has a canonical structure of similarity $LM_n$-algebra, where the similarity $S_F : L/F \times L/F \to L/F$ is defined by $S_F(x/F, y/F) := S(x,y)/F$, for every $x,y \in L$, where $x/F$ is the congruence class of $x$ with respect to $\sim_F$ .

The canonical surjection $x \longmapsto x/F$ is a similarity $LM_n$-algebra homomorphism.

**Definition 3.4.** A similarity $LM_n$-algebra is called *representable* if it is a subdirect product of similarity $LM_n$-chains.

**Lemma 3.2.** *If $x \vee y = 1$ then $x \longrightarrow y = \varphi_1(y)$ and $y \longrightarrow x = \varphi_1(x)$.*

**Proof.** If $x \vee y = 1$ then $\varphi_i(x) \vee \varphi_i(y) = 1$ for every $i = 1, ..., n-1$. Then, for $i \in \{1, ..., n-1\}$ we have:

$$\begin{aligned} N\,\varphi_i(x) &= N\varphi_i(x) \wedge 1 = N\,\varphi_i(x) \wedge (\,\varphi_i(x) \vee \,\varphi_i(y)) \\ &= (N\,\varphi_i(x) \wedge \,\varphi_i(x)) \vee (N\,\varphi_i(x) \wedge \,\varphi_i(y)) = \\ &= 0 \vee (N\,\varphi_i(x) \wedge \,\varphi_i(y)) = N\,\varphi_i(x) \wedge \,\varphi_i(y). \end{aligned}$$

Hence $N\,\varphi_i(x) \leq \,\varphi_i(y)$ for every $i = 1, ..., n-1$, so, $x \longrightarrow y = \bigwedge_{i=1}^{n-1} (N\,\varphi_i(x) \vee \varphi_i(y)) = \bigwedge_{i=1}^{n-1} \varphi_i(y) = \varphi_1(y)$. Therefore it follows that $y \longrightarrow x = \varphi_1(x)$. ■

**Theorem 3.1.** *For a similarity $LM_n$-algebra $(L,S)$, the following are equivalent:*

(1) *$(L,S)$ is representable,*

(2) *$S(x \longrightarrow y, 1) \vee (y \longrightarrow x) = 1$ for every $x,y \in L$,*

(3) $x \vee y = 1$ *implies* $x \vee S(y, 1) = 1$,
(4) *Any prime n-filter is an S-filter.*

**Proof.**(1) $\Rightarrow$ (2). Because $(L, S)$ is representable, we can consider $x \leq y$ or $y \leq x$.
If $x \leq y$ then $x \longrightarrow y = 1$, hence $S(x \longrightarrow y, 1) \vee (y \longrightarrow x) = 1$.
If $y \leq x$ then $y \longrightarrow x = 1$, hence $S(x \longrightarrow y, 1) \vee (y \longrightarrow x) = 1$.

(2) $\Rightarrow$ (3). By Lemma 3.2 we have that $x \longrightarrow y = \varphi_1(y)$ and $y \longrightarrow x = \varphi_1(x)$, therefore, from (2) we obtain $\varphi_1(x) \vee S(\varphi_1(y), 1) = 1$. But $S(\varphi_1(y), 1) = S(y \longleftrightarrow 1, 1) \leq S(y, 1)$(by Remark 3.3 ) so, $1 = \varphi_1(x) \vee S(\varphi_1(y), 1) \leq x \vee S(y, 1)$, hence $x \vee S(y, 1) = 1$.

(3) $\Rightarrow$ (4). Let $F \subset L$ be a prime $n$-filter and $x \in F$. Since $N\varphi_1(x) \vee x = 1$(by Lemma 2.3,($c_4$)), from (3) we deduce that $N\varphi_1(x) \vee S(x, 1) = 1$. If we suppose that $N\varphi_1(x) \in F$, because $\varphi_1(x) \in F$ we obtain that $0 = N\varphi_1(x) \wedge \varphi_1(x) \in F$, which is impossible because $F$ is prime, hence proper. Since $1 \in F$ and $F$ is prime, it follows that $S(x, 1) \in F$. By Proposition 3.4 we deduce that $F$ is an $S$-filter.

(4) $\Rightarrow$ (1). In the representation of Corollary 2.2, every prime $n$-filter $F$ is an $S$-filter of $(L, S)$, so $(L/F, S_F)$ is a similarity $LM_n$-algebra by Remark 3.4, and the inclusion mapping $i$ is a morphism of similarity $LM_n$-algebras, therefore it is a representation of $(L, S)$ as a subdirect product of the family $\{(L/F, S_F) : F \in Spec_n(L)\}$.■

**Remark 3.5.** The similarity $LM_n$-algebra $(L, E)$ in **Example 1** is representable.

Indeed, $E(x \longrightarrow y, 1) \vee (y \longrightarrow x) = ((x \longrightarrow y) \longleftrightarrow 1) \vee (y \longrightarrow x) = \varphi_1(x \longrightarrow y) \vee (y \longrightarrow x) = (x \longrightarrow y) \vee (y \longrightarrow x) = 1$.

**Lemma 3.3.** *For any $LM_n$-algebra $L$, the similarity $LM_n$-algebra $(L, \Delta)$ is representable iff $L$ is an $LM_n$-chain.*

**Proof.** " $\Rightarrow$ ". Let $x, y \in L$. If $x \nleq y$ then $x \longrightarrow y \neq 1$, so $\Delta(x \longrightarrow y, 1) = 0$. But $(L, \Delta)$ is representable, hence, from Theorem 3.1,(2), it follows that $\Delta(x \longrightarrow y, 1) \vee (y \longrightarrow x) = 1$, so $y \longrightarrow x = 1$, hence $y \leq x$. Therefore, $L$ is an $LM_n$-chain.

" $\Leftarrow$ ". Now, let $L$ be an $LM_n$-chain and $x, y \in L$. If $x \leq y$ then $x \longrightarrow y = 1$ and if $y \leq x$ then $y \longrightarrow x = 1$, hence in both cases, $\Delta(x \longrightarrow y, 1) \vee (y \longrightarrow x) = 1$. Therefore $(L, \Delta)$ is a representable similarity $LM_n$-algebra. ■

**Remark 3.6.** As a consequence of the previous lemma, there exist similarity $LM_n$-algebras which are not representable: for example, $(L, \Delta)$ where $L$ is not an $LM_n$-chain.

**Proposition 3.6.** *If $(L, S)$ is a representable similarity $LM_n$-algebra, then the following are equivalent:*
(1) $x \leq y$ *implies* $S(x, 1) \leq S(y, 1)$,
(2) $S(x \vee y, 1) = S(x, 1) \vee S(y, 1)$.

**Proof.** (1) $\Rightarrow$ (2). Without loss of generality, we can suppose $x \leq y$. Then $S(x, 1) \leq S(y, 1)$, hence $S(x \vee y, 1) = S(y, 1) = S(x, 1) \vee S(y, 1)$.

(2) $\Rightarrow$ (1). Obvious. ■

**Definition 3.5.** If $L$ is an $LM_n$-algebra and $S$ is a similarity on $L$, we will say that $S$ is *isotone* if

$$x \leq y \text{ implies } S(x, 1) \leq S(y, 1), \text{ for any } x, y \in L.$$

*Open problem:* Find an example of a similarity operation which is not isotone.

## 4. Strong similarity $\mathbf{LM}_n$-algebra

For every $L \in \mathbf{LM}_n$ I consider the following implication (which is the generalization of the residuation considered by Moisil [1965]):

$$\begin{aligned} x \quad \Rightarrow \quad y &= y \vee N\varphi_{n-1}(x) \vee (\varphi_{n-1}(x) \wedge N\varphi_{n-2}(x) \wedge \varphi_{n-2}(y)) \vee ... \\ &... \vee (\varphi_2(x) \wedge N\varphi_1(x) \wedge \varphi_1(y)) \vee (\varphi_1(x) \wedge \varphi_1(y)). \end{aligned}$$

**Lemma 4.1.** *([3]) In every $LM_n$-algebra:*

$$x \Rightarrow y = y \vee \bigwedge_{i=1}^{n-1} (N\varphi_i(x) \vee \varphi_i(y)).$$

Let $x \Leftrightarrow y = (x \Rightarrow y) \wedge (y \Rightarrow x)$.

**Remark 4.1.** By Definition 3.1 we have $x \Rightarrow y = y \vee (x \longrightarrow y)$, hence $x \longrightarrow y \leq x \Rightarrow y, x \longleftrightarrow y \leq x \Leftrightarrow y$ for every $x, y \in L$.

**Remark 4.2.** For every $x, y \in L$ it follows that $\varphi_1(x \Rightarrow y) = x \longrightarrow y$.

Indeed,

$$\begin{aligned} \varphi_1(x \Rightarrow y) &= \varphi_1(y \vee \bigwedge_{i=1}^{n-1} (N\varphi_i(x) \vee \varphi_i(y))) = \varphi_1(y) \vee \bigwedge_{i=1}^{n-1} (N\varphi_i(x) \vee \varphi_i(y)) \\ &= \bigwedge_{i=1}^{n-1} (\varphi_1(y) \vee N\varphi_i(x) \vee \varphi_i(y)) = \bigwedge_{i=1}^{n-1} (N\varphi_i(x) \vee \varphi_i(y)) = x \longrightarrow y. \end{aligned}$$

Then $\varphi_1(x \Leftrightarrow y) = x \longleftrightarrow y$.

**Definition 4.1.** A pair $(L, \Rightarrow)$, where $L$ is a lattice with 0 and $\Rightarrow$ satisfies $x \wedge y \leq z$ iff $y \leq x \Rightarrow z$ is called a *Heyting algebra.*

Cignoli proved in 1975 that:

**Theorem 4.1.** *If $L \in \mathbf{LM}_n$ then $(L, \Rightarrow)$ is a Heyting algebra.*

We note that $x \Rightarrow y$ is a good generalization of the Boolean implication $\bar{x} \vee y$.

**Proposition 4.1.** *([3]) If $L \in \mathbf{LM}_n$ then*
*(i) If $x \in C(L)$ then $x \Rightarrow y = Nx \vee y$,*
*(ii) If $y \in C(L)$ then $x \Rightarrow y = N\varphi_{n-1}(x) \vee y$.*

**Definition 4.2.** A Heyting algebra satisfying the identity

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1$$

is called a *linear Heyting algebra* .

**Proposition 4.2.** *([3]) If $L \in \mathbf{LM}_n$ then $L$ is a linear Heyting algebra.*

**Lemma 4.2.** *([1]) In every Heyting algebra we have*
(1) *$x \wedge (x \Rightarrow y) \leq y$,*
(2) *$y \leq x \Rightarrow y$,*
(3) *$1 \Rightarrow x = x$,*
(4) *$x \leq y$ iff $x \Rightarrow y = 1$ (hence $x \Rightarrow x = x \Rightarrow 1 = 1$ and $0 \Rightarrow y = 1$),*

The fact that every $LM_n$-algebra is a linear Heyting algebra has important consequences:

**Proposition 4.3.** *([3]) In every $LM_n$-algebra*
(5) $x \Rightarrow (y \vee z) = (x \Rightarrow y) \vee (x \Rightarrow z)$,
(6) $(x \wedge y) \Rightarrow z = (x \Rightarrow z) \vee (y \Rightarrow z)$,
(7) $x \vee y = ((x \Rightarrow y) \Rightarrow y) \wedge ((y \Rightarrow x) \Rightarrow x)$.

**Remark 4.3.** $(i)$. We have that $x \Leftrightarrow y = 1$ iff $x = y$,

(ii). If $x, y \in \{0, 1\}$ then $x \Leftrightarrow y = \{ \begin{array}{l} 1, x = y \\ 0, x \neq y. \end{array}$

**Lemma 4.3.** *If $L$ is an $LM_n$-algebra then*
(8) $1 \Rightarrow x = x \Leftrightarrow 1 = x$,
(9) $(x \Rightarrow y) \wedge (y \Rightarrow z) \leq x \Rightarrow z$ *(hence* $(x \Leftrightarrow y) \wedge (y \Leftrightarrow z) \leq x \Leftrightarrow z$*),*
(10) $x \Leftrightarrow y \leq (x \wedge z) \Leftrightarrow (y \wedge z)$,
(11) $x \Leftrightarrow y \leq (x \Leftrightarrow z) \Leftrightarrow (y \Leftrightarrow z)$.

**Proof.** (8). Immediate.

(9). The relation is equivalent with $x \wedge (x \Rightarrow y) \wedge (y \Rightarrow z) \leq z$. But $x \wedge (x \Rightarrow y) \leq y$, hence $x \wedge (x \Rightarrow y) \wedge (y \Rightarrow z) \leq y \wedge (y \Rightarrow z) \leq z$ (by Lemma 4.2,(1)).

It follows that $(z \Rightarrow y) \wedge (y \Rightarrow x) \leq z \Rightarrow x$, hence $(x \Leftrightarrow y) \wedge (y \Leftrightarrow z) \leq x \Leftrightarrow z$.

(10) It is sufficient to study the case $x \leq y$ (by Corollary 2.3). Then $x \wedge z \leq y \wedge z$, hence $x \Rightarrow y = 1$ and $(x \wedge z) \Rightarrow (y \wedge z) = 1$.

We only have to prove that $y \Rightarrow x \leq (y \wedge z) \Rightarrow (x \wedge z)$. This relation is equivalent with $(y \wedge z) \wedge (y \Rightarrow x) \leq x \wedge z$, which is true because $(y \wedge z) \wedge (y \Rightarrow x) \leq y \wedge (y \Rightarrow x) \leq x$ and $(y \wedge z) \wedge (y \Rightarrow x) \leq z$.

(11). By (9) it follows that $(x \Leftrightarrow y) \wedge (y \Leftrightarrow z) \leq x \Leftrightarrow z$, hence $x \Leftrightarrow y \leq (y \Leftrightarrow z) \Rightarrow (x \Leftrightarrow z)$, and similarly $(y \Leftrightarrow x) \wedge (x \Leftrightarrow z) \leq y \Leftrightarrow z$, hence $y \Leftrightarrow x \leq (x \Leftrightarrow z) \Rightarrow (y \Leftrightarrow z)$.

Therefore $x \Leftrightarrow y \leq (x \Leftrightarrow z) \Leftrightarrow (y \Leftrightarrow z)$. ■

Let $F$ be an $n$-filter of $L$.

I recall from the previous section the relation

$$x \sim_F y \text{ iff } x \longleftrightarrow y \in F.$$

**Remark 4.4.** By Remark 4.2 we have that $\varphi_1(x \Leftrightarrow y) = x \longleftrightarrow y$, hence

$$x \sim_F y \text{ iff } x \longleftrightarrow y \in F \text{ iff } \varphi_1(x \Leftrightarrow y) \in F \text{ iff } x \Leftrightarrow y \in F.$$

I recall that in the previous section by $L/F$ I denoted the $LM_n$-algebra $L/\sim_F$ and by $x/F$ the congruence class of $x$ with respect to $\sim_F$(we have that $x/F = 1/F$ iff $x \sim_F 1$ iff $x \Leftrightarrow 1 \in F$ iff $x \in F$).

**Remark 4.5.** The quotient $LM_n$-algebra $L/F$ is also a Heyting algebra, hence $x/F \leq y/F$ iff $x/F \Rightarrow y/F = 1/F$ iff $x \Rightarrow y \in F$.

**Definition 4.3.** A *strong similarity $LM_n$-algebra* is a pair $(L, S)$ where $L$ is an $LM_n$-algebra and $S : L \times L \to L$ is a binary operation on $L$ such that the following properties hold for every $x, y, z \in L$ :
$(S_1)$ $S(x, x) = 1$,
$(S_2)$ $S(x, y) = S(y, x)$,
$(S_3)$ $S(x, y) \wedge S(y, z) \leq S(x, z)$,
$(S_4')$ $x \wedge S(x, y) \leq y$,
$(S_5')$ $S(x \Leftrightarrow y, 1) \leq S(x, z) \Leftrightarrow S(y, z)$.

An operator $S$ which satisfies $S_1$-$S_3, S_4', S_5'$ will be called a *strong similarity operation* on $L$ (or, simply, a *strong similarity* on $L$).

The relation " $\leq$ " between two strong similarities is defined as in the case of similarities (see **Section 3**).

The notions of subalgebra and homomorphism are also defined as usual.

**Remark 4.6.** From $S_5'$ and Lemma 4.3,(11) we deduce that

$S(x \Leftrightarrow y, 1) \leq S(x,y) \Leftrightarrow S(y,y) = S(x,y) \Leftrightarrow 1 = S(x,y)$ for every $x, y \in L$.

**Examples:**

**1**. On every $LM_n$-algebra $L$, the operation $\overline{E}(x,y) = x \Leftrightarrow y$ is a strong similarity. Indeed, $\overline{E}(x,x) = 1$ and $\overline{E}(x,y) = \overline{E}(y,x)$.

The condition $S_3$ results from Lemma 4.3,(9).

For $S_4'$ we have that $x \wedge (x \Rightarrow y) \leq y$ (by Lemma 4.2,(1)), hence $x \wedge (x \Leftrightarrow y) \leq x \wedge (x \Rightarrow y) \leq y$.

The condition $S_5'$ results from Lemma 4.3,(8) and (11):
$\overline{E}(x \Leftrightarrow y, 1) = (x \Leftrightarrow y) \Leftrightarrow 1 = x \Leftrightarrow y \leq (x \Leftrightarrow z) \Leftrightarrow (y \Leftrightarrow z)$.

**2**. The operation $\Delta : L \times L \to L$ defined in **Example 2** of **Section 3** is a strong similarity.

The conditions $S_1$-$S_3$ were proved in **Section 3**.

It is obvious that $x \wedge \Delta(x,y) = \{ \begin{array}{c} x, x = y \\ 0, x \neq y \end{array} \leq y$.

For $S_5'$ just replace $\longleftrightarrow$ by $\Leftrightarrow$ in the proof of $S_5$ in **Example 2** of the previous section and use Remark 4.3, $(ii)$.

Therefore, $\Delta(x \Leftrightarrow y, 1) = \{ \begin{array}{c} 1, x = y \\ 0, x \neq y \end{array}$ (by Remark 4.3,$(i)$).

Also, $\Delta(x,z) \Leftrightarrow \Delta(y,z) = \{ \begin{array}{cc} 1, \Delta(x,z) = \Delta(y,z) \\ 0, \quad \text{otherwise} \end{array}$ (by Remark 4.3,$(ii)$).

Therefore $\Delta(x \Leftrightarrow y, 1) \leq \Delta(x,z) \Leftrightarrow \Delta(y,z)$.

**Remark 4.7.** For any strong similarity $S$ on $L$ we have that $\Delta \leq S \leq \overline{E}$.

Indeed, the condition $\Delta \leq S$ is obvious and from $S_4'$ we deduce that $S(x,y) \leq x \Rightarrow y$, hence, with $S_2$, $S(x,y) \leq x \Leftrightarrow y$.

**Proposition 4.4.** *For any strong similarity $S$ on $L$ the following conditions are equivalent:*
$(i)$ $S = \overline{E}$,
$(ii)$ $S(x,1) = x$, *for every* $x \in L$.

**Proof.** $(i) \Rightarrow (ii)$. By Lemma 4.3,(8).

$(ii) \Rightarrow (i)$. We only have to prove that $x \Leftrightarrow y \leq S(x,y)$. But $x \Leftrightarrow y = S(x \Leftrightarrow y, 1) \leq S(x,y)$ (by Remark 4.6). Therefore $\overline{E} \leq S$. ■

The notion of $S$-*filter* is defined as in **Section 3** (see Definition 3.3).

**Proposition 4.5.** *Let $(L,S)$ be a strong similarity $LM_n$-algebra and $F$ an n-filter. Then $F$ is an $S$-filter iff $S(x,1) \in F$ for every $x \in F$.*

**Proof.** " $\Rightarrow$ ". Because $1 \in F$ we have that $S(x,1) \in F$ for every $x \in F$.

" $\Leftarrow$ ". If $x, y \in F$, then $x \longleftrightarrow y \in F$ (by Remark 2.4). Since $x \longleftrightarrow y \leq x \Leftrightarrow y$ (by Remark 4.1) it follows that $x \Leftrightarrow y \in F$, hence $S(x \Leftrightarrow y, 1) \in F$.

But, $S(x \Leftrightarrow y, 1) \leq S(x,y)$ (by Remark 4.6), hence $S(x,y) \in F$.

Therefore, $F$ is an $S$-filter. ■

**Proposition 4.6.** *If $(L,S)$ is a strong similarity $LM_n$-algebra and $F \subseteq L$ is an $S$-filter, then $\sim_F$ is a congruence with respect to the strong similarity $LM_n$-algebra $(L,S)$.*

**Proof.** As in Proposition 3.5 we only have to prove that $\sim_F$ is compatible with $S$. Suppose that $x_1, x_2, y_1, y_2 \in L$ such that $x_1 \sim_F x_2$ and $y_1 \sim_F y_2$.

Hence $x_1 \Leftrightarrow x_2 \in F$ and $y_1 \Leftrightarrow y_2 \in F$, so $S(x_1 \Leftrightarrow x_2, 1) \in F$ and $S(y_1 \Leftrightarrow y_2, 1) \in F$. But

$$\begin{aligned} S(x_1 \Leftrightarrow x_2, 1) &\leq S(x_1, y_1) \Leftrightarrow S(y_1, x_2) \text{ and} \\ S(y_1 \Leftrightarrow y_2, 1) &\leq S(y_1, x_2) \Leftrightarrow S(x_2, y_2), \end{aligned}$$

hence $S(x_1, y_1) \Leftrightarrow S(y_1, x_2) \in F$ and $S(y_1, x_2) \Leftrightarrow S(x_2, y_2) \in F$. So, $S(x_1, y_1) \sim_F S(y_1, x_2)$ and $S(y_1, x_2) \sim_F S(x_2, y_2)$, hence $S(x_1, y_1) \sim_F S(x_2, y_2)$. ■

As I said in the previous section for the class of similarity $LM_n$-algebras, the class of strong similarity $LM_n$-algebras is also equational, hence:

**Remark 4.8.** If $(L, S)$ is a strong similarity $LM_n$-algebra and $F \subseteq L$ is an $S$-filter of $L$, then the quotient $LM_n$-algebra $L/F$ has a canonical structure of strong similarity $LM_n$-algebra, where the strong similarity $S_F : L/F \times L/F \to L/F$ is defined by $S_F(x/F, y/F) := S(x, y)/F$, for every $x, y \in L$.

The canonical surjection $x \longmapsto x/F$ is a strong similarity $LM_n$-algebra homomorphism.

**Definition 4.4.** A strong similarity $LM_n$-algebra is called *representable* if it is a subdirect product of strong similarity $LM_n$-chains.

**Lemma 4.4.** *If $x \vee y = 1$ then $x \Rightarrow y = y$ and $y \Rightarrow x = x$.*

**Proof.** If $x \vee y = 1$ then $N\ \varphi_i(x) \leq\ \varphi_i(y)$ for every $i = 1, ..., n-1$ (see the proof of Lemma 3.2). Then $x \Rightarrow y = y \vee \bigwedge_{i=1}^{n-1} (N\ \varphi_i(x) \vee\ \varphi_i(y)) = y \vee \bigwedge_{i=1}^{n-1} \varphi_i(y) = y \vee \varphi_1(y) = y$. Therefore we obtain that $y \Rightarrow x = x$. ■

**Theorem 4.2.** *For a strong similarity $LM_n$-algebra $(L, S)$, the following are equivalent:*

(1) *$(L, S)$ is representable,*
(2) *$S(x \Rightarrow y, 1) \vee (y \Rightarrow x) = 1$ for every $x, y \in L$,*
(3) *$x \vee y = 1$ implies $x \vee S(y, 1) = 1$,*
(4) *Any prime n-filter is an S-filter.*

**Proof.** (1) $\Rightarrow$ (2). Because $(L, S)$ is representable, we can consider $x \leq y$ or $y \leq x$.

If $x \leq y$ then $x \Rightarrow y = 1$, hence $S(x \Rightarrow y, 1) \vee (y \Rightarrow x) = 1$.

If $y \leq x$ then $y \Rightarrow x = 1$, hence $S(x \Rightarrow y, 1) \vee (y \Rightarrow x) = 1$.

(2) $\Rightarrow$ (3). From (2) and Lemma 4.4 it follows that $x \vee S(y, 1) = 1$.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) follow in the same way as in the Theorem 3.1, but using Proposition 4.5. ■

**Remark 4.9.** The strong similarity $LM_n$-algebra $(L, \overline{E})$ in **Example 1** is representable.

Indeed, $\overline{E}(x \Rightarrow y, 1) \vee (y \Rightarrow x) = ((x \Rightarrow y) \Leftrightarrow 1) \vee (y \Rightarrow x) = (x \Rightarrow y) \vee (y \Rightarrow x) = 1$ (by Proposition 4.2).

**Lemma 4.5.** *For any $LM_n$-algebra $L$, the strong similarity $LM_n$-algebra $(L, \Delta)$ is representable iff $L$ is an $LM_n$-chain.*

**Proof.** As for Lemma 3.3, but using Theorem 4.2. ■

**Remark 4.10.** From the previous lemma it follows that there exist strong similarity $LM_n$-algebras which are not representable: $(L, \Delta)$ where $L$ is not an $LM_n$-chain.

**Proposition 4.7.** *If* $(L, S)$ *is a representable strong similarity* $LM_n$*-algebra, then the following are equivalent:*
(1) $x \leq y$ *implies* $S(x, 1) \leq S(y, 1)$,
(2) $S(x \vee y, 1) = S(x, 1) \vee S(y, 1)$.

**Proof.** (1) $\Rightarrow$ (2). Without loss of generality we can suppose $x \leq y$. Then $S(x, 1) \leq S(y, 1)$, hence $S(x \vee y, 1) = S(x, 1) \vee S(y, 1)$.

(2) $\Rightarrow$ (1). Obvious. ■

As in the previous section, an open problem appears: to find a strong similarity which is not isotone.

## 5. A general theory of similarity

Let $(L, \leq)$ be a bounded lattice and $\circledast$ a commutative binary operation on $L$ such that the following relations are verified:
$(l_1)$ $x \circledast 1 = x$,
$(l_2)$ $x \circledast 0 = 0$,
$(l_3)$ $x \circledast y \leq x \wedge y$.

Also, I consider two functions $\varphi, \Phi : L \to L$ such that $\varphi(x) \leq x$ and $x \leq \Phi(x)$ for every $x \in L$ (it follows that $\varphi(0) = 0$ and $\Phi(1) = 1$).

Let "$\longrightarrow$" be another binary operation on $L$ such that
$(l_4)$ $1 \longrightarrow x = \varphi(x)$ for all $x \in L$ or $1 \longrightarrow x = x$ for all $x \in L$,
$(l_5)$ $(x \wedge y) \longrightarrow y = 1$.

**Definition 5.1.** By an *L-algebra* we understand a bounded lattice $(L, \leq)$ with all the operations and properties mentioned above. If the relation $\leq$ is total, $L$ will be called an *L-chain.*

In the following, by $L$ we will understand an $L$-algebra.

**Remark 5.1.** Since inequalities can be written as equalities in any (semi)lattice, the class of $L$-algebras is the union of two equational classes, defined by identities $1 \longrightarrow x = \varphi(x)$ and $1 \longrightarrow x = x$, respectively.

Now let us consider the following axioms:
$(l_3^{'})$ $x \circledast y = 1$ iff $x = y = 1$,
$(l_5^{'})$ If $x \leq y$ then $x \longrightarrow y = 1$.

**Remark 5.2.** The axiom $(l_3)$ implies the axiom $(l_3^{'})$ and the axioms $(l_5)$ and $(l_5^{'})$ are equivalent.

Indeed, let us consider that $x \circledast y \leq x \wedge y$. If $x \circledast y = 1$ then $x \wedge y = 1$, hence $x = y = 1$ and if $x = y = 1$ then $x \circledast y = 1$ (by $(l_1)$). For the second affirmation, if $x \leq y$, then $(l_5)$ induces $x \longrightarrow y = 1$ and conversely, we have that $x \wedge y \leq y$, hence $(l_5^{'})$ implies $(x \wedge y) \longrightarrow y = 1$.

In the following we will use $(l_3^{'})$ and $(l_5^{'})$ frequently. Thus e.g.:

**Remark 5.3.** We have that $x \longrightarrow x = 1$ and $x \longrightarrow 1 = 1$.

For $x, y \in L$ I define

$$(l_6)\ x \longleftrightarrow y = (x \longrightarrow y) \circledast (y \longrightarrow x).$$

**Remark 5.4.** It follows that
$(i)$ $x \longleftrightarrow y = 1$ iff $x \longrightarrow y = y \longrightarrow x = 1$ (hence $x \longleftrightarrow x = 1$),

*(ii)* $x \longleftrightarrow 1 = (x \longrightarrow 1) \circledast (1 \longrightarrow x) = 1 \circledast (1 \longrightarrow x) = 1 \longrightarrow x$, hence, by $(l_4)$, we have $x \longleftrightarrow 1 = \varphi(x)$ or $x \longleftrightarrow 1 = x$.

**Definition 5.2.** A *similarity $L$-algebra* is a pair $(L, S)$ where $L$ is an $L$-algebra and $S : L \times L \to L$ is a binary operation on $L$ such that the following properties hold for every $x, y, z \in L$ :
$(S_1)$ $S(x, x) = 1$,
$(S_2)$ $S(x, y) = S(y, x)$,
$(S_3)$ $S(x, y) \circledast S(y, z) \leq S(x, z)$,
$(S_4)$ $x \circledast S(x, y) \leq \Phi(y)$,
$(S_5)$ $S(x \longleftrightarrow y, 1) \leq S(x, z) \longleftrightarrow S(y, z)$.

The operation $S$ will be called, simply, a *similarity* on $L$.

**Example.** Let's consider the binary operation $\Delta : L \times L \to L$ defined by

$$\Delta(x, y) = \left\{ \begin{array}{l} 1, x = y \\ 0, x \neq y \end{array} \right., \text{ for any } x, y \in L.$$

Then, this operation satisfies the condition $(S_1) - (S_4)$ from the above definition. Indeed:
$(S_1)$ $\Delta(x, x) = 1$ and
$(S_2)\Delta(x, y) = \Delta(y, x)$ are obvious.

$(S_3)$ $\Delta(x, y) \circledast \Delta(y, z) = \left\{ \begin{array}{l} 1 \circledast \Delta(y, z), x = y \\ 0 \circledast \Delta(y, z), x \neq y \end{array} \right. = \left\{ \begin{array}{l} 1, x = y = z \\ 0, \text{otherwise} \end{array} \right. \leq \Delta(x, z).$

$(S_4)$ $x \circledast \Delta(x, y) = \left\{ \begin{array}{l} x \circledast 1, x = y \\ x \circledast 0, x \neq y \end{array} \right. = \left\{ \begin{array}{l} x, x = y \\ 0, x \neq y \end{array} \right. \leq \Phi(y).$

If moreover,
$(l_5^{''})$ $x \longrightarrow y = 1$ implies $x \leq y$,
which in view of $(l_5^{'})$, $(l_3^{'})$ and $(l_6)$ is equivalent to
$(l_5^{'''})$ $x \longleftrightarrow y = 1$ iff $x = y$,
then condition $(S_5)$ holds as well.

Indeed, we have that $\Delta(x \longleftrightarrow y, 1) = \left\{ \begin{array}{l} 1, x \longleftrightarrow y = 1 \\ 0, x \longleftrightarrow y \neq 1 \end{array} \right. = \left\{ \begin{array}{l} 1, x = y \\ 0, x \neq y \end{array} \right..$

Also, $\Delta(x, z) \longleftrightarrow \Delta(y, z) = \left\{ \begin{array}{ll} 1, \Delta(x, z) = \Delta(y, z) \\ 0, & \text{otherwise} \end{array} \right..$

Therefore $\Delta(x \longleftrightarrow y, 1) \leq \Delta(x, z) \longleftrightarrow \Delta(y, z)$.

**Remark 5.5.** From $(S_5)$ and Remark 5.4,(ii) we obtain that:

$$S(x \longleftrightarrow y, 1) \leq S(x, y) \longleftrightarrow S(y, y) = S(x, y) \longleftrightarrow 1 = \varphi(S(x, y)) \text{ or } S(x, y),$$

but in both cases we obtain that $S(x \longleftrightarrow y, 1) \leq S(x, y)$.

**Definition 5.3.** A nonempty subset $F$ of $L$ is called an *$L$-filter* if it verifies the following conditions:
$(f_1)$ $x \in F$ and $x \leq y$ implies $y \in F$,
$(f_2)$ $x, y \in F$ implies $x \circledast y \in F$,
$(f_3)$ $x \in F$ implies $\varphi(x) \in F$.

**Remark 5.6.** *It is obvious that for any $L$-filter $F$ of an $L$-algebra condition $(f_1)$ shows that $1 \in F$ (because $F \neq \emptyset$ and $x \leq 1$ for every $x \in F$).*

**Definition 5.4.** An $L$-filter $F$ of $L$ is called an *$S$-filter* if $S(x, y) \in F$ for every $x, y \in F$.

**Proposition 5.1.** *Let $(L, S)$ be a similarity $L$-algebra. Then an $L$-filter $F$ is an $S$-filter iff $S(x, 1) \in F$ for every $x \in F$.*

**Proof.** "$\Rightarrow$". Because $1 \in F$ we have that $S(x, 1) \in F$ for every $x \in F$.
"$\Leftarrow$". If $x, y \in F$, then $S(x, 1), S(y, 1) \in F$, hence $S(x, 1) \circledast S(y, 1) \in F$.
But $S(x, 1) \circledast S(y, 1) \leq S(x, y)$ (by $(S_2)$ and $(S_3)$), then $S(x, y) \in F$.
Therefore, $F$ is an $S$-filter. ■

For an $L$-filter $F$ I consider the relation

$$x \sim_F y \text{ iff } x \longleftrightarrow y \in F$$

and the following axiom

$$(P_1) \sim_F \textit{ is a congruence on } L.$$

In view of Remark 5.1, if $L$ satisfies the axiom $(P_1)$, then the quotient algebra $L/\sim_F$ is an $L$-algebra, denoted simply by $L/F$ (by $x/F$ we denote the congruence class of $x \in L$ relative to $\sim_F$).

**Proposition 5.2.** *If $L$ satisfies the axiom $(P_1)$, $S$ is a similarity on $L$ and $F$ is an $S$-filter, then $\sim_F$ is a congruence on the similarity $L$-algebra $(L, S)$.*

**Proof.** We have only to study the compatibility of $\sim_F$ with $S$. Suppose that $x_1, x_2, y_1, y_2 \in L$ such that $x_1 \sim_F x_2$ and $y_1 \sim_F y_2$.
It follows that $x_1 \longleftrightarrow x_2, y_1 \longleftrightarrow y_2 \in F$. Hence $S(x_1 \longleftrightarrow x_2, 1), S(y_1 \longleftrightarrow y_2, 1) \in F$.

But

$$\begin{aligned} S(x_1 \longleftrightarrow x_2, 1) &\leq S(x_1, y_1) \longleftrightarrow S(y_1, x_2) \text{ and} \\ S(y_1 \longleftrightarrow y_2, 1) &\leq S(y_1, x_2) \longleftrightarrow S(x_2, y_2), \end{aligned}$$

hence $S(x_1, y_1) \longleftrightarrow S(y_1, x_2), S(y_1, x_2) \longleftrightarrow S(x_2, y_2) \in F$. So, $S(x_1, y_1) \sim_F S(y_1, x_2)$ and $S(y_1, x_2) \sim_F S(x_2, y_2)$, hence $S(x_1, y_1) \sim_F S(x_2, y_2)$. ■

**Remark 5.7.** In this case, the corresponding similarity on the quotient algebra $L/F$

$$S_F : L/F \times L/F \to L/F \text{ is defined by } S_F(x/F, y/F) = S(x, y)/F,$$

and $(L/F, S_F)$ is a similarity $L$-algebra.

**Definition 5.5.** A proper $L$-filter $F$ is called *prime* if $x \vee y \in F$ implies $x \in F$ or $y \in F$.

**Definition 5.6.** By a *maximal (minimal) L-filter* is meant a maximal (minimal) element in the family of proper $L$-filters ordered by set inclusion.

By a *maximal (minimal) prime L-filter* is meant a maximal (minimal) element in the family of prime $L$-filters ordered by set inclusion.

I consider now the following four axioms:

$(P_2)$ *If $F$ is a prime $L$-filter then $L/F$ is an $L$-chain*;

$(P_3)$ *If $x \vee y = 1$ then either $x \longrightarrow y = \varphi(y)$ and $y \longrightarrow x = \varphi(x)$, or $x \longrightarrow y = y$ and $y \longrightarrow x = x$*;

$(P_4)$ *If $F$ is a minimal prime $L$-filter and $x \in F$ then there is $y \in L \backslash F$ such that $x \vee y = 1$;*

$(P_5)$ *There is a family $\mathcal{F}$ of prime $L$-filters of $L$ such that $L$ is the subdirect product (as an $L$-algebra) of the family $\{L/F : F \in \mathcal{F}\}$.*

**Definition 5.7.** A similarity $L$-algebra $(L, S)$ is called *representable* if it is a subdirect product of similarity $L$-chains.

**Theorem 5.1.** *If $(L,S)$ is a similarity $L$-algebra and $L$ satisfies the axioms $(P_1)-(P_5)$, then the following conditions are equivalent:*
(1) *$(L,S)$ is representable,*
(2) *$S(x \longrightarrow y,1) \vee (y \longrightarrow x) = 1$ for every $x,y \in L$,*
(3) *$x \vee y = 1$ implies $x \vee S(y,1) = 1$,*
(4) *Any minimal prime $L$-filter is an $S$-filter.*

**Proof.**(1) $\Rightarrow$ (2). Because $(L,S)$ is representable we can consider $x \leq y$ or $y \leq x$.
If $x \leq y$ then $x \longrightarrow y = 1$, hence $S(x \longrightarrow y,1) \vee (y \longrightarrow x) = 1$.
If $y \leq x$ then $y \longrightarrow x = 1$, hence $S(x \longrightarrow y,1) \vee (y \longrightarrow x) = 1$.
(2) $\Rightarrow$ (3). Let $x,y \in L$ such that $x \vee y = 1$. By $(P_3)$ we have two cases:
*a*) If $x \longrightarrow y = \varphi(y)$ and $y \longrightarrow x = \varphi(x)$, from (2) we obtain that $S(\varphi(y),1) \vee \varphi(x) = 1$. According to $(l_4)$ we have two subcases:
$a_1$) If $1 \longrightarrow t = \varphi(t)$ for every $t \in L$ we obtain (via Remark 5.4,(ii)) that $S(\varphi(y),1) = S(y \longleftrightarrow 1,1) \leq S(y,1)$ (by Remark 5.5), then $1 = S(\varphi(y),1) \vee \varphi(x) \leq S(y,1) \vee x$, hence $x \vee S(y,1) = 1$.
$a_2$) If $1 \longrightarrow t = t$ by Remark 5.3 we obtain that $t \longleftrightarrow 1 = t$ for all $t \in L$ and by Remark 5.4 we obtain that $t \longleftrightarrow 1 = \varphi(t)$ for all $t \in L$. Hence $\varphi$ is the identical map. Therefore this second subcase leads us to the case *b*):
*b*) If $x \longrightarrow y = y$ and $y \longrightarrow x = x$, from (2) we obtain that $S(y,1) \vee x = 1$.
(3) $\Rightarrow$ (4). Let $F \subset L$ be a minimal prime $L$-filter and $x \in F$. Since $L$ has the property $(P_4)$, there is $y \in L \backslash F$ such that $x \vee y = 1$, hence $y \vee S(x,1) = 1$(by (3)). Because $1 \in F, y \notin F$ and $F$ is prime, it follows that $S(x,1) \in F$.
Therefore $F$ is an $S$-filter (by Proposition 5.1).
(4) $\Rightarrow$ (1). The axiom $(P_5)$ shows that there exists a family $\mathcal{F}$ of prime $L$-filters such that $L$ is the subdirect product of the family $\{L/F : F \in \mathcal{F}\}$. By condition (4), Remark 5.7 and axiom $(P_2)$ we deduce that $L$ is a subdirect product of the family of similarity $L$ chains $\{L/F : F \in \mathcal{F}\}$. Therefore $(L,S)$ is representable. ■

**Lemma 5.1.** *If $L$ is a chain and $(L,\Delta)$ is a similarity $L$-algebra, then it is representable.*

**Proof.** Let $x,y \in L$. If $x \leq y$ then $x \longrightarrow y = 1$ and if $y \leq x$ then $y \longrightarrow x = 1$, hence in both cases $\Delta(x \longrightarrow y,1) \vee (y \longrightarrow x) = 1$. Therefore $(L,\Delta)$ is a representable similarity $L$-algebra. ■

*Open problem*: If $(L,\Delta)$ is representable then $L$ is a chain?

**Proposition 5.3.** *If $(L,S)$ is a representable similarity $L$-algebra, then the following are equivalent:*
(1) *$x \leq y$ implies $S(x,1) \leq S(y,1)$,*
(2) *$S(x \wedge y,1) = S(x,1) \vee S(y,1)$.*

**Proof.** (1) $\Rightarrow$ (2). Without loss of generality, we can suppose $x \leq y$. Then $S(x,1) \leq S(y,1)$, hence $S(x \vee y,1) = S(x,1) \vee S(y,1)$.
(2) $\Rightarrow$ (1). Obvious. ■

**Applications**

Now we consider three particular situations:
**1.** If $L$ is an $LM_n$-algebra, we take

$$\circledast = \wedge, \varphi = \varphi_1, \Phi = \varphi_{n-1} \text{ and } x \longrightarrow y = \bigwedge_{i=1}^{n-1} (N\varphi_i(x) \vee \varphi_i(y)) (\text{ see Definition 3.1}),$$

then we obtain the theory of *similarity $LM_n$-algebra* presented in **Section 3**.

We know that $L$ is a bounded lattice; this fact, together with Proposition 3.2 (via Remark 5.2) and Lemma 3.1, proves that $L$ is an $L$-algebra.

Moreover, axiom $(l_5^{''})$ is fulfilled by Proposition 3.2.

The axiom $(P_1)$ holds in every $LM_n$-algebra according to Proposition 3.1.

For the axiom $(P_2)$ we have that:

**Proposition 5.4.** *If $L$ is an $LM_n$-algebra and $F$ is a prime $n$-filter then $L/F$ is an $LM_n$-chain.*

**Proof.** Let $x, y \in L$. We know that $(L, \Rightarrow)$ is a linear Heyting algebra (see Proposition 4.2 from **Section 3**), hence $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$. But $F$ is prime and $1 \in F$, then $x \Rightarrow y \in F$ or $y \Rightarrow x \in F$. Then $(x \Rightarrow y)/F = 1/F$ or $(y \Rightarrow x)/F = 1/F$, hence $x/F \Rightarrow y/F = 1/F$ or $y/F \Rightarrow x/F = 1/F$. Therefore, because $L/F$ is a Heyting algebra, $x/F \leq y/F$ or $y/F \leq x/F$, that is, $L/F$ is an $LM_n$-chain. ■

In our case the axiom $(P_3)$ becomes

$$(P_3)\textit{If } x \vee y = 1 \textit{ then } x \longrightarrow y = \varphi_1(y) \textit{ and } y \longrightarrow x = \varphi_1(x),$$

which is true by Lemma 3.2.

For the axiom $(P_4)$ we need an important result from the theory of $LM_n$-algebras, namely: the minimal prime $n$-filters coincides with the prime $n$-filters (see Theorem 4.3 and Remark 4.4 from [4]) and the fact that in any $LM_n$-algebra, $x \vee N\varphi_1(x) = 1$ (see Lemma 2.3, $(c_4)$). If $F$ is a prime $n$-filter then it is proper, hence $x \in F$ implies $\varphi_1(x) \in F$, so $N\varphi_1(x) \notin F$ (otherwise, $0 = \varphi_1(x) \wedge N\varphi_1(x) \in F$- a contradiction).

The axiom $(P_5)$ is nothing else but Corollary 2.2 from **Section 2**.

In this case, Lemma 3.3 is stronger than Lemma 5.1 because we have an "iff" condition, Remark 3.6 gives an answer of the open problem that appeared in the general case.

**2.** If $L$ is an $LM_n$-algebra, we take

$$\circledast = \wedge, \varphi = \varphi_1, \Phi = 1_L$$

and

$$\begin{aligned} x \quad \Rightarrow \quad y &= y \vee N\varphi_{n-1}(x) \vee (\varphi_{n-1}(x) \wedge N\varphi_{n-2}(x) \wedge \varphi_{n-2}(y)) \vee ... \\ &\quad ... \vee (\varphi_2(x) \wedge N\varphi_1(x) \wedge \varphi_1(y)) \vee (\varphi_1(x) \wedge \varphi_1(y)) \\ &= y \vee \bigwedge_{i=1}^{n-1} (N\varphi_i(x) \vee \varphi_i(y)) (\text{ see Lemma 4.1}), \end{aligned}$$

then we obtain the theory of *strong similarity $LM_n$-algebra* presented in **Section 4**.

We have that $L$ is a bounded lattice, hence, by Lemma 4.2, (3) and (4)(via Remark 5.2), we deduce that $L$ is an $L$-algebra.

Moreover, axiom $(l_5^{''})$ is fulfilled by Lemma 4.2, (4).

For an $n$-filter $F$ we have the relation

$$x \sim_F y \text{ iff } x \Leftrightarrow y \in F.$$

Remark 4.4 shows that $x \sim_F y$ is a congruence on $L$, hence the axiom $(P_1)$ is satisfied.

The axioms $(P_2), (P_4), (P_5)$ are exactly as in the case **1**.

The axiom $(P_3)$ is different from the case **1** because now we have

$$(P_3) \textit{ If } x \vee y = 1 \textit{ then } x \Rightarrow y = y \textit{ and } y \Rightarrow x = x,$$

which is true by Lemma 4.4.

As in the first case, Lemma 4.5 is stronger than Lemma 5.1 because we also have an "iff" condition, Remark 3.6 gives an answer of the open problem that appeared in the general case.

**3.** Now I consider the case of $MV$-algebras. Starting from the general aspects presented above, I obtain the theory of similarity $MV$-algebra from [11].

In the following, I remind some important definitions and proprieties of $MV$-algebras which I'll use in my presentation.

**Definition 5.8.** ([5], [10]) An *$MV$-algebra* is an algebra $\mathcal{L} = (L, \oplus, ^*, 0)$ of type $(2, 1, 0)$ satisfying the following equations:

$(mv_1)$ $(L, \oplus, 0)$ is a commutative monoid,
$(mv_2)$ $x^{**} = x$,
$(mv_3)$ $x \oplus 0^* = 0^*$,
$(mv_4)$ $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

In order to simplify the notation, an $MV$-algebra $\mathcal{L} = (L, \oplus, ^*, 0)$ will be denoted by its support set, $L$. For an $MV$-algebra $L$ the constant 1 and the auxiliary operations $\odot$ are defined as follows :

$$(mv_5)\ \ 1 = 0^*,$$

$$(mv_6)\ \ x \odot y = (x^* \oplus y^*)^*,$$

for any $x, y \in L$.

**Lemma 5.2.** *([10]) For $x, y \in L$, the following conditions are equivalent:*

$(i)$ $x^* \oplus y = 1$,
$(ii)$ $x \odot y^* = 0$,
$(iii)$ $y = x \oplus (y^* \odot x)$,
$(iv)$ *There is an element $z \in A$ such that $x \oplus z = y$,*
$(v)$ *There is an element $t \in A$ such that $x = y \odot t$.*

For any two elements $x, y \in L$ we write $x \leq y$ iff $x$ and $y$ satisfy one of the equivalent conditions $(i)$-$(v)$ from the above lemma and we have that $\leq$ is an order relation on $L$ (which is called the *natural order* on $L$).

We will say that $L$ is an *$MV$-chain* if it is linearly ordered relative to natural order.

**Proposition 5.5.** *([10], Proposition 1.1.5, p.10) The natural order determines on $L$ a structure of bounded distributive lattice, namely, the join $x \vee y$ and the meet $x \wedge y$ of the elements $x$ and $y$ are given by:*

$$x \vee y = x \odot y^* \oplus y = y \odot x^* \oplus x,$$

$$x \wedge y = (x^* \vee y^*)^* = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

*Clearly, $x \odot y \leq x \wedge y \leq x, y \leq x \vee y \leq x \oplus y$ and $x \wedge x^* \leq y \vee y^*$.*

**Remark 5.8.** *([5], p.468)* It is clear that as in the case of Boolean algebras, there is a duality involving elements 0 and 1, the operations $\oplus$ and $\odot$, and the operations $\vee$ and $\wedge$. Thus any theorem will have its dual as a consequence from the axioms.

**Lemma 5.3.** *([10], p.8 and Lemma 1.1.4, p.10) If $x, y, z \in L$ then we have:*

(1) $1^* = 0, 0^* = 1$
(2) $x \oplus y = (x^* \odot y^*)^*$,
(3) $x \oplus 1 = 1, x \odot 1 = x$,
(4) $x \odot 0 = 0$,

(5) $x \oplus x^* = 1, x \odot x^* = 0$,
(6) $x \leq y$ *iff* $y^* \leq x^*$,
(7) $x \odot z \leq y$ *iff* $z \leq x^* \oplus y$,
(8) $x \odot (y \odot z) = (x \odot y) \odot z$.

**Proposition 5.6.** *(*[10]*, Proposition 1.1.6, p.11) The following equations hold in every $MV$-algebra:*
(9) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(10) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$.

**Remark 5.9.** We have that $x \odot y = 1$ iff $x = y = 1$.

Indeed, it is clear that if $x = y = 1$ then $x \odot y = 1$. Conversely, if $x \odot y = 1$ then $x^* \oplus y^* = 0$, so $x = x \oplus 0 = x \oplus (x^* \oplus y^*) = (x \oplus x^*) \oplus y^* = 1 \oplus y^* = 1$ and similarly, $y = 1$.

**Theorem 5.2.** *(*[16]*, Theorem 3.2,* [5]*,Theorem 3.3,* [2]*, Theorem 1) In every $MV$-algebra we have:*
$(i)$ $x \oplus y = (x \vee y) \oplus (x \wedge y)$,
$(ii)$ $x \odot y = (x \vee y) \odot (x \wedge y)$,
$(iii)$ $(x^* \odot y) \wedge (y^* \odot x) = 0$,
$(iv)$ $(x^* \oplus y) \vee (y^* \oplus x) = 1$.

**Remark 5.10.** By Theorem 5.2 we deduce that

$$\begin{aligned}(x^* \odot y) \oplus (y^* \odot x) &= (x^* \odot y) \vee (y^* \odot x) \text{ and} \\ (x^* \oplus y) \odot (y^* \oplus x) &= (x^* \oplus y) \wedge (y^* \oplus x).\end{aligned}$$

Also, for an $MV$-algebra one defines the operations $\longrightarrow$ and $\longleftrightarrow$ by

$$(mv_7) \;\; x \longrightarrow y = x^* \oplus y$$

and

$(mv_8)$ $x \longleftrightarrow y = (x \longrightarrow y) \odot (y \longrightarrow x) = (x^* \oplus y) \wedge (y^* \oplus x)$(by Remark 5.10).

**Remark 5.11.** It is easy to see that:
$(i)$ $1 \longrightarrow x = x$ and $x \longleftrightarrow 1 = x$,
$(ii)$ By Lemma 5.2, we have that $x \leq y$ iff $x \longrightarrow y = 1$ ( hence, $x \longrightarrow x = 1$ and $x \longrightarrow 1 = 1$). Therefore $x \longleftrightarrow y = 1$ iff $x = y$.

**Remark 5.12.** By Lemma 5.3,(7) we have that $x \odot y \leq z$ iff $x \leq y \longrightarrow z$.

**Definition 5.9.** *(*[10]*)* An *$MV$-filter* of $L$ is a nonempty subset $F \subseteq L$ which verifies the following conditions:
$(f_1)$ $x \in F$ and $x \leq y$ implies $y \in F$,
$(f_2)$ $x, y \in F$ implies $x \odot y \in F$.

**Remark 5.13.** Every $MV$-filter contains the element 1.

**Definition 5.10.** *(*[10]*)* An *$MV$-ideal* of $L$ is a nonempty subset $I \subseteq L$ which verifies the following conditions:
$(i_1)$ $x \in I$ and $y \leq x$ implies $y \in I$,
$(i_2)$ $x, y \in I$ implies $x \oplus y \in I$.

**Proposition 5.7.** *(*[10]*, Proposition 1.2.6, p.15) Let $I$ be an ideal of the $MV$-algebra $L$. Then the binary relation*

$$x \sim_I y \text{ iff } (x^* \odot y) \oplus (y^* \odot x) \in I$$

*is a congruence relation on $L$.*

**Remark 5.14.** According to Remark 5.10 we have that

$$x \sim_I y \text{ iff } (x^* \odot y) \vee (y^* \odot x) \in I.$$

Dually (by Remark 5.8), for an $MV$-filter $F$ we have the congruence relation on $L$

$$x \sim_F y \text{ iff } (x^* \oplus y) \wedge (y^* \oplus x) \in F,$$

that is (by $(mv_8)$),

$$x \sim_F y \text{ iff } x \longleftrightarrow y \in F.$$

A *prime filter* in an $MV$-algebra is defined as in Definition 5.5.

**Definition 5.11.** ([10]) A proper MV-ideal $I$ of $L$ is called *prime* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

**Proposition 5.8.** ([6]) *For an MV-ideal $I$ of an MV-algebra $L$ the following are equivalent:*
(*i*) *$I$ is prime,*
(*ii*) *$L/I$ is a non-trivial MV-chain.*

**Lemma 5.4.** *If $x \vee y = 1$ then $x \longrightarrow y = y$ and $y \longrightarrow x = x$.*

**Proof.** By Proposition 5.6,(10), we have

$$\begin{aligned} x &= x \oplus 0 = x \oplus (x \vee y)^* = x \oplus (x^* \wedge y^*) = (x \oplus x^*) \wedge (x \oplus y^*) \\ &= 1 \wedge (x \oplus y^*) = x \oplus y^* = y \longrightarrow x \end{aligned}$$

and similarly, $x \longrightarrow y = y$. ■

**Proposition 5.9.** ([10], *Theorem 6.1.5, p.114*) *If $I$ is a minimal prime ideal and $x \in I$ then there is $y \in L \backslash I$ such that $x \wedge y = 0$.*

The following representation theorem, due to Chang, is well known:

**Theorem 5.3.** ([10]) *Every nontrivial MV-algebra $L$ is a subdirect product of MV-chains (arising as quotient MV-algebras over prime ideals).*

**Definition 5.12.** ([11]) A *similarity MV-algebra* is a pair $(L, S)$ where $L$ is an $MV$-algebra and $S : L \times L \to L$ is a binary operation on $L$ such that the following properties hold for every $x, y, z \in L$ :
$(S_1)$ $S(x,x) = 1$,
$(S_2)$ $S(x,y) = S(y,x)$,
$(S_3)$ $S(x,y) \odot S(y,z) \leq S(x,z)$,
$(S_4)$ $x \odot S(x,y) \leq y$,
$(S_5)$ $S(x \longleftrightarrow y, 1) \leq S(x,z) \longleftrightarrow S(y,z)$.

The operation $S$ will be called, as usual, *a similarity* on $L$.

We take: $\circledast = \odot, \Phi = \varphi = 1_L, x \longrightarrow y = x^* \oplus y$.

Because the operation $\oplus$ is commutative, from relation $(mv_6)$ it follows that $\odot$ is commutative.

By Proposition 5.5, Lemma 5.3,(3) and (4) and Remark 5.11 (via Remark 5.2), the axioms $(l_1) - (l_5)$ are satisfied, so every $MV$-algebra is an $L$-algebra which satisfies $(l_5'')$.

The axiom $(P_1)$ results by Remark 5.14.

By the dual of Proposition 5.8, the axiom $(P_2)$ is satisfied in every $MV$-algebra.

The axiom $(P_3)$ is in this case

$$(P_3) \text{ If } x \vee y = 1 \text{ then } x \longrightarrow y = y \text{ and } y \longrightarrow x = x,$$

which is true by Lemma 5.4.

The axiom ($P_4$) is obtained by the dual of Proposition 5.9.

By the dual of Theorem 5.3 we get that every nontrivial $MV$-algebra $L$ is a subdirect product of $MV$-chains (arising as quotient $MV$-algebras over prime filters). Hence the axiom ($P_5$) is satisfied (so, every nontrivial $MV$-algebra is representable).

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