Annals of the University of Craiova, Math. Comp. Sci. Ser.
Volume 35, 2008, Pages 76–77
ISSN: 1223-6934

On the integral form of the triangle inequality

Raluca Ciurcea

Abstract. We prove a formula concerning the precision in the triangle inequality.

2000 Mathematics Subject Classification. 26D10; 26D15; 26A24; 26A45.
Key words and phrases. Leibniz-Newton Formula, absolutely continuous function, integrable function.

The discrepancy in the integral form of the triangle inequality can be easily estimated in terms of variance. Precisely, if $f : [a, b] \to \mathbb{R}$ is a square integrable function, then

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b |f(x)|\,dx - \left|\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{1}{b-a}\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| dx \\ &\le \left(\frac{1}{b-a}\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right|^2 dx\right)^{1/2} = \sqrt{\operatorname{Var}(f)}. \end{aligned}$$

If f is Lipschitz, with Lipschitz constant

$$\operatorname{Lip}(f) = \sup_{x \neq y}\left|\frac{f(x) - f(y)}{x - y}\right|,$$

then we may take into account the Ostrowski inequality (cf. [3], p. 63),

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left(\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right)(b-a)\operatorname{Lip}(f),$$

in order to conclude that

$$\sqrt{\operatorname{Var}(f)} \le \sqrt{\frac{7}{60}}(b-a)\operatorname{Lip}(f) \approx 0.341\,57(b-a)\operatorname{Lip}(f).$$

The aim of this paper is to show that a better estimate is available.

Theorem 0.1. *If $f : [a, b] \to \mathbb{R}$ is a Lipschitz function, then*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b |f(x)|\,dx - \left|\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \min\left\{\sqrt{\operatorname{Var}(f)}, \frac{\operatorname{Lip}(f)}{3}(b-a)\right\}. \end{aligned}$$

Received: 10 November 2008.

In the particular case of continuously differentiable functions we may use the equality $\operatorname{Lip}(f) = \sup_{x\in[a,b]} |f'(x)|$.

Proof. According to the discussion above it suffices to prove that

$$\frac{1}{b-a}\int_a^b |f(x)|\,dx - \left|\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{\operatorname{Lip}(f)}{3}(b-a). \tag{L}$$

For this, notice first that f is an absolutely continuous function whose derivative f' belongs to $L^\infty([a,b])$. Clearly, $\|f'\|_{L^\infty} = \operatorname{Lip}(f)$. By the Leibniz-Newton Formula (for absolutely continuous functions, see [2]) we infer that

$$(b-a)\,f(x) = \int_a^b f(t)dt + \int_a^x (t-a)\,f'(t)dt - \int_x^b (b-t)f'(t)dt,$$

which yields

$$\begin{aligned}(b-a)\,|f(x)| &\leq \left|\int_a^b f(t)dt\right| + \int_a^x (t-a)\,|f'(t)|\,dt + \int_x^b (b-t)f'(t)dt \\ &\leq \left|\int_a^b f(t)dt\right| + \operatorname{Lip}(f)\cdot\frac{(x-a)^2+(b-x)^2}{2}.\end{aligned}$$

By integrating against $[a,b]$ we arrive at the inequality (L). ∎

Corollary 0.1. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function with Lipschitz derivative. Then*

$$\bigvee_a^b f \leq \left|\frac{f(b)-f(a)}{b-a}\right| + \frac{\operatorname{Lip}(f')}{3}(b-a).$$

Proof. In fact, if $v:[a,b]\to\mathbb{R}$ is differentiable and its derivative is integrable, then v has bounded variation and

$$\bigvee_a^b v = \int_a^b |v'(t)|\,dt.$$

See [1], p. 104. ∎

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(Raluca Ciurcea) Department of Mathematics, University of Craiova,
Al.I. Cuza Street, No. 13, Craiova RO-200585, Romania, Tel. & Fax: 40-251412673
E-mail address: r.ciurcea@yahoo.com