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Continuous family of eigenvalues concentrating in a small neighborhood at the right of the origin for a class of discrete boundary value problems

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Abstract. In this paper, we prove the existence of a continuous spectrum that lies in a neighborhood at the right of the origin for some nonlinear difference operators. Our proofs rely essentially on the Banach fixed point theorem and a minimization technique.

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1. Introduction and main results

This paper is concerned with the study of the existence of solutions for the discrete boundary value problem

$$\begin{cases} -\Delta\left(\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)\right) = \lambda u(k), & k \in \mathbb{Z}[1,N] \\ u(0) = u(N+1) = 0 \end{cases} \tag{1}$$

where $N \geq 2$ is a positive integer and Δ denotes the forward difference operator with step 1, that is $\Delta u(k) = u(k+1) - u(k)$. Here and hereafter, we denote by $\mathbb{Z}[a,b]$ the discrete interval $\{a, a+1, \ldots, b\}$ where a and b are integers and $a < b$. Throughout this paper we assume that $\alpha_{k-1} : \mathbb{R} \to \mathbb{R}$ are given functions and for any $k \in \mathbb{Z}[1, N+1]$, α_{k-1} is of the class C^1 on $\mathbb{R}$, while λ is a positive constant. Moreover, we assume that m is a positive constant which satisfies

$$|\alpha_{k-1}(t)| \leq m, \text{ and } |\alpha'_{k-1}(t)| \leq m, \quad \forall\, t \in \mathbb{R},\ \forall\, k \in \mathbb{Z}[1, N+1].$$

Remark 1.1. *We point out the fact that the functions* $\alpha_{k-1}(t) = \cos t$ *or* $\alpha_{k-1}(t) = \sin t$ *satisfy the the above assumptions for* $m = 1$. *We remark that there exist also other functions which satisfy those assumptions. An example can be* $\alpha_{k-1}(t) = e^{-|t|}\sin(\eta t)$ *(*$\eta > 0$*) and* $m = \eta + 1$.

The study of discrete boundary value problems has captured special attention in the last years. We just refer to the recent results of Agarwal [1], Agarwal *et al.* [2, 3], Cai and Yu [5], Yu and Guo [10], Zhang and Liu [12], Mihăilescu-Rădulescu-Tersian [7] and the references therein. The studies regarding discrete boundary problems are highly motivated by their applicability in various fields like mathematical physics, nonlinear partial differential equations (for a comprehensive treatment of this theory we recommend the new book [8]) and numerical analysis. Finally, we remember that a problem similar to (1) was recently analyzed by Costea and Mihăilescu [6] in the continuous case.

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We are interested in finding weak solutions of for problems of type (1). For this we define the function space

$$H = \{\, u : \mathbb{Z}[0, N+1] \to \mathbb{R};\ \text{such that } u(0) = u(N+1) = 0\}.$$

Clearly, H is a N-dimensional Hilbert space (see [3]) with respect to the inner product

$$\langle u, v\rangle = \sum_{k=1}^{N+1} \Delta u(k-1)\Delta v(k-1), \quad \forall\, u, v \in H,$$

and the associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{N+1} |\Delta u(k-1)|^2\right)^{1/2}.$$

On the other hand, it useful to introduce other norms on H, namely

$$|u|_p = \left(\sum_{k=1}^{N} |u(k)|^p\right)^{1/p}, \quad \forall\, u \in H \text{ and } p \geq 2.$$

Since H is a N-dimensional Hilbert space, using the fact that any two norms on a finite dimensional space are equivalent, there exist two positive constants C_1 and C_2 such that

$$C_1\|u\| \leq |u|_2 \leq C_2\|u\|, \quad \forall\, u \in H. \tag{2}$$

Throughout this paper we assume that the constants C_1, C_2 defined above are sharp, that is, C_1 is the largest and C_2 is the smallest constant that satisfy (2).

Definition 1.1. *We say that $\lambda > 0$ is an eigenvalue of problem (1) if there exists $u \in H \setminus \{0\}$ such that*

$$\sum_{k=1}^{N+1} [\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)]\, \Delta v(k-1) = \lambda \sum_{k=1}^{N} u(k)v(k), \forall\, v \in H. \tag{3}$$

Moreover a function $u \in H \setminus \{0\}$ which satisfies (3) for a fixed $\lambda > 0$ is called a *weak solution* of problem (1).

We prove that the above operators possess a *continuous* family of positive eigenvalues, excepting the case when $\alpha_0(0) = \alpha_1(0) = \ldots = \alpha_N(0)$. However, even in that case we can prove the existence of at least one positive eigenvalue. The main results of our study are given by the following theorems:

Theorem 1.1. *Assume that there exist $j, k \in \mathbb{Z}[0, N]$ such that $\alpha_k(0) \neq \alpha_j(0)$. Then any $\lambda \in (0, 1/C_2^2)$ is an eigenvalue of problem (1), where C_2 is the positive constant defined in (2). Moreover, for each $\lambda \in (0, 1/C_2^2)$ there exist an unique corresponding weak solution u_λ.*

Theorem 1.2. *Assume that $\alpha_0(0) = \alpha_1(0) = \ldots = \alpha_N(0)$ and for each $k \in \mathbb{Z}[1, N+1]$ the function α_{k-1} admits a bounded primitive $\Gamma_{k-1} : \mathbb{R} \to \mathbb{R}$. Then there exists at least one positive eigenvalue λ of problem (1), such that $\lambda \geq (m+1)/C_2^2$, where C_2 is the positive constant defined in (2).*

2. Proof of Theorem 1.1

In order to prove Theorem 1.1 we use a method borrowed from the proof of a nonlinear version of the Lax-Milgram Theorem (see Zeidler [11], Section 2.15). Our proof will use as main tool the Banach fixed point theorem (see Zeidler [11], Section 1.6).

First, we define the operators $a : H \times H \to \mathbb{R}$ by

$$a(u,v) = \sum_{k=1}^{N+1} \left[\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)\right] \Delta v(k-1), \quad \forall\, u,v \in H,$$

and $b_\lambda : H \times H \to \mathbb{R}$ by

$$b_\lambda(u,v) = \lambda \sum_{k=1}^{N} u(k)v(k), \quad \forall\, u,v \in H.$$

It is enough to show that for any $\lambda \in (0, 1/C_2^2)$ there exists $u \in H \setminus \{0\}$ such that

$$a(u,v) = b_\lambda(u,v), \quad \forall v \in H.$$

We point out certain properties of the operators a respectively b_λ.

Proposition 2.1. *The operator a satisfies the following properties:*
(1) *for any $u \in H$ the application $v \mapsto a(u,v)$ is linear and continuous on H;*
(2) $a(u,u-v) - a(v,u-v) \geq \|u-v\|^2, \quad \forall\, u,v \in H;$
(3) $|a(u,w) - a(v,w)| \leq (2m+1)\|u-v\| \cdot \|w\|, \quad \forall\, u,v,w \in H.$

Proof. (1) We fix $u \in H$. It is clear that the application $v \mapsto a(u,v)$ is linear. On the the other hand,

$$\begin{aligned}
|a(u,v)| &= \left| \sum_{k=1}^{N+1} \left[\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)\right] \Delta v(k-1) \right| \\
&\leq (m+1)\, |\langle u,v \rangle| + \sum_{k=1}^{N+1} |\alpha_{k-1}(\Delta u(k-1))| \cdot |\Delta v(k-1)| \\
&\leq (m+1)\|u\| \cdot \|v\| + \sum_{k=1}^{N+1} m|\Delta v(k-1)| \\
&\leq (m+1)\|u\| \cdot \|v\| + m \left(\sum_{k=1}^{N+1} 1^2 \right)^{1/2} \left(\sum_{k=1}^{N+1} |\Delta v(k-1)|^2 \right)^{1/2} \\
&= \left[(m+1)\|u\| + m(N+1)^{1/2} \right] \|v\|.
\end{aligned}$$

It follows that $v \mapsto a(u,v)$ is continuous.
(2) We have

$$\begin{aligned}
a(u,u-v) - a(v,u-v) = &\sum_{k=1}^{N+1} \left[\alpha_{k-1}(\Delta u(k-1)) - \alpha_{k-1}(\Delta v(k-1))\right] \Delta(u-v)(k-1) \\
&+ (m+1) \sum_{k=1}^{N+1} |\Delta(u-v)(k-1)|^2
\end{aligned}$$

Using the mean value theorem and taking into account that $|\alpha'_{k-1}(t)| \leq m$ for all $t \in \mathbb{R}$ and all $k \in \mathbb{Z}[1, N+1]$ we deduce that

$$a(u, u-v) - a(v, u-v) = \sum_{k=1}^{N+1} \alpha'_{k-1}(\theta(k-1))|\Delta(u-v)(k-1)|^2 + (m+1)\|u-v\|^2$$

$$\geq -m\|u-v\|^2 + (m+1)\|u-v\|^2 = \|u-v\|^2,$$

where $\theta(k-1) = \mu(k-1)\Delta u(k-1) + [1-\mu(k-1)]\,\Delta v(k-1)$ for all $k \in \mathbb{Z}[1, N+1]$, with $\mu(k-1) \in [0,1]$ for all $k \in \mathbb{Z}[1, N+1]$.

(3) Using the same arguments and notations as above we have

$$\begin{aligned}
|a(u,w) - a(v,w)| &\leq \left| \sum_{k=1}^{N+1} \left[\alpha_{k-1}(\Delta u(k-1)) - \alpha_{k-1}(\Delta v(k-1))\right] \Delta w(k-1) \right| \\
&\quad + (m+1) \left| \sum_{k=1}^{N+1} \Delta(u-v)(k-1)\Delta w(k-1) \right| \\
&\leq \sum_{k=1}^{N+1} |\alpha'_{k-1}(\theta(k-1))| \cdot |\Delta(u-v)(k-1)| \cdot |\Delta w(k-1)| \\
&\quad + (m+1)|\langle u-v, w\rangle| \\
&\leq m \sum_{k=1}^{N+1} |\Delta(u-v)(k-1)||\Delta w(k-1)| + (m+1)\|u-v\| \cdot \|w\| \\
&\leq (2m+1)\|u-v\| \cdot \|w\|.
\end{aligned}$$

□

Proposition 2.2. *For any $\lambda \in (0, 1/C_2^2)$ the operator b_λ satisfies the following properties:*

(1) *b_λ is bilinear and continuous on $H \times H$;*
(2) *$b_\lambda(u,u) \geq 0$ for all $u \in H$;*
(3) *$|b_\lambda(u,w) - b_\lambda(v,w)| \leq \lambda C_2^2 \|u-v\| \cdot \|w\|, \quad \forall\, u, v, w \in H$;*

Proof. (1) It is clear that b_λ is a bilinear operator on $H \times H$. Using (2) we obtain

$$|b_\lambda(u,v)| = \lambda \left| \sum_{k=1}^{N} u(k)v(k) \right| \leq \lambda \sum_{k=1}^{N} |u(k)| \cdot |v(k)| \leq \lambda |u|_2 \cdot |v|_2 \leq \lambda C_2^2 \|u\| \cdot \|v\|.$$

The above argument shows that b_λ is continuous.

(2) For any $u \in H$ we have $b_\lambda(u,u) = \lambda \sum\limits_{k=1}^{N} |u(k)|^2 \geq 0$.

(3) For any $u, v, w \in H$ we have

$$\begin{aligned}
|b_\lambda(u,w) - b_\lambda(v,w)| &= \lambda \left| \sum_{k=1}^{N} (u-v)(k)\, w(k) \right| \\
&\leq \lambda \sum_{k=1}^{N} |(u-v)(k)| \cdot |w(k)| \\
&\leq \lambda |u-v|_2 \cdot |v|_2 \\
&\leq \lambda C_2^2 \|u-v\| \cdot \|w\|.
\end{aligned}$$

The proof of Proposition 2.2 is now complete. □

Proof of Theorem 1.1 Let $\lambda \in (0, 1/C_2^2)$ be arbitrary but fixed. By Proposition 2.1 (1) and the Riesz theorem (see e.g. Brezis [4], Theorem V.5) we deduce that for each $u \in H$ there exists an element called $Au \in H$ such that

$$a(u,v) = \langle Au, v\rangle, \quad \forall v \in H.$$

Thus we can define the operator $A : H \to H$. By Proposition 2.1 (2) and (3) it follows that A satisfies the properties

$$\|u-v\|^2 \leq \langle Au, u-v\rangle - \langle Av, u-v\rangle, \quad \forall u,v \in H \tag{4}$$

that is, A is strongly monotone, and

$$|\langle Au, w\rangle - \langle Av, w\rangle| \leq (2m+1)\|u-v\| \cdot \|w\|, \quad \forall u,v,w \in H. \tag{5}$$

Relation (5) implies

$$\|Au - Av\| = \sup_{\|w\|\leq 1} |\langle Au - Av, w\rangle| \leq (2m+1)\|u-v\|, \quad \forall u,v \in H. \tag{6}$$

On the other hand, by Proposition 2.2 (1) and the Riesz theorem we deduce that for each $u \in H$ there exists an element called $B_\lambda u \in H$ such that

$$b_\lambda(u,v) = \langle B_\lambda u, v\rangle \quad \forall\, v \in H.$$

In this way we can define an operator $B_\lambda : H \to H$. By Proposition 2.2 it follows that B_λ is a linear operator which satisfies the properties

$$\langle B_\lambda u, u-v\rangle - \langle B_\lambda v, u-v\rangle = b_\lambda(u-v, u-v) \leq \lambda C_2^2 \|u-v\|^2, \quad \forall\, u,v \in H \tag{7}$$

and

$$\begin{aligned} \|B_\lambda u - B_\lambda v\| &= \sup_{\|w\|\leq 1} |\langle B_\lambda u - B_\lambda v, w\rangle & (8)\\ &= \sup_{\|w\|\leq 1} |b_\lambda(u-v, w)| & (9)\\ &\leq \lambda C_2^2 \|u-v\|, & (10) \end{aligned}$$

for all $u, v \in H$.

We define the operator $S : H \to H$ by

$$Su = u - t(Au - B_\lambda u)$$

where $t \in \left(0, \frac{2(1-\lambda C_2^2)}{(\lambda C_2^2 + 2m+1)^2}\right)$. The relations (4) and (6-10) show that for each $u, v \in H$ the following inequalities hold true

$$\begin{aligned} \|Su - Sv\|^2 &= \langle Su - Sv, Su - Sv\rangle \\ &= \|u-v\|^2 - 2t\langle Au - Av, u-v\rangle + 2t\langle B_\lambda u - B_\lambda v, u-v\rangle \\ &\quad + t^2\|Au - Av\|^2 - 2t^2\langle Au - Av, B_\lambda u - B_\lambda v\rangle + t^2\|B_\lambda u - B_\lambda v\|^2 \\ &\leq \|u-v\|^2 - 2t\|u-v\|^2 + 2t\lambda C_2^2\|u-v\|^2 + t^2(2m+1)^2\|u-v\|^2 \\ &\quad + 2t^2\|Au - Av\| \cdot \|B_\lambda u - B_\lambda v\| + t^2(\lambda C_2^2)^2\|u-v\|^2 \\ &\leq \left[1 - 2t\left(1 - \lambda C_2^2\right) + 2(2m+1)\lambda C_2^2 t^2 + (2m+1)^2 t^2\right]\|u-v\|^2 \\ &\quad + (\lambda C_2^2)^2 t^2 \|u-v\|^2 \\ &= \beta\|u-v\|^2 \end{aligned}$$

where $\beta = 1 - 2\left(1 - \lambda C_2^2\right) t + (\lambda C_2^2 + 2m + 1)^2 t^2 \geq 0$. If $t = 0$ or $t = \frac{2(1-\lambda C_2^2)}{(\lambda C_2^2+2m+1)^2}$ then $\beta = 1$. This implies that $\sqrt{\beta} < 1$ for all $t \in \left(0, \frac{2(1-\lambda C_2^2)}{(\lambda C_2^2+2m+1)^2}\right)$. Therefore,

$$\|Su - Sv\| \leq \sqrt{\beta}\|u - v\|, \quad \forall u, v \in H$$

that is, S is $\sqrt{\beta}$-contractive with $\sqrt{\beta} < 1$. By the Banach fixed point theorem (see Zeidler [11], Section 1.6) it follows that the problem

$$u = Su$$

has an unique solution $u_\lambda \in H$, that is, the problem

$$Au_\lambda = B_\lambda u_\lambda$$

has an unique solution $u_\lambda \in H$. It follows that

$$\langle Au_\lambda, v\rangle = \langle B_\lambda u_\lambda, v\rangle, \quad \forall\, v \in H,$$

that is,

$$\sum_{k=1}^{N+1} \left[\alpha_{k-1}(\Delta u_\lambda(k-1)) + (m+1)\Delta u_\lambda(k-1)\right] \Delta v(k-1) = \lambda \sum_{k=1}^{N} u_\lambda(k) v(k), \forall\, v \in H. \tag{11}$$

Finally we have to prove that u_λ is non-trivial. Arguing by contradiction, we assume that $u_\lambda(k) = 0, \forall\, k \in \mathbb{Z}[1, N]$. We obtain that

$$\sum_{k=1}^{N+1} \alpha_{k-1}(0)\Delta v(k-1) = 0, \quad \forall\, v \in H. \tag{12}$$

Taking $v_i \in H$ such that $v_i(j) = \delta_{ij}$ for $i \in \mathbb{Z}[1, N]$ and using (12) we deduce that $\alpha_0(0) = \ldots = \alpha_N(0)$ which is a contradiction.

Thus we have proved that any $\lambda \in (0, 1/C_2^2)$ is an eigenvalue of problem (1).

3. Proof of Theorem 1.2

First we point out the fact that under the hypotheses of Theorem 1.2 the conclusion of Theorem 1.1 does not hold. Indeed, in that case we have $\alpha_0(0) = \ldots = \alpha_N(0)$ and thus the non-triviality of the solution obtained by applying the Banach fixed point theorem cannot be stated. However, we can prove the existence of a positive eigenvalue of problem (1) under the hypotheses of Theorem 1.2 using a minimization technique. Such techniques are usually used in finding principal eigenvalues (see e.g. Szulkin-Willem [9]).

Since every function α_{k-1} admits a bounded primitive Γ_{k-1}, we can assume that there exist $M > 0$ such that for any $k \in \mathbb{Z}[1, N+1]$ we have

$$|\Gamma_{k-1}(t)| \leq M \quad \forall\, t \in \mathbb{R}.$$

We define the functional $I : H \to \mathbb{R}$,

$$I(u) = \sum_{k=1}^{N+1} \Gamma_{k-1}(\Delta u(k-1)) + \frac{m+1}{2}(\Delta u(k-1))^2.$$

Standard arguments assure that $I \in C^1(H, \mathbb{R})$ and

$$\langle I'(u), v\rangle = \sum_{k=1}^{N+1} \left[\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)\right] \Delta v(k-1).$$

We consider the minimization problem
(P) minimize $I(u)$ under conditions $u \in H$ and $\sum_{k=1}^{N} |u(k)|^2 = 1$.

We point out the fact that problem **(P)** is well defined. Indeed for all $u \in H$ with $\sum_{k=1}^{N} |u(k)|^2 = 1$ we have

$$\begin{aligned} I(u) &= \sum_{k=1}^{N+1} \Gamma_{k-1}(\Delta u(k-1)) + \frac{m+1}{2}\|u\|^2 \\ &\geq -M(N+1) + \frac{m+1}{2C_2^2}|u|_2^2 \\ &= -M(N+1) + \frac{m+1}{2C_2^2} > -\infty. \end{aligned}$$

Proposition 3.1. *The functional I is continuous on H.*

Proof. Let $\{u_n\}$ a sequence in H such that u_n converges to u in H. We observe that for all $k \in \mathbb{Z}[1, N+1]$

$$|\Delta u_n(k-1) - \Delta u(k-1)| \leq \left(\sum_{j=1}^{N+1} |\Delta u_n(j-1) - \Delta u(j-1)|^2 \right)^{1/2} = \|u_n - n\| \to 0.$$

The above relation implies that $\Delta u_n(k-1) \to \Delta u(k-1)$ as $n \to \infty$ for all $k \in \mathbb{Z}[1, N+1]$. Using the fact that each Γ_{k-1} is continuous we obtain

$$|I(u_n) - I(u)| \leq \sum_{k=1}^{N+1} |\Gamma_{k-1}(\Delta u_n(k-1)) - \Gamma_{k-1}(\Delta u(k-1))| + \frac{m+1}{2}\left| \|u_n\|^2 - \|u\|^2 \right|.$$

Since the the term on the right-hand side of the above relation converges to 0 as $n \to \infty$, we conclude that $I(u_n)$ converges to $I(u)$ and the proof of Proposition 3.1 is now complete. □

Proof of Theorem 1.2 Due to the fact that problem **(P)** is well defined there exists $\Lambda \in \mathbb{R}$ such that

$$\Lambda = \inf_{u \in H,\, |u|_2^2 = 1} I(u).$$

There exists $\{u_n\}$ a minimizing sequence in H, that is,

$$I(u_n) \to \Lambda$$

and $\sum_{k=1}^{N} |u_n(k)|^2 = 1$ for all n. We point out that the sequence $\{u_n\}$ is bounded in H. Indeed, the above information shows that

$$\begin{aligned} \|u_n\|^2 &= \frac{2}{m+1}\left[I(u_n) - \sum_{k=1}^{N+1} \Gamma_{k-1}(\Delta u_n(k-1)) \right] \\ &\leq \frac{2}{m+1}\left[I(u_n) + M(N+1) \right] \\ &\leq \frac{2}{m+1}\left[\Lambda + c + M(N+1) \right], \quad \forall n \geq 1, \end{aligned}$$

where c is a positive constant.

The fact that $\{u_n\}$ is bounded in H and H is a finite dimensional space implies that there exists a subsequence, still denoted by $\{u_n\}$, that converges to an element $u \in H$. It can be easily shown that $|u|_2^2 = 1$. By Proposition 3.1 we have $I(u_n) \to I(u) = \Lambda$. Thus we obtain that u is a solution of problem **(P)**.

Let $v \in H$ be arbitrary but fixed. Then for all ε in a suitable neighborhood of the origin the function

$$g(\varepsilon) = I\left(\frac{u+\varepsilon v}{|u+\varepsilon v|_2}\right)$$

is well defined and possess a minimum in $\varepsilon = 0$. Then it is clear that $g'(0) = 0$. A simple computation shows that

$$\begin{aligned} g'(\varepsilon) &= \sum_{k=1}^{N+1}\left[\alpha_{k-1}\left(\frac{\Delta(u+\varepsilon v)(k-1))}{|u+\varepsilon v|_2}\right) + (m+1)\frac{\Delta(u+\varepsilon v)(k-1)}{|u+\varepsilon v|_2}\right] \\ & \frac{\Delta v(k-1)|u+\varepsilon v|_2^2 - \Delta(u+\varepsilon v)(k-1)\sum\limits_{k=1}^{N+1} u(k)v(k)+\varepsilon v^2(k)}{|u+\varepsilon v|_2^3} \end{aligned}$$

Since $|u|_2^2 = 1$ we get

$$\begin{aligned} g'(0) &= \sum_{k=1}^{N+1}\left[\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)\right]\Delta v(k-1) \\ &\quad - \sum_{k=1}^{N+1}\left[\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)\right]\Delta u(k-1)\sum_{k=1}^{N} u(k)v(k) \end{aligned}$$

and thus

$$\sum_{k=1}^{N+1}\left[\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)\right]\Delta v(k-1) = \lambda\sum_{k=1}^{N} u(k)v(k)$$

where

$$\begin{aligned} \lambda &= \sum_{k=1}^{N+1}\left[\alpha_{k-1}(\Delta u(k-1)) + (m+1)\Delta u(k-1)\right]\Delta u(k-1) \\ &\geq -m\sum_{k=1}^{N+1}\Delta u(k-1) + (m+1)\|u\|^2 \\ &= (m+1)\|u\|^2 \geq \frac{m+1}{C_2^2}|u|_2^2 = \frac{m+1}{C_2^2} \end{aligned}$$

We conclude that $\lambda \geq (m+1)/C_2^2$ is an eigenvalue for problem (1). The proof of Theorem 1.2 is complete.

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