Annals of the University of Craiova, Math. Comp. Sci. Ser.
Volume 35, 2008, Pages 87–96
ISSN: 1223-6934

Convex Functions on Time Scales

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Abstract. We define the notion of a convex function on time scales. Some results connecting this notion with the notion of convex function on a classic interval and convex sequences are also included. We also define the subdifferential of a convex function on time scale and present some properties regarding it.

2000 Mathematics Subject Classification. 26D15; 39A13.
Key words and phrases. Time scales, convex function, dynamic derivatives.

1. Introduction

The theory of time scales was introduced by Stefan Hilger in his PhD thesis [6] in 1988. The calculus and applications of dynamic derivatives on time scales provide an unification and an extension of traditional differential and difference equations. In the same time, they can be seen as an unification of the discrete theory with the continuous theory. Also, it is a crucial tool in many computational and numerical applications.

Many results from the continuous case are carried over the discrete very easy, but others seem to be completely different. The study on time scales comes to reveal such discrepancies and to make us understand the difference between the two cases.

Also, once we have proved a result for a general time scale, by choosing $\mathbb{R}$ we get a continuous result, while, if we choose $\mathbb{Z}$, we get the same result in the discrete case. Since this are not the only time scales, the result thus obtained is much more general. And so, the time scale calculus brings an unification and also an extension of the calculus.

The applications of this calculus are substantial and they have received a lot of attention in the last years. The most important ones include the dynamic equations, which include both differential equations and difference equations. Also, the applications in biology, engineering, economics, physics, neural networks, social sciences and others have lately came to light. For more details and the basic rules of calculus associated with the dynamic derivatives and integrals, see [1], [2], [4], [6], [10] and [11].

2. Preliminaries

A *time scale* (or *measure chain*) is any nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$ (together with the topology of subspace of $\mathbb{R}$).

Throughout this paper $\mathbb{T}$ will denote a time scale and, for any I interval of $\mathbb{R}$ (open or closed), $I_{\mathbb{T}} = I \cap \mathbb{T}$ a time scale interval.

Received: 27 November 2008.

For all $t \in \mathbb{T}$, we define the *forward jump operator* σ and the *backward jump operator* ρ by the formulas:

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}.$$

We make the convention:

$$\inf \emptyset := \sup \mathbb{T}, \quad \sup \emptyset := \inf \mathbb{T}.$$

If $\sigma(t) > t$, then t is said to be *right-scattered*, and if $\rho(t) < t$, then t is said to be *left-scattered*. The points that are simultaneously right-scattered and left-scattered are called *isolated*. If $\sigma(t) = t$, then t is said to be *right dense*, and if $\rho(t) = t$, then t is said to be *left dense*. The points that are simultaneously right-dense and left-dense are called *dense*.

The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by

$$\mu(t) := \sigma(t) - t$$

and

$$\nu(t) := t - \rho(t)$$

are called, respectively, the *forward* and *backward graininess functions*.

If $\mathbb{T}$ has a right-scattered minimum m, then define $\mathbb{T}_\kappa = \mathbb{T} \backslash \{m\}$; otherwise $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, then define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Finally, put $\mathbb{T}^\kappa_\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

Definition 2.1. *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, one defines the delta derivative of f in t, to be the number denoted by $f^\Delta(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

for all $s \in U_\mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, one defines the nabla derivative of f in t, to be the number denoted by $f^\nabla(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| < \varepsilon|\rho(t) - s|,$$

for all $s \in V_\mathbb{T}$.

We say that f is *delta differentiable* on $\mathbb{T}^\kappa$, provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and that f is *nabla differentiable* on $\mathbb{T}_\kappa$, provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

For $\mathbb{T} = \mathbb{R}$, we have

$$f^\Delta(t) = f^\nabla(t) = f'(t).$$

while, for $\mathbb{T} = \mathbb{Z}$, we get

$$f^\Delta(t) = f(t+1) - f(t)$$

is the forward difference operator, while

$$f^\nabla(t) = f(t) - f(t-1)$$

is the backward difference operator.

For all $t \in \mathbb{T}^\kappa$, we have the following properties:

(i) If f is delta differentiable at t, then f is continuous at t.
(ii) If f is left continuous at t and t is right-scattered, then f is delta differentiable at t with $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$.
(iii) If t is right-dense, then f is delta differentiable at t, if and only if, the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^\Delta(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is delta differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.

Also, for all $t \in \mathbb{T}_\kappa$ we have the following properties:

(i) If f is nabla differentiable at t, then f is continuous at t.

(ii) If f is right continuous at t and t is left-scattered, then f is nabla differentiable at t with $f^{\nabla}(t) = \frac{f(t)-f(\rho(t))}{\nu(t)}$.

(iii) If t is left-dense, then f is nabla differentiable at t, if and only if, the limit $\lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case, $f^{\nabla}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$.

(iv) If f is nabla differentiable at t, then $f(\rho(t)) = f(t) - \nu(t)f^{\nabla}(t)$.

Definition 2.2. *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. We denote by C_{rd} the set of all rd-continuous functions.*

A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous, if it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits are finite at all right-dense points in $\mathbb{T}$. We denote by C_{ld} the set of all ld-continuous functions.

Obviously, the set of continuous functions on $\mathbb{T}$ contains both C_{rd} and C_{ld}.

Definition 2.3. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}^\kappa$. Then, we define the delta integral by $\int_a^t f(s)\Delta s = F(t) - F(a)$.*

A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $G^{\nabla}(t) = f(t)$, for all $t \in \mathbb{T}_\kappa$. Then, we define the nabla integral by $\int_a^t f(s)\Delta s = G(t) - G(a)$.

According to [2, Theorem 1.74], every rd-continuous function has a delta antiderivative and every ld-continuous function has a nabla antiderivative.

Theorem 2.1. ([2, Theorem 1.75])

(i) If $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$, then

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t).$$

(ii) If $f \in C_{ld}$ and $t \in \mathbb{T}^\kappa$, then

$$\int_{\rho(t)}^{t} f(s)\nabla s = \nu(t)f(t).$$

Theorem 2.2. ([2], Theorem 1.76) *If $f^{\Delta} \geq 0$ or $f^{\nabla} \geq 0$, then f is nondecreasing.*

Theorem 2.3. ([2, Theorem 1.77]) *If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$ and $f, g \in C_{rd}$, then*

(i) $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$

(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$

(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$

(vii) $\int_a^a f(t)\Delta t = 0;$

(viii) *if $f(t) \geq 0$ for all t, then $\int_a^b f(t)\Delta t \geq 0$;*

A similar theorem works for the nabla antiderivative (for $f, g \in C_{ld}$).

3. Convex functions

Now we define a convex function on a time scale and we give some properties of convex functions in relation with the properties from the continuous case and the discrete case.

Definition 3.1. *A function $f : \mathbb{T} \to \mathbb{R}$ is called* convex *on $I_{\mathbb{T}}$, if*

$$f(\lambda t + (1-\lambda)s) \leq \lambda f(t) + (1-\lambda) f(s), \tag{1}$$

for all $t, s \in I_{\mathbb{T}}$ and all $\lambda \in [0,1]$ such that $\lambda t + (1-\lambda)s \in I_{\mathbb{T}}$.

The function f is strictly convex *on $I_{\mathbb{T}}$ if the inequality (1) is strict for distinct $t, s \in I_{\mathbb{T}}$ and $\lambda \in (0,1)$.*

The function f is concave *(respectively,* strictly concave*) on $I_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).*

A function that is both convex and concave on $I_{\mathbb{T}}$ is called affine *on $I_{\mathbb{T}}$.*

Remark 3.1. *If $\mathbb{T} = \{a, b\}$ (it consists only of two points) or $\mathbb{T} = \{a\}$ (only one point), then any function $f : \mathbb{T} \to \mathbb{R}$ is affine.*

Remark 3.2. *Definition 3.1 is equivalent to Definition 14 from* [7]*. If $t_1, t_2 \in I_{\mathbb{T}}$, with $t_1 < t_2$ and $t \in I_{\mathbb{T}}$ such that $t_1 \leq t \leq t_2$ and $t = \lambda t_1 + (1-\lambda)t_2$, then we get $\lambda = \frac{t_2 - t}{t_2 - t_1}$ and $1 - \lambda = \frac{t - t_1}{t_2 - t_1}$. Thus, the inequality 1 can be written as*

$$(t_2 - t) f(t_1) + (t_1 - t_2) f(t) + (t - t_1) f(t_2) \geq 0, \tag{2}$$

that is inequality (7) from [7, Definition 14]*.*

Remark 3.3. (i) *For $\mathbb{T} = \mathbb{R}$, Definition 3.1 is exactly the definition of a convex or concave function, because $\lambda t + (1-\lambda)s \in I$, for all $t, s \in I$.*

(ii) *For $\mathbb{T} = \mathbb{N}$, Definition 3.1 gives us the definition of convex sequences (see, for example,* [13]*). A real sequence $(a_i)_{i=1}^n$ is said to be convex if*

$$a_i + a_{i+2} \geq a_{i+1} \quad \text{for all} \quad i \in \mathbb{N},\ i + 2 \leq n.$$

Indeed, a real sequence can be seen as a function defined on the time scale $\mathbb{T} = \mathbb{N}$. If $i, i+2 \in \mathbb{N}$ then

$$\frac{1}{2} i + \frac{1}{2}(i+2) = i + 1 \in \mathbb{N}$$

and we get the convexity of a sequence, from the Definition 3.1.

The next theorem can be found also in [7].

Theorem 3.1. *Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be a delta differentiable function on $I_{\mathbb{T}}^{\kappa}$. If f^{Δ} is nondecreasing (nonincreasing) on $I_{\mathbb{T}}^{\kappa}$, then f is convex (concave) on $I_{\mathbb{T}}$.*

Proof. As noticed in Remark 3.1, if $I_{\mathbb{T}}^{\kappa}$ has only one point or two points, then any f is convex and concave on $I_{\mathbb{T}}^{\kappa}$.

Let $s < t < u \in I_{\mathbb{T}}^{\kappa}$. Using Theorem 1.14 from [3], we get the existence of points $s_1, s_2 \in [s, t)_{\mathbb{T}}$ and $u_1, u_2 \in [t, u)_{\mathbb{T}}$ such that

$$f^{\Delta}(s_1) \leq \frac{f(t) - f(s)}{t - s} \leq f^{\Delta}(s_2) \text{ and } f^{\Delta}(u_1) \leq \frac{f(u) - f(t)}{u - t} \leq f^{\Delta}(u_2). \tag{3}$$

Since $s < s_2 < u_1$, from (3) we get

$$\frac{f(t) - f(s)}{t - s} \leq f^{\Delta}(s_2) \leq f^{\Delta}(u_1) \leq \frac{f(u) - f(t)}{u - t}, \tag{4}$$

for nondecreasing f^{Δ} and

$$\frac{f(t)-f(s)}{t-s} \geq f^{\Delta}(s_2) \geq f^{\Delta}(u_1) \geq \frac{f(u)-f(t)}{u-t}, \tag{5}$$

for nonincreasing f^{Δ}.

The condition 4 is equivalent with the condition 1 and with the convexity of f, while the condition 5 is equivalent with the reversed condition from (1) and with its concavity.

□

Obviously, the nabla version of the Theorem 3.1 holds too.

Theorem 3.2. *Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be a nabla differentiable function on $I_{\mathbb{T}_\kappa}$. If f^{∇} is nondecreasing (nonincreasing) on $I_{\mathbb{T}_\kappa}$, then f is convex (concave) on $I_{\mathbb{T}}$.*

Corollary 3.1. *Let $f : I_{\mathbb{T}} \to \mathbb{R}$ be a continuous function.*

(i) *If $f^{\Delta\Delta}$ exists on $I_{\mathbb{T}}^{\kappa^2}$ and $f^{\Delta\Delta} \geq 0$ ($f^{\Delta\Delta} \geq 0$) for all $t \in I_{\mathbb{T}}^{\kappa^2}$, then f is convex (concave).*

(ii) *If $f^{\nabla\nabla}$ exists on $I_{\mathbb{T}_{\kappa^2}}$ and $f^{\nabla\nabla} \geq 0$ ($f^{\nabla\nabla} \geq 0$) for all $t \in I_{\mathbb{T}_{\kappa^2}}$, then f is convex (concave).*

Proof. Using Theorem 2.2, f^{Δ} is increasing and using Theorem 3.1 it follows the convexity (concavity) of f. In the same manner, Theorems 2.2 and 3.2 prove the conclusion from (ii). □

It is well known that, in the continuous case, a convex function on $[a,b]$ is continuous on (a,b) (see, for example, [8]). A similar result is valid for a time scale.

Theorem 3.3. *A convex function on $[a,b]_{\mathbb{T}}$ is continuous on $(a,b)_{\mathbb{T}}$.*

Proof. Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ a convex function and $t \in (a,b)_{\mathbb{T}}$. If there is a real interval $[c,d]$ such that $[c,d] \subset [a,b]_{\mathbb{T}}$ and $t \in (c,d)$, then, since f is convex on $[c,d]$, it is continuous on (c,d), as in the continuous case.

Suppose t is left-scattered and right dense. We have two cases: (1) there exists a $t_1 \in [a,b]_{\mathbb{T}}$ such that $[t,t_1] \subset [a,b]_{\mathbb{T}}$, or (2) t is the limit of a nonincreasing sequence $t_1 > t_2 > ... > t$.

(1) From the continuous case, we know that f is continuous on (t,t_1) and $f(t) > \lim_{s \searrow t} f(s) = f(t+)$.

Since $t \in (a,b)_{\mathbb{T}}$, we can find $x < t < y \in [a,b]_{\mathbb{T}}$ such that $t = \lambda x + (1-\lambda)y$, for a $\lambda \in [0,1]$. Since f is convex, we have

$$f(t) \leq \lambda f(x) + (1-\lambda) f(y).$$

Fixing arbitrarily x and making y to tend to t (this is possible, since t is right dense), then λ will tend to 0 and we get

$$f(t) \leq \lim_{s \searrow t} f(s) = f(t+),$$

and so, f is continuous in t.

(2) Since t_i are isolated points, for $i \in \mathbb{N}$ then f is delta differentiable in all t_i and, using Theorem 3.1 we have

$$f^{\Delta}(t_i) \geq f^{\Delta}(t_{i+1}), \text{ for all } i \in \mathbb{N},$$

and, thus $\left(f^{\Delta}(t_i)\right)_i$ is a nondecreasing sequence of real numbers. From the existence of $x \in [a,b]_{\mathbb{T}}$, $x < t$ and $\sigma(x) = t$, we get the existence of $f^{\Delta}(x) = \frac{f(t)-f(x)}{t-x}$ as a real number and

$$f^{\Delta}(x) \geq f^{\Delta}(t_i), \text{ for all } i \in \mathbb{N}$$

that is, the convergence of the sequence $\left(f^{\Delta}(t_i)\right)_i$ and the existence of $f^{\Delta}(t)$. The delta differentiability of f in t gives us the continuity in t.

The same arguments will assure us the continuity in t, a right-scattered and left dense point of $(a, b)_{\mathbb{T}}$.

If t is a dense point, but there is no real interval $[c, d]$ such that $[c, d] \subset [a, b]_{\mathbb{T}}$ and $t \in (c, d)$, then we have two monotone sequences $(s_i)_{i\in\mathbb{N}}$ and $(t_i)_{i\in\mathbb{N}}$ such that $s_1 \leq ...s_n \leq ... \leq t \leq ... \leq t_n \leq ... \leq t_1$ and t is the limit of both sequences. Arguing as above, we get the convergence of $\left(f^{\Delta}(s_i)\right)_i$ and $\left(f^{\Delta}(t_i)\right)_i$. If

$$\lim_{i\to\infty} f^{\Delta}(s_i) = \lim_{i\to\infty} f^{\Delta}(t_i)$$

then we get the delta differentiability of f in t, and thus the continuity. Elsewhere, modifying f in all the points s_i to get the above equality, will lead us to a continuous function in t, but since we did not change the value in t nor in t_i we have the right-continuity of f in t. Analogously, we get the left left-continuity in t, and thus, f is continuous in t.

Finally, if t is isolated, then f is continuous in t and, since this are all the possible cases, we get the conclusion. □

As it can be noticed, the definition of convexity on time scales is much similar with the definition of convexity on $\mathbb{R}$. The connection between those notions is better observed from the following theorem.

Theorem 3.4. *A function $f : \mathbb{T} \to \mathbb{R}$ is convex on $I_{\mathbb{T}} = I \cap \mathbb{T}$ if and only if there exists a convex function $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}(t) = f(t)$, for all $t \in I_{\mathbb{T}}$.*

Proof. The sufficiency is clear, since if it exists a convex function $\tilde{f}$ defined on I such that $\tilde{f}(t) = f(t)$, for all $t \in I_{\mathbb{T}}$, then

$$\tilde{f}((1-\lambda)t + \lambda s) \leq (1-\lambda)\tilde{f}(t) + \lambda\tilde{f}(s),$$

for all $t, s \in I$ and all $\lambda \in [0, 1]$. When $t, s, (1-\lambda)t+\lambda s \in \mathbb{T}$, then we get the inequality from (1), that is the convexity of f on $I_{\mathbb{T}}$.

For the necessity, we define $\tilde{f}$ using f and uniting all scattered points by lines. More exactly

$$\tilde{f}(t) = \begin{cases} f(t), & \text{if} \quad t \in I_{\mathbb{T}} \\ \\ f(s) + \frac{f(\sigma(s)) - f(s)}{\mu(s)}(t-s), & \text{if} \quad t \in (s, \sigma(s)),\ s \in I_{\mathbb{T}},\ s \text{ right-scattered.} \end{cases}$$

Now, we shall prove that this function is convex on I. For that, it suffices to prove that for any $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$\tilde{f}(\lambda x + (1-\lambda)y) \leq \lambda\tilde{f}(x) + (1-\lambda)\tilde{f}(y). \tag{6}$$

If $I_{\mathbb{T}} = \{s, \sigma(s)\}$, (that is, $I_{\mathbb{T}}$ has only two points) then $\tilde{f}$ is an affine function, and so, it is convex. If $x, y, \lambda x + (1-\lambda)y \in I_{\mathbb{T}}$ then the inequality 6 comes from the convexity of f, while for $x, y \in I_{\mathbb{T}}$ and $y = \sigma(x)$, we get equality in (6), since $\tilde{f}$ is affine on $[x, y]$.

For the case $x, y \in I_{\mathbb{T}}$ and $y > \sigma(x)$, we need to notice that the chord uniting the points $(x, f(x))$ and $(y, f(y))$ is above all the points $(z, f(z))$, with $z \in I_{\mathbb{T}}$ (using the convexity of f on $I_{\mathbb{T}}$). Since the graph of $\tilde{f}$ on $[x, y]$ is obtained by uniting this points

$(z, f(z))$, then it will be under the chord uniting $(x, f(x))$ and $(y, f(y))$ and thus, we have inequality 6.

If $x \in I_{\mathbb{T}}$ and $y \in I \setminus \mathbb{T}$, with $y \leq \sigma(x)$, then $(y, f(y))$ is on the chord from $(x, f(x))$ to $(\sigma(x), f(\sigma(x)))$ and so are all the points $(\lambda x + (1-\lambda)y, f(\lambda x + (1-\lambda)y))$ and we get again equality in (6). If $y > \sigma(x)$, then we can find $z \in I_{\mathbb{T}}$ such that $x < z$ and $z < y < \sigma(z)$. Using the convexity of f in $x, z, \sigma(z)$ we get

$$\frac{f(x) - f(z)}{x - z} \leq \frac{f(x) - f(\sigma(z))}{x - \sigma(z)},$$

while, from the affinity of $\tilde{f}$ on $[z, \sigma(z)]$ we have

$$\frac{f(x) - f(z)}{x - z} \leq \frac{f(x) - \tilde{f}(y)}{x - y} \leq \frac{f(x) - f(\sigma(z))}{x - \sigma(z)},$$

that is, the chord from $(x, f(x))$ to $(y, f(y))$ is between the chord from $(x, f(x))$ to $(z, f(z))$ and the chord from $(x, f(x))$ to $(\sigma(z), f(\sigma(y)))$. If $\lambda x + (1-\lambda)y \in [z, y]$ the the conclusion follows from the affinity of $\tilde{f}$, while if $\lambda x + (1-\lambda)y \in [x, z]$, then the conclusion comes by using the same arguments as for the case $x, y \in I_{\mathbb{T}}$ and $y > \sigma(x)$.

Some symmetric arguments will solve the case $y \in I_{\mathbb{T}}$ and $x \in I \setminus \mathbb{T}$ and also the case $x, y \in I \setminus \mathbb{T}$ and thus, the theorem is completed proved. □

Remark 3.4. *Since we did not use the continuity of f on $(a, b)_{\mathbb{T}}$ during the proof of Theorem 3.4, we can give another proof of Theorem 3.3. For any convex function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ there is a convex function $\tilde{f} : [a, b] \to \mathbb{R}$ such that $\tilde{f}(t) = f(t)$ for all $t \in [a, b]_{\mathbb{T}}$. Since $\tilde{f}$ is continuous on (a, b), it follows the continuity of f on $(a, b)_{\mathbb{T}}$.*

4. The subdifferential

Before defining the subdifferential of a function on a time scale, we define the left nabla derivative and the right delta derivative of function.

For a dense point $t \in \mathbb{T}$, we know that

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

when that limit exists. The same discussion holds for $f^{\nabla}(t)$. The nonexistence of the limit may be caused by the different lateral limits in the above relation. Thus, for a dense point $t \in \mathbb{T}$, we define

$$f'_{-} = \lim_{s \to t,\ s < t} \frac{f(t) - f(s)}{t - s}$$

and

$$f'_{+} = \lim_{s \to t,\ s > t} \frac{f(t) - f(s)}{t - s}.$$

For a function $f : \mathbb{T} \to \mathbb{R}$ and a point $t \in \mathbb{T}^{\kappa}$, such that $f^{\Delta}(t)$ or f'_{+} exist, we define

$$f_{+}^{\Delta}(t) = \begin{cases} f^{\Delta}(t), & \text{if } \quad t \text{ is scatered} \\ f'_{+}, & \text{if } \quad t \text{ is dense.} \end{cases}$$

If $f^{\nabla}(t)$ or f'_{-} exist for a point $t \in \mathbb{T}_{\kappa}$, we define

$$f_-^\nabla(t) = \begin{cases} f^\nabla(t), & \text{if} \quad t \text{ is scatered} \\ f'_-, & \text{if} \quad t \text{ is dense.} \end{cases}$$

We call $f_-^\nabla(t)$ the left nabla derivative of f in t and $f_+^\Delta(t)$ the right delta derivative of f in t. We say that f is *right delta differentiable* on $\mathbb{T}^\kappa$, provided $f_+^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$ and that f is *left nabla differentiable* on $\mathbb{T}_\kappa$, provided $f_-^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

Regarding the properties of the left nabla derivative and the right delta derivative of a point t, it is easy to see that if t is scattered, then $f_-^\nabla(t) = f^\nabla(t)$ and $f_+^\Delta = f^\Delta$, while if t is dense and $f_-^\nabla(t) = f_+^\Delta$, then we have

$$f_-^\nabla(t) = f^\nabla(t) = f_+^\Delta = f^\Delta,$$

elsewhere, the function f is not delta nor nabla differentiable in t.

Remark 4.1. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and a point $t \in \mathbb{T}_\kappa^\kappa$ such that there exist finite $f_-^\nabla(t)$ and $f_+^\Delta(t)$. Then the function f is continuous.*

Indeed, if f is scattered, then the existence of $f_-^\nabla(t)$ and $f_+^\Delta(t)$ is the existence of $f^\nabla(t)$ and $f^\Delta(t)$ and that implies the continuity of f in t. If t is dense, then we have $f'_-(t)$ and $f'_+(t)$ as finite numbers. If $\lim_{s\to t, s<t} f(s) \neq f(t)$ or $\lim_{s\to t, s>t} f(s) \neq f(t)$, then $f'_-(t)$ and $f'_+(t)$ are infinite.

Using this notations, we give the next theorem, which is a time scale variant of a Theorem of Stolz (see [12]).

Theorem 4.1. *Let $f : [a, b]_\mathbb{T} \to \mathbb{R}$ be a convex function. Then for all $x, y \in (a, b)_\mathbb{T}$, with $x < y$ we have*

$$f_-^\nabla(x) \leq f_+^\Delta(x) \leq f_-^\nabla(y) \leq f_+^\Delta(y)$$

and thus both f_-^∇ and f_+^Δ exist and they are nondecreasing on $(a, b)_\mathbb{T}$.

Proof. For every convex function f, from the Definition 3.1, we get

$$\frac{f(t) - f(x)}{t - x} \leq \frac{f(s) - f(x)}{s - x} \leq \frac{f(u) - f(x)}{u - x}$$

for every $t \leq s < x < u \in [a, b]_\mathbb{T}$.

Now, if x is a left dense point and right-scattered, then making s to tend to x, we shall get

$$f^\nabla(x) \leq \frac{f(u) - f(x)}{u - x}$$

noticing that $S : [a, b]_\mathbb{T} \to \mathbb{R}$, $S(s) = \frac{f(s)-f(x)}{s-x}$ is nondecreasing and bonded above as a function of s. If we put $u = \sigma(x)$, we get $f^\nabla(x) \leq f^\Delta(x)$. A symmetric argument will yield the same conclusion for x right dense point and left-scattered. If x is an isolated point, we just have to put $s = \rho(x)$ and $u = \sigma(x)$ and get the same inequality. If t is dense, then making s and u to tend to x and using the nondecreasing function S, we get

$$\lim_{s\to t,\ s<t} \frac{f(s) - f(x)}{s - x} \leq \lim_{s\to t,\ u>t} \frac{f(u) - f(x)}{u - x}.$$

So, for every $x \in (a, b)_\mathbb{T}$ we have

$$f_-^\nabla(x) \leq f_+^\Delta(x).$$

On the other hand, for $x < u \le v < y$ we have

$$\frac{f(u)-f(x)}{u-x} \le \frac{f(v)-f(x)}{v-x} \le \frac{f(v)-f(y)}{v-y}$$

and we get $f_+^{\Delta}(x) \le f_-^{\nabla}(y)$.

□

Remark 4.2. *Using Remark 4.1 and Theorem 4.1, we can find again the continuity of a convex function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ in $(a, b)_{\mathbb{T}}$, since every $t \in (a, b)_{\mathbb{T}}$ has both $f_+^{\Delta}(t)$ and $f_-^{\nabla}(t)$.*

Now, we are able to present an extension of the subdifferential of a function.

A function $f : I_{\mathbb{T}} \to \mathbb{R}$ admits a *support line* at $t \in I_{\mathbb{T}}$ if there exists a $\lambda \in \mathbb{R}$ such that

$$f(s) \ge f(t) + \lambda(s-t)$$

for every $s \in I_{\mathbb{T}}$.

The set of all such λ is called the *subdifferential of f at t* and it is denoted by $\partial f(t)$. The geometric interpretation of the subdifferential is that it provides the slopes of the lines that "touches" the graph of f, but they do not separate the points of its graph. The subdifferential is always a convex set, since if $\lambda_1, \lambda_2 \in \partial f(t)$, then any convex combination of λ_1 and λ_2 belongs to $\partial f(t)$, but it may be empty. We shall prove that the convex functions have the property that $\partial f(t) \ne \emptyset$ at all points $t \in (a, b)_{\mathbb{T}}$.

Theorem 4.2. *Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be a convex function on a time scale. Then $\partial f(t) \ne \emptyset$ for all $t \in (a, b)_{\mathbb{T}}$. Moreover, every function $\varphi : [a, b]_{\mathbb{T}} \to \mathbb{R}$ for which $\varphi(t) \in \partial f(t)$, verifies the double inequality*

$$f_-^{\nabla}(t) \le \varphi(t) \le f_+^{\Delta}(t)$$

for all $t \in (a, b)_{\mathbb{T}}$ and thus it is nondecreasing on $(a, b)_{\mathbb{T}}$.

Proof. Let $t < u \le s \in [a, b]_{\mathbb{T}}$. Then

$$\frac{f(u)-f(t)}{u-t} \le \frac{f(s)-f(t)}{s-t}$$

If u tends to t, then $\frac{f(u)-f(t)}{u-t}$ tends to $f'_+(t)$, for t right dense or to $f^{\Delta}(t)$, for t right-scattered. Thus, it yields

$$f(s) \ge f(t) + f_+^{\Delta}(t)(s-t).$$

Reasoning the same for $s \le u < t \in [a, b]_{\mathbb{T}}$ we get

$$f(s) \ge f(t) + f_-^{\nabla}(t)(s-t).$$

Using Theorem 4.1, we have $f_-^{\nabla}(t) \le f_+^{\Delta}(t)$ and, together with $s - t \le 0$, it yields

$$f(s) \ge f(t) + f_+^{\Delta}(t)(s-t), \text{ for all } t \in (a, b)_{\mathbb{T}} \text{ and } s \in [a, b]_{\mathbb{T}}.$$

In the same manner, we can prove that $f_-^{\nabla}(t) \in \partial f(t)$ and, since both $f_+^{\Delta}(t)$ and $f_-^{\nabla}(t)$ exist for a convex function f and $t \in (a, b)_{\mathbb{T}}$, we have that every $\varphi(t) \in [f_-^{\nabla}(t), f_+^{\Delta}(t)]$ belongs to $\partial f(t)$.

Using again Theorem 4.1, we get the fact that f is nondecreasing. □

Theorem 4.2 is also true for a and b, provided $f_+^{\Delta}(a)$ and $f_-^{\nabla}(b)$ exist as finite numbers.

Remark 4.3. *If $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a convex function and $\varphi : \mathbb{T} \to \mathbb{R}$ is a function such that $\varphi(x) \in \partial f(x)$, for all $x \in \mathbb{T}^{\kappa}_{\kappa}$, then*

$$f(s) = \sup\{f(t) + (s-t)\varphi(t) | t \in [a,b]_{\mathbb{T}}\} \quad \text{for all } s \in (a,b)_{\mathbb{T}}.$$

Moreover, if f is also continuous, then the above relation works for all $s \in [a,b]_{\mathbb{T}}$.

As in the continuous case, only the convex functions on time scales have $\partial f(t)$ nonempty for all $t \in (a,b)_{\mathbb{T}}$.

Theorem 4.3. *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a function such that $\partial f(t) \neq \emptyset$ for all $t \in (a,b)_{\mathbb{T}}$. Then f is convex.*

Proof. Let $t, s \in (a,b)_{\mathbb{T}}$ and $\gamma \in (0,1)$ such that $\gamma s + (1-\gamma)t \in (a,b)_{\mathbb{T}}$. For every $\lambda \in \partial(\gamma s + (1-\gamma)t)$ we have

$$f(t) \geq f(\gamma s + (1-\gamma)t) + \gamma(t-s)\lambda,$$
$$f(s) \geq f(\gamma s + (1-\gamma)t) - (1-\gamma)(t-s)\lambda.$$

By multiplying the first inequality by $1-\gamma$, the second one by γ and then summing them up, we get

$$\gamma f(s) + (1-\gamma)f(t) \geq f(\gamma s + (1-\gamma)t),$$

and that is the convexity of f. □

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