About Bernstein polynomials

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Abstract. In this article we want to determinate a recursive formula for Bernstein polynomials associated to the functions \( e_p(x) = x^p, \ p \in \mathbb{N}, \) and an expresion for the central moments of the Bernstein polynomials.

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1. Introduction

In this section we recall some notions and results which we will use in this paper.

In the following, we note by \( \mathbb{N} \) the set of positive integer and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( B_m : C[0,1] \to C[0,1], \ m \in \mathbb{N} \) be the Bernstein operators defined for any function \( f \in C[0,1] \) by

\[
(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left( \frac{k}{m} \right),
\]

where \( p_{m,k}(x) \) are the fundamental polynomials, defined by

\[
p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \quad k \in \{0,1,\ldots,m\}
\]

and \( x \in [0,1] \).

For the bidimensional case, we have

\[
(B_m f)(x,y) = \sum_{k+j\geq 0, k+j\leq m} p_{m,k,j}(x,y) f\left( \frac{k}{m}, \frac{j}{m} \right),
\]

where \( p_{m,k,j}(x,y) \) are the bivariate Bernstein polynomials, defined by

\[
p_{m,k,j}(x,y) = \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j}.
\]

The operators \( B_m, m \geq 1 \), are named the Bernstein bivariate polynomials, see [2].

For \( x \in \mathbb{R}, \ k \in \mathbb{N}_0 \), let \( x^{[k]} = x(x-1)\ldots(x-k+1), \ x^{[0]} = 1 \). It is well known that (see [4])

\[
x^k = \sum_{\nu=1}^{k} S(k,\nu)x^{[\nu]}, \quad x \in \mathbb{R}, \ k \in \mathbb{N}^*,
\]

and

\[
x^{[k]} = \sum_{\nu=1}^{k} s(k,\nu)x^{[\nu]}, \quad x \in \mathbb{R}, \ k \in \mathbb{N}^*,
\]
where $S(k, \nu), \nu \in \{1, 2, \ldots, k\}$ are the Stirling numbers of second kind, and $s(k, \nu), \nu \in \{1, 2, \ldots, k\}$ are the Stirling numbers of first kind. These numbers verify the relations

$$S(p, k) = kS(p-1, k) + S(p-1, k-1), \quad S(1, 1) = 1,$$

for $p \in \mathbb{N}, p \geq 3, k \in \{2, 3, \ldots, p-1\}$, and

$$s(p, k) = s(p-1, k-1) - (p-1)s(p-1, k), \quad s(1, 1) = 1,$$

for $p \geq 3, k \in \{2, 3, \ldots, p-1\}$. We note $S(p, k) = 0$ and $s(p, k) = 0$, from definition, if $p, k \in \mathbb{N}, p < k$, or if $k = 0$.

It is not difficult to prove that

**Proposition 1.1.** If $m, p \in \mathbb{N}, p \geq 3$, then

$$S(p, p-1) = \frac{p(p-1)}{2}, s(p, p - 1) = -\frac{p(p-1)}{2},$$

$$S(p, p - 2) = \frac{(p-2)(p-1)p(3p-5)}{24},$$

$$s(p, p - 2) = \frac{(p-2)(p-1)p(3p-1)}{24}.$$

In the paper [3] we proved that

**Proposition 1.2.** If $m, p \in \mathbb{N},$ then

$$(B_m e_p)(x) = \frac{1}{m^p} \sum_{k=1}^{p} m^{|k|} S(p, k) x^k,$$

where $e_p(x) = x^p, x \in [0, 1]$

For the bidimensional case, we have that (see [1])

**Proposition 1.3.** If $m, p, q \in \mathbb{N},$ then

$$(B_m e_{pq})(x, y) = \frac{1}{mp + q} \sum_{i=1}^{p} \sum_{j=1}^{q} m^{i+j} S(p, i) S(q, j)x^i y^j,$$

where $e_{pq}(x, y) = x^p y^q, x, y \in \Delta_2$.

2. Main results

**Theorem 2.1.** We have the formula

$$(B_m e_{p+1})(x) = x (B_m e_p)(x) + \frac{x(1-x)}{m} (B_m e_p)'(x)$$

for $m, p \in \mathbb{N}, x \in [0, 1]$.

**Proof.** From the relation (10), we can write:

$$B_m e_{p+1})(x) = \frac{1}{m^{p+1}} \sum_{k=1}^{p+1} m^{|k|} S(p+1, k) x^k = \frac{1}{m^{p+1}} \left( \sum_{k=1}^{p} km^{|k|} S(p, k) x^k + \sum_{k=1}^{p} S(p, k) \cdot m^{|k|}(m-k)x^{k+1} \right) .$$

For $m, p \in \mathbb{N}, x \in [0, 1]$. 

**Proof.** From the relation (10), we can write:

$$B_m e_{p+1})(x) = \frac{1}{m^{p+1}} \sum_{k=1}^{p+1} m^{|k|} S(p+1, k) x^k = \frac{1}{m^{p+1}} \left( \sum_{k=1}^{p} km^{|k|} S(p, k) x^k + \sum_{k=1}^{p} S(p, k) \cdot m^{|k|}(m-k)x^{k+1} \right) .$$
\[ \cdot \frac{\partial}{\partial x} (B_m e_p p \cdot x^k) = \frac{x(1 - x)}{m} \sum_{k=1}^{m} \frac{km^{[k]} S(p, k) x^{k-1}}{m^{p+1}} + \frac{m x}{m^{p+1}} \sum_{k=1}^{p} m^{[k]} S(p, k) x^k = \]
\[ = \frac{x(1 - x)}{m} (B_m e_p p \cdot x) + x(B_m e_p p) (x). \]

For the bidimensional case, we have

**Theorem 2.2.** If \( m, p, q \in \mathbb{N} \) and \((x, y) \in \Delta_2\), then
\[ (B_m e_p p \cdot 1_q \cdot (x, y)) = \frac{1}{m^{p+q+1}} \sum_{i=1}^{p+q} \sum_{j=1}^{q} m^{[i+j]} S(p+1, i) S(q, j) x^{i} y^{j} = \frac{1}{m^{p+q+1}} \cdot \]
\[ \sum_{i=1}^{p+q} \sum_{j=1}^{q} m^{[i+j]} i S(p, i) S(q, j) x^{i-1} y^{j} + \frac{1}{m^{p+q+1}} \sum_{i=1}^{p+q} \sum_{j=1}^{q} m^{[i+j]} S(p, i-1) S(q, j) x^{i} y^{j} = \]
\[ = \frac{x}{m} \cdot \frac{1}{m^{p+q+1}} \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} i S(p, i) S(q, j) x^{i-1} y^{j} + \frac{1}{m^{p+q+1}} \sum_{i=1}^{p+q} \sum_{j=1}^{q} m^{[i+j]} (m - (i - 1) - j) S(p, i-1) S(q, j) x^{i} y^{j} = \]
\[ \cdot x^{i} y^{j} - \frac{1}{m^{p+q+1}} \left( x \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} S(p, i) S(q, j) i x^{i-1} y^{j} + y \sum_{i=1}^{q} \sum_{j=1}^{q} m^{[i+j]} S(p, i) S(q, j) x^{i} y^{j} \right) \cdot \]
\[ \cdot x^{i} y^{j} - \frac{1}{m^{p+q+1}} \left( x \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} S(p, i) S(q, j) i x^{i-1} y^{j} + y \sum_{i=1}^{q} \sum_{j=1}^{q} m^{[i+j]} S(p, i) S(q, j) x^{i} y^{j} \right) \cdot \]
\[ \cdot y^{j-1} = \frac{x}{m} \frac{\partial}{\partial x} (B_m e_p p \cdot x, y) + x B_m e_p p \cdot x, y) - \frac{x^2}{m} \frac{\partial}{\partial x} (B_m e_p p \cdot (x, y) - \frac{xy}{m} \cdot \frac{\partial}{\partial y} (B_m e_p p \cdot (x, y). \]

We can prove the equality (2.3) analogously.

In the sequel, we will find an expression for the central moments of the Bernstein polynomials.

**Lemma 2.1.** We have
\[ (B_m e_p p) (x) = \frac{a_{p,0} (x) + a_{p,1} (x) m + \ldots + a_{p,p-1} (x) m^{p-1}}{m^{p-1}}, \]
where
\[ a_{p,k} (x) = \sum_{j=k+1}^{p} S(p, j) s(j, k+1) x^j, \]
for \( k \in \{0, 1, \ldots, p - 1\}. \)
Proof. We have

\[(B_m e_p)(x) = \frac{1}{mp^{-1}} \sum_{j=1}^{p} m[j]S(p, j)x^j = \frac{1}{mp^{-1}} \sum_{j=1}^{p} \left( \sum_{k=1}^{j} s(j, k) m^k \right) S(p, j)x^j = \]

\[= \frac{1}{mp^{-1}} \sum_{k=0}^{p-1} \left( \sum_{j=k+1}^{p} S(p, j)s(j, k + 1)x^j \right) m^k. \]

\[\square\]

Theorem 2.3. The central moments of the Bernstein polynomials admit the representation

\[(B_m(* - x)^p)(x) = b_{p,0}(x) + b_{p,1}(x)m + \ldots + b_{p,p-1}(x)m^{p-1} \] (17)

where

\[b_{p,k}(x) = \sum_{j=0}^{k} (-1)^j \left( \begin{array}{c} p \vspace{1mm} \\ j \end{array} \right) x^j a_{p-j,k-j}(x), \] (18)

\[k \in \{0, 1, \ldots, p - 2\} \] and

\[b_{p,p-1}(x) = \sum_{j=0}^{p-1} (-1)^j \left( \begin{array}{c} p \vspace{1mm} \\ j \end{array} \right) x^j a_{p-j,p-j-1}(x) \] + \((-1)^p x^p, \) from where the conclusion results.

\[\square\]

Lemma 2.2. We have \(b_{p,p-1} = 0, p \geq 1, b_{p,p-2} = 0, p \geq 3, b_{p,p-3} = 0, p \geq 5.\)

Proof. From the relation (18), we have that

\[b_{p,p-1}(x) = \sum_{j=0}^{p-1} (-1)^j \left( \begin{array}{c} p \vspace{1mm} \\ j \end{array} \right) x^j a_{p-j,p-j-1}(x) + (-1)^p x^p, \]

\[b_{p,p-2}(x) = \sum_{j=0}^{p-2} (-1)^j \left( \begin{array}{c} p \vspace{1mm} \\ j \end{array} \right) x^j a_{p-j,p-j-1}(x) \]

and after (16) and (9) we can write

\[a_{p-j,p-j-1}(x) = S(p-j, p-j)s(p-j, p-j)x^{p-j} = x^{p-j}, \]

\[a_{p-j,p-j-2}(x) = \frac{(p-j)(p-j-1)}{2} x^{p-j-1} - \frac{(p-j)(p-j-2)}{2} x^{p-j} \]

\[= - \frac{(p-j)(p-j-1)}{2} x^{p-j-1}, \quad a_{p-j,p-j-3}(x) = \sum_{\nu=p-j-2}^{p-j} S(p-j, \nu)s(\nu, p-j-2)x^{p-j-1} = \quad \frac{(p-j)(p-j-1)(p-j-2)}{24} x^{p-j-1} \]

\[+ \frac{(p-j-2)(p-j-1)(p-j)(3p-3j-1)}{24} x^{p-j-1} = \frac{(p-j)(p-j-1)(p-j)(3p-3j-1)}{24} x^{p-j-1}. \]
\begin{equation}
(3p-3j-5)x^{p-j-2}-6(p-j-1)x^{p-j-1}+(3p-3j-1)x^{p-j}) \text{, so that, using the relation}
\end{equation}
\begin{equation}
\sum_{k=1}^{n} (-1)^k k^m \binom{n}{k} = 0, m, n \in \mathbb{N}, 0 \leq m < n, \text{ we have}
\end{equation}
\begin{equation}
b_{p,p-1}(x) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} x^j = 0,
\end{equation}
\begin{equation}
b_{p,p-2}(x) = \sum_{j=0}^{p-2} (-1)^j \binom{p}{j} x^j = \frac{(p-j)(p-j-1)}{2} \left(x^{p-j} - x^{p-j-1}\right) = \frac{x^p - x^{p-1}}{2},
\end{equation}
\begin{equation}
b_{p,p-3}(x) = \sum_{j=0}^{p-3} (-1)^j \binom{p}{j} \frac{(p-j)(p-j-1)(p-j-2)}{24} \left(x^{p-j} - x^{p-j-1}\right) = 0.
\end{equation}

Theorem 2.4. We have
\begin{equation}
\lim_{m \to \infty} (B_m(\ast - x)^p)(x) = 0, p \geq 1
\end{equation}
\begin{equation}
\lim_{m \to \infty} m (B_m(\ast - x)^p)(x) = 0, p \geq 3
\end{equation}
and
\begin{equation}
\lim_{m \to \infty} m^2 (B_m(\ast - x)^p)(x) = 0, p \geq 4
\end{equation}

Proof. The proof follows immediately from the Lemma 2.2.

References


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