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# About Bernstein polynomials

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Abstract. In this article we want to determinate a recursive formula for Bernstein polynomials associated to the functions $e_p(x) = x^p$, $p \in \mathbb{N}$, and an expresion for the central moments of the Bernstein polinomyals.

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## 1. Introduction

In this section we recall some notions and results which we will use in this paper. In the following, we note by $\mathbb{N}$ the set of positive integer and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $B_m : C[0,1] \to C[0,1]$, $m \in \mathbb{N}$ be the Bernstein operators defined for any function $f \in C[0,1]$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1}$$

where $p_{m,k}(x)$ are the fundamental polynomials, defined by

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \quad k \in \{0,1,\ldots,m\} \tag{2}$$

and $x \in [0,1]$.

For the bidimensional case, we have

$$(B_m f)(x,y) = \sum_{k+j \geq 0, k+j \leq m} p_{m,k,j}(x,y) f\left(\frac{k}{m}, \frac{j}{m}\right), \tag{3}$$

$(x,y) \in \Delta_2 = \{(x,y) \in \mathbb{R} \times \mathbb{R} / x, y \geq 0, x+y \leq 1\}$, where

$$p_{m,k,j}(x,y) = \frac{m!}{k!j!(m-k-j)!} x^k y^j (1-x-y)^{m-k-j}. \tag{4}$$

The operators $B_m, m \geq 1$, are named the Bernstein bivariate polynomials, see [2].

For $x \in \mathbb{R}$, $k \in \mathbb{N}_0$, let $x^{[k]} = x(x-1)\ldots(x-k+1)$, $x^{[0]} = 1$. It is well known that (see [4])

$$x^k = \sum_{\nu=1}^{k} S(k,\nu) x^{[\nu]}, \quad x \in \mathbb{R},\ k \in \mathbb{N}^*, \tag{5}$$

and

$$x^{[k]} = \sum_{\nu=1}^{k} s(k,\nu) x^{\nu}, \quad x \in \mathbb{R},\ k \in \mathbb{N}^*, \tag{6}$$

where $S(k,\nu)$, $\nu \in \{1,2,\dots,k\}$ are the Stirling numbers of second kind, and $s(k,\nu)$, $\nu \in \{1,2,\dots,k\}$ are the Stirling numbers of first kind. These numbers verify the relations

$$S(p,k) = kS(p-1,k) + S(p-1,k-1),\ S(1,1) = 1, \tag{7}$$
$$S(2,1) = S(2,2) = 1,\ S(p,1) = S(p,p) = 1,$$

for $p \in \mathbb{N}$, $p \geq 3$, $k \in \{2,3,\dots,p-1\}$, and

$$s(p,k) = s(p-1,k-1) - (p-1)s(p-1,k),\ s(1,1) = 1, \tag{8}$$
$$s(2,1) = -1, s(2,2) = 1,$$

for $p \geq 3$, $k \in \{2,3,\dots,p-1\}$. We note $S(p,k) = 0$ and $s(p,k) = 0$, from definition, if $p,k \in \mathbb{N}, p < k$, or if $k = 0$.

It is not difficult to prove that

**Proposition 1.1.** *If $m,p \in \mathbb{N}$, $p \geq 3$, then*

$$S(p,p-1) = \frac{p(p-1)}{2}, s(p,p-1) = -\frac{p(p-1)}{2}, \tag{9}$$
$$S(p,p-2) = \frac{(p-2)(p-1)p(3p-5)}{24},$$
$$s(p,p-2) = \frac{(p-2)(p-1)p(3p-1)}{24}.$$

In the paper [3] we proved that

**Proposition 1.2.** *If $m,p \in \mathbb{N}$, then*

$$(B_m e_p)(x) = \frac{1}{m^p}\sum_{k=1}^{p} m^{[k]} S(p,k)x^k, \tag{10}$$

*where $e_p(x) = x^p$, $x \in [0,1]$*

For the bidimensional case, we have that (see [1])

**Proposition 1.3.** *If $m,p,q \in \mathbb{N}$, then*

$$(B_m e_{pq})(x,y) = \frac{1}{m^{p+q}}\sum_{i=1}^{p}\sum_{j=1}^{q} m^{[i+j]} S(p,i)S(q,j)x^i y^j, \tag{11}$$

*where $e_{pq}(x,y) = x^p y^q$, $x,y \in \Delta_2$.*

## 2. Main results

**Theorem 2.1.** *We have the formula*

$$(B_m e_{p+1})(x) = x(B_m e_p)(x) + \frac{x(1-x)}{m}(B_m e_p)'(x) \tag{12}$$

*for $m,p \in \mathbb{N}$, $x \in [0,1]$.*

*Proof.* From the relation (10), we can write:

$$B_m e_{p+1})(x) = \frac{1}{m^{p+1}}\sum_{k=1}^{p+1} m^{[k]} S(p+1,k)x^k = \frac{1}{m^{p+1}}\left(\sum_{k=1}^{p} km^{[k]}S(p,k)x^k + \sum_{k=1}^{p} S(p,k)\cdot\right.$$
$$\left.\cdot m^{[k]}(m-k)x^{k+1}\right) = \frac{1}{m^{k+1}}\left(\sum_{k=1}^{p} km^{[k]}S(p,k)x^k + mx\sum_{k=1}^{p} m^{[k]}S(p,k)x^k - x\sum_{k=1}^{p} km^{[k]}\cdot\right.$$

$$\left.\cdot S(p,k)x^k\right) = \frac{x(1-x)}{m^{p+1}} \sum_{k=1}^{p} k m^{[k]} S(p,k) x^{k-1} + \frac{mx}{m^{p+1}} \sum_{k=1}^{p} m^{[k]} S(p,k) x^k =$$

$$= \frac{x(1-x)}{m} (B_m e_p)'(x) + x (B_m e_p)(x). \qquad \square$$

For the bidimensional case, we have

**Theorem 2.2.** *If $m, p, q \in \mathbb{N}$ and $(x,y) \in \Delta_2$, then*

$$(B_m e_{p+1q})(x,y) = \frac{x(1-x)}{m} \frac{\partial}{\partial x} (B_m e_{pq})(x,y) + \tag{13}$$

$$+x (B_m e_{pq})(x,y) - \frac{xy}{m} \frac{\partial}{\partial y} (B_m e_{pq})(x,y),$$

$$(B_m e_{pq+1})(x,y) = \frac{y(1-y)}{m} \frac{\partial}{\partial y} (B_m e_{pq})(x,y) + \tag{14}$$

$$+y (B_m e_{pq})(x,y) - \frac{xy}{m} \frac{\partial}{\partial x} \left(B_m e_{pq}\right)(x,y).$$

*Proof.* We have

$$(B_m e_{p+1q})(x,y) = \frac{1}{m^{p+q+1}} \sum_{i=1}^{p+1} \sum_{j=1}^{q} m^{[i+j]} S(p+1,i) S(q,j) x^i y^j = \frac{1}{m^{p+q+1}} \cdot$$

$$\cdot \sum_{i=1}^{p+1} \sum_{j=1}^{q} m^{[i+j]} i S(p,i) S(q,j) x^i y^j + \frac{1}{m^{p+q+1}} \sum_{i=1}^{p+1} \sum_{j=1}^{q} m^{[i+j]} S(p,i-1) S(q,j) x^i y^j =$$

$$= \frac{x}{m} \cdot \frac{1}{m^{p+1}} \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} i S(p,i) S(q,j) x^{i-1} y^j + \frac{1}{m^{p+q+1}} \sum_{i=1}^{p+1} \sum_{j=1}^{q} m^{[i-1+j]} (m - (i -$$

$$- 1) - j) S(p,i-1) S(q,j) x^i y^j = \frac{x}{m} \frac{\partial}{\partial x} (B_m e_{pq})(x,y) + \frac{1}{m^{p+q+1}} \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} (m -$$

$$- i - j) S(p,i) S(q,j) x^{i+1} y^j = \frac{x}{m} \frac{\partial}{\partial x} (B_m e_{pq})(x,y) + \frac{x}{m^{p+q}} \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} S(p,i) S(q,j) \cdot$$

$$\cdot x^i y^j - \frac{x}{m^{p+q+1}} \left( x \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} S(p,i) S(q,j) i x^{i-1} y^j + y \sum_{i=1}^{p} \sum_{j=1}^{q} m^{[i+j]} S(p,i) S(q,j) x^i \cdot \right.$$

$$\left. \cdot j y^{j-1} \right) = \frac{x}{m} \frac{\partial}{\partial x} l!eft(B_m e_{pq}(x,y) + x B_m e_{pq}(x,y) - \frac{x^2}{m} \frac{\partial}{\partial x} (B_m e_{pq})(x,y) - \frac{xy}{m} \cdot$$

$$\cdot \frac{\partial}{\partial y} (B_m e_{pq})(x,y) = \frac{x(1-x)}{m} \frac{\partial}{\partial x} (B_m e_{pq})(x,y) + x (B_m e_{pq})(x,y) - \frac{xy}{m} \frac{\partial}{\partial y} (B_m e_{pq})(x,y).$$

We can prove the equality (2.3) analogously. $\square$

In the sequel, we will find an expression for the central moments of the Bernstein polynomials.

**Lemma 2.1.** *We have*

$$(B_m e_p)(x) = \frac{a_{p,0}(x) + a_{p,1}(x) m + \ldots + a_{p,p-1}(x) m^{p-1}}{m^{p-1}}, \tag{15}$$

*where*

$$a_{p,k}(x) = \sum_{j=k+1}^{p} S(p,j) s(j,k+1) x^j, \tag{16}$$

*for $k \in \{0, 1, \ldots, p-1\}$.*

*Proof.* We have

$$(B_m e_p)(x) = \frac{1}{m^{p-1}} \sum_{j=1}^{p} m^{[j]} S(p,j) x^j = \frac{1}{m^{p+1}} \sum_{j=1}^{p} \left( \sum_{k=1}^{j} s(j,k) m^k \right) S(p,j) x^j =$$

$$= \frac{1}{m^{p-1}} \sum_{k=0}^{p-1} \left( \sum_{j=k+1}^{p} S(p,j) s(j,k+1) x^j \right) m^k.$$

□

**Theorem 2.3.** *The central moments of the Bernstein polynomials admit the representation*

$$(B_m(* - x)^p)(x) = \frac{b_{p,0}(x) + b_{p,1}(x)m + \ldots + b_{p,p-1}(x)m^{p-1}}{m^{p-1}} \tag{17}$$

*where*

$$b_{p,k}(x) = \sum_{j=0}^{k} (-1)^j \binom{p}{j} x^j a_{p-j,k-j}(x), \tag{18}$$

$k \in \{0, 1, \ldots, p-2\}$ *and*

$$b_{p,p-1}(x) = \sum_{j=0}^{p-1} (-1)^j \binom{p}{j} x^j a_{p-j,p-j-1}(x) + (-1)^p x^p.$$

*Proof.* We have $(B_m(* - x)^p)(x) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} x^j (B_m e_{p-j})(x) = \sum_{j=0}^{p-1} (-1)^j \binom{p}{j} x^j \cdot$ $\cdot \sum_{\nu=0}^{p-j-1} \frac{a_{p-j,\nu}(x) m^\nu}{m^{p-j-1}} + (-1)^p \binom{p}{p} x^p = \frac{1}{m^{p-1}} \sum_{j=0}^{p-1} (-1)^j \binom{p}{j} x^j m^j \sum_{\nu=0}^{p-j-1} a_{p-j,\nu}(x) m^\nu +$ $+ (-1)^p x^p$, from where the conclusion results. □

**Lemma 2.2.** *We have* $b_{p,p-1} = 0, p \geq 1$, $b_{p,p-2} = 0, p \geq 3$ $b_{p,p-3} = 0, p \geq 5$.

*Proof.* From the relation (18), we have that $b_{p,p-1}(x) = \sum_{j=0}^{p-1} (-1)^j \binom{p}{j} x^j a_{p-j,p-j-1}(x) + (-1)^p x^p$, $b_{p,p-2}(x) = \sum_{j=0}^{p-2} (-1)^j \binom{p}{j} x^j \cdot$ $\cdot a_{p-j,p-j-2}(x)$, $b_{p,p-3}(x) = \sum_{j=0}^{p-3} (-1)^j \binom{p}{j} x^j a_{p-j,p-j-3}(x)$ and after (16) and (9) we can write $a_{p-j,p-j-1}(x) = S(p-j,p-j) s(p-j,p-j) x^{p-j} = x^{p-j}$, $a_{p-j,p-j-2}(x) = \frac{(p-j)(p-j-1)}{2} x^{p-j-1} - \frac{(p-j)(p-j-1)}{2} x^{p-j} =$ $= -\frac{(p-j)(p-j-1)}{2}(x^{p-j} - x^{p-j-1})$, $a_{p-j,p-j-3}(x) = \sum_{\nu=p-j-2}^{p-j} S(p-j,\nu) s(\nu, p-$ $-j-2) x^\nu = S(p-j,p-j-2) s(p-j-2,p-j-2) x^{p-j-2} + S(p-j,p-j-1) s(p-j-1,p-j-$ $-2) x^{p-j-1} + S(p-j,p-j) s(p-j,p-j-2) x^{p-j} = \frac{(p-j-2)(p-j-1)(p-j)}{24} \cdot$ $\cdot (3p-3j-5) x^{p-j-2} - \frac{(p-j-1)(p-j)}{2} \frac{(p-j-1)(p-j-2)}{2} x^{p-j-1} +$ $+ \frac{(p-j-2)(p-j-1)(p-j)(3p-3j-1)}{24} x^{p-j} = \frac{(p-j-2)(p-j-1)(p-j)}{24} \cdot$

$\cdot((3p-3j-5)x^{p-j-2}-6(p-j-1)x^{p-j-1}+(3p-3j-1)x^{p-j})$, so that, using the relation $\sum_{k=1}^{n}(-1)^k k^m \binom{n}{k} = 0, m, n \in \mathbb{N}, 0 \leq m < n$, we have $b_{p,p-1}(x) = \sum_{j=0}^{p}(-1)^j \binom{p}{j} x^p = 0$,
$b_{p,p-2}(x) = \sum_{j=0}^{p-2}(-1)^j \binom{p}{j} x^j \frac{-(p-j)(p-j-1)}{2}(x^{p-j} - x^{p-j-1}) = -\frac{x^p - x^{p-1}}{2} \cdot$
$\cdot\sum_{j=0}^{p-2}(-1)^j (p-j)(p-j-1)\binom{p}{j} = 0$ and $b_{p,p-3}(x) = \sum_{j=0}^{p-3}(-1)^j \binom{p}{j} \frac{(p-j-2)(p-j-1)}{24} \cdot$
$\cdot\, (p-j)((3p-3j-1)x^p - 6(p-j-1)x^{p-1} + (3p-3j-5)x^{p-2}) = 0.$ □

**Theorem 2.4.** *We have*

$$\lim_{m\to\infty}(B_m(*-x)^p)(x) = 0, p \geq 1$$

$$\lim_{m\to\infty} m(B_m(*-x)^p)(x) = 0, p \geq 3$$

*and*

$$\lim_{m\to\infty} m^2(B_m(*-x)^p)(x) = 0, p \geq 4$$

*Proof.* The proof follows immediately from the Lemma 2.2. □

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