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Sublinear convection elliptic equations with singular nonlinearity

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Abstract. We present some existence results for the classical solutions to singular elliptic problems of the form

$$\begin{cases} -\Delta u = \lambda \dfrac{|\nabla u|^p}{f(u)} + \beta & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, while $p \in (0,1)$, $\lambda > 0$, $\beta \geq 0$, $f^{'} > 0$ on $(0,\infty)$ and $f(0) = 0$. Our analysis combines monotonicity arguments with elliptic estimates.

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1. Introduction

In a series of recent works have been studied the problems with gradient terms. For example, when trying to find solutions for the model equation

$$\begin{cases} -\Delta_p u + g(u)|\nabla u|^p = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

growth at infinity of $g(s)$ and the regular or singular nature of μ play a crucial role. Removable singularity results were proved by H. Brezis and L. Nirenberg in [2] for $p = 2$, showing that if $sg(s) \geq \gamma s^{-2}$ with $\gamma > 1$, then any compact set of zero capacity (the standard Newtonian capacity) is removable. In [1] it can be found a study of the existence of bounded solutions of boundary value problems of the type

$$\begin{cases} -\mathrm{div}(A(x,u)\nabla u) + \dfrac{1}{2}A^{'}(x,u)|\nabla u|^2 = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^q(\Omega), q > \frac{N}{2}, A$ is a bounded, smooth function.

Elliptic equations involving a gradient term appear in many fields. For instance, Bellman's dynamic programming principle arising in optimal stochastic control problem, indicates that the Bellman function u which minimizes the cost functional is also a solution of the nonlinear elliptic equation

$$-\frac{\Delta u}{2} + \frac{|\nabla u|^p}{p} + \lambda u = f(u) \qquad \text{in } \Omega.$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < p \leq 2$, $\lambda > 0$ denotes the discount factor and f is a smooth or singular nonlinearity. As remarked by many authors (see Serrin [11], Choquet-Bruhat and Leray [3], Kazdan and Warner [8]), the requirement that the nonlinearity $|\nabla u|^p$ grows at most quadratically is natural in order to apply the maximum principle.

If we consider the well-known example $\Delta\omega = \omega^p$ in Ω, $\omega > 0$ in Ω and $\omega = \infty$ on $\partial\Omega$ then the function $\eta = \omega^{-1}$ satisfies

$$\begin{cases} -\Delta\eta = \eta^{2-p} - \dfrac{2}{\eta}|\nabla\eta|^2 & \text{in } \Omega \\ \eta > 0 & \text{in } \Omega \\ \eta = 0 & \text{on } \partial\Omega \end{cases}$$

The above equation contains both singular nonlinearities (η^{2-p} and $\eta^{-1}, p > 2$) and a convection term ($|\nabla\eta|^2$).

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Consider the nonlinear singular problem

$$\begin{cases} -\Delta u = \lambda|\nabla u|^p + u^{-\alpha} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda \in \mathbb{R}, p \in (0,1), \alpha > 0$. From [4] we have the result that the problem has at least one classical solution for all $\lambda \in \mathbb{R}$.

In this paper we try to answer the below questions :
i) what can be said when the singularity $u^{-\alpha}$ is near $|\nabla u|^p$?
ii) what happens if near $|\nabla u|^p$ we have other kind of functions than $u^{-\alpha}$?

2. The Main Results

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain, $p \in (0,1)$, $\lambda > 0$, $\beta \geq 0$ and $f \in C^{0,\gamma}[0,\infty)$ $(0 \leq \gamma \leq 1)$ is a function which satisfies :

(f1) $f(0) = 0, f > 0$ and $f' > 0$ on $(0,\infty)$.

(f2) there exists $t_0 > 0$ such that $f(t) > 1$ on (t_0,∞).

We are concerned with the following boundary value problem

$$(1) \qquad \begin{cases} -\Delta u = \lambda\dfrac{|\nabla u|^p}{f(u)} + \beta & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, while $p \in (0,1)$, $\lambda > 0$, $\beta \geq 0$, $f(0) = 0$ and $f' > 0$ on $(0,\infty)$.

In the particular case $f(u) = u^\alpha$ $(\alpha > 0)$ we obtain

$$(2) \qquad \begin{cases} -\Delta u = \lambda\, u^{-\alpha}|\nabla u|^p + \beta & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, while $p \in (0,1)$, $\lambda > 0$, $\beta \geq 0$, $f(0) = 0$ and $f' > 0$ on $(0,\infty)$. Notice that the above hypotheses are quite natural. Typical

examples are $f(t) = t^\alpha$ $(\alpha > 0)$, $f(t) = t + \ln(t+1)$, $f(t) = e^t - 1$ and a counterexample is $f(t) = \frac{t}{t+1}$.
The problem (2) for $p = 2$, was studied by Rădulescu in [10].

We first extend an auxiliary result (we refer to [6, Lemma 2.1] for a complete proof). This proof following an idea given in [7].

Lemma 2.1 *Let $0 < p < 1$, $\beta \geq 0$, $\lambda > 0, f$ satisfies $(f1)$ and $(f2)$. Assume that there exist ω_1, $\omega_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ such that*

(i) $\Delta\omega_2 + \lambda\frac{|\nabla\omega_2|^p}{f(\omega_2)} + \beta \leq 0 \leq \Delta\omega_1 + \lambda\frac{|\nabla\omega_1|^p}{f(\omega_1)} + \beta \qquad \text{in } \Omega.$

(ii) $\omega_1, \omega_2 > 0$ in Ω and $\omega_1 \leq \omega_2 \qquad$ on $\partial\Omega$.

Then $\omega_1 \leq \omega_2$ in Ω.

Proof: Assume by contradiction that $\omega_1 \leq \omega_2$ does not hold throughout Ω and let $\eta = \frac{\omega_1}{\omega_2}$. Because $\eta \leq 1$ on $\partial\Omega$, η achieves its maximum on Ω. Also,

$$\omega_2^2\eta_{x_i} = \omega_{1x_i}\omega_2 - \omega_{2x_i}\omega_1 \qquad \text{for all i} \in \overline{1,N}.$$

Therefore,

$$\nabla\omega_2^2\nabla\eta + \omega_2^2\Delta\eta = \omega_2\Delta\omega_1 - \omega_1\Delta\omega_2. \qquad (I)$$

Let $x_M \in \Omega$ denote a maximum point of η. Therefore, in particular we have

$$\nabla\eta(x_M) = 0, \qquad -\Delta\eta(x_M) \geq 0.$$

Using (I) we obtain

$$(\omega_1\Delta\omega_2 - \omega_2\Delta\omega_1)(x_M) \geq 0 \qquad (II)$$

and

$$(\omega_1\Delta\omega_2 - \omega_2\Delta\omega_1)(x_M) + \lambda\frac{\omega_1}{f(\omega_2)}|\nabla\omega_2|^p(x_M) - \lambda\frac{\omega_2}{f(\omega_1)}|\nabla\omega_1|^p(x_M) +$$
$$+\beta(\omega_1(x_M) - \omega_2(x_M)) \leq 0.$$

From (II), $\omega_2(x_M) < \omega_1(x_M)$ and the fact that f is increasing (see (f1)) on $(0, \infty)$ we also have

$$\frac{\lambda}{f(\omega_1)}[\omega_1|\nabla\omega_2|^p - \omega_2|\nabla\omega_1|^p](x_M) + \beta(\omega_1(x_M) - \omega_2(x_M)) < 0.$$

But we know that

$$|\nabla\omega_2|^p(x_M) = (\frac{\omega_2|\nabla\omega_1|}{\omega_1})^p(x_M).$$

Therefore, we find

$$\frac{\lambda|\nabla\omega_1|^p\omega_2^p}{f(\omega_1)}[\underbrace{\omega_1^{1-p} - \omega_2^{1-p}}_{>0}](x_M) + \beta\underbrace{(\omega_1(x_M) - \omega_2(x_M))}_{>0} < 0.$$

which contradicts $\omega_2(x_M) - \omega_1(x_M) < 0$. Therefore we will have $\omega_1 \leq \omega_2$ in Ω.

□

Our main result is the following.

Theorem 2.2 *Assume that $0 < p < 1$ and conditions (f1)-(f2) are fulfilled. Then the problem (1) has at least one classical solution for all $\lambda > 0$ and $\beta \geq 0$.*

Proof: Let us first consider the problem

$$\begin{cases} -\Delta v = \beta & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

It is easy to see that the unique solution v of the above problem verifies

$$-\Delta v = \beta \leq \lambda \frac{|\nabla v|^p}{f(v)} + \beta$$

It follows that $\underline{u} := v$ is a sub-solution of (1).

We now focus on finding a super-solution $\overline{u}$ of (1) such that $\underline{u} \leq \overline{u}$ in Ω.
For this purpose, let $\Gamma : [0, 1) \to [0, \infty)$ defined by

$$\Gamma(t) = \int_0^t \frac{1}{\sqrt{2\int_s^1 \frac{1}{f(x)}dx}} ds \qquad 0 \leq t < 1$$

From $(f1) - (f2)$, $0 < s < 1$ we remark that Γ is well defined.
We claim that Γ is bijective. Indeed,

$$\Gamma^{'}(t) = \frac{1}{\sqrt{2\int_t^1 \frac{1}{f(x)}dx}} > 0 \quad \text{therefore } \Gamma \text{ is increasing}$$

,$\Gamma(0) = 0$ and $\Gamma(t) \geq Ct$ with $C > 0$ from where we obtain $\lim_{t\to\infty} \Gamma(t) = \infty$ and the claim follows.
Set $\ell := \lim_{t\nearrow 1} \Gamma(t)$ and let $\zeta : [0, \ell) \to [0, 1)$ be the inverse of Γ.

Since ζ is the inverse of Γ we have $\zeta(0) = 0$ and $\zeta \in C^1(0, \ell)$ with $\Gamma(\zeta(t)) = t$ we find

$$\zeta^{'} = \sqrt{2\int_{\zeta(t)}^1 \frac{1}{f(x)}dx} \quad \text{for all } 0 < t < \ell.$$

This yields

$$(S) \qquad \begin{cases} -\zeta^{''}(t) = \dfrac{1}{f(\zeta(t))} & \text{for all t} \in (0, \ell) \\ \zeta(t), \zeta^{'}(t) > 0 & \text{for all t} \in (0, \ell) \\ \zeta(0) = 0 \end{cases}$$

Let φ_1 denote the first eigenfunction of the Laplace operator in $H_0^1(\Omega)$. The existence of a super-solution of (1) is obtained in the following result

Lemma 2.3 *There exist two positive constants $M_\lambda > 0$ and $c > 0$ such that $\overline{u} := M_\lambda \zeta(c\varphi_1)$ is a super-solution of (1).*

Proof: By the strong maximum principle there exist $\omega \subset\subset \Omega$ and $\delta > 0$ such that

$$|\nabla\varphi_1| > \delta \quad \text{in } \Omega\backslash\omega \text{ and } \varphi_1 > \delta \text{ in } \omega.$$

Also, we have

$$\overline{u}_{x_1} = M_\lambda c\zeta^{'}(c\varphi_1)\varphi_{1x_1}$$

$$\overline{u}_{x_1x_1} = M_\lambda c^2\zeta^{''}(c\varphi_1)\varphi_{1x_1}^2 + M_\lambda c\zeta^{'}(c\varphi_1)\varphi_{1x_1x_1}$$

Therefore

$$-\Delta\overline{u} = \frac{M_\lambda c^2|\nabla\varphi_1|^2}{f(\zeta(c\varphi_1))} + M_\lambda c\lambda_1\varphi_1\zeta^{'}(c\varphi_1)$$

Thus, since (f2), (S), the fact that ζ is concave, $\zeta^{'}(c\varphi_1) > 0$ in $\bar{\omega}$, $p \in (0,1)$ we can choose $M_\lambda > 1$ such that

$$M_\lambda c\lambda_1\varphi_1\zeta^{'}(c\varphi_1) \geq \frac{2\lambda(M_\lambda c\zeta^{'}(c\varphi_1)|\nabla\varphi_1|)^p}{f(M_\lambda\zeta(c\varphi_1))}$$

Also, we easy to see that

$$M_\lambda c\lambda_1\varphi_1\zeta^{'}(c\varphi_1) \geq 2\beta \quad \text{in } \bar{\omega}$$

Thus,

$$M_\lambda c\lambda_1\varphi_1\zeta^{'}(c\varphi_1) \geq \frac{|\nabla\overline{u}|^p}{f(\overline{u})} + \beta \qquad \text{in } \bar{\omega} \quad (I)$$

Next, from (f1), $p \in (0,1)$ and $M_\lambda > 1$ we find

$$\frac{M_\lambda c^2|\nabla\varphi_1|^2}{f(\zeta(c\varphi_1))} \geq \frac{M_\lambda c^2|\nabla\varphi_1|^2}{f(M_\lambda\zeta(c\varphi_1))} \geq \frac{M_\lambda c^2\delta^2}{f(M_\lambda\zeta(c\varphi_1))} \geq$$

$$\geq \lambda\frac{(M_\lambda c\zeta^{'}(c\varphi_1)|\nabla\varphi|)^p}{f(M_\lambda\zeta(c\varphi_1))} = \lambda\frac{|\nabla\overline{u}|^p}{f(\overline{u})} \qquad \text{in } \Omega\backslash\omega$$

Therefore, we showed that in $\Omega\backslash\omega$ there holds

$$\frac{M_\lambda c^2|\nabla\varphi_1|^2}{f(\zeta(c\varphi_1))} \geq \lambda\frac{|\nabla\overline{u}|^p}{f(\overline{u})} + \beta. \qquad (II)$$

Finally, from (I) and (II) we derive

$$-\Delta\overline{u} \geq \lambda\frac{|\nabla\overline{u}|^p}{f(\overline{u})} + \beta. \qquad \text{in } \Omega$$

This ends the proof.

□

Let us come back to the proof of Theorem 2.2. So far we have constructed a sub-solution $\underline{u} := v$ and a super-solution $\overline{u} := M_\lambda\zeta(c\varphi_1)$ such that

$$\Delta\overline{u} + \lambda\frac{|\nabla\overline{u}|^p}{f(\overline{u})} + \beta \leq 0 \leq \Delta\underline{u} + \lambda\frac{|\nabla\underline{u}|^p}{f(\underline{u})} + \beta \qquad \text{in } \Omega$$

$$\overline{u}, \underline{u} > 0 \qquad \text{in } \Omega$$

$$M_\lambda\zeta(c\varphi_1) = \overline{u} > \underline{u} = 0 \qquad \text{on } \partial\Omega.$$

By Lemma 2.1 we obtain $\overline{u} \geq \underline{u}$ in Ω. The conclusion follows now by the sub and supersolution method for the pair $(\underline{u}, \overline{u})$.

□

The linear case.

In the linear case the problem (1) becomes

$$(3)\qquad \begin{cases} -\Delta u = \lambda \dfrac{|\nabla u|^p}{f(u)} + \beta u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda > 0$, $\beta > 0$. We obtain

Theorem 2.4 *Assume that $0 < p < 1$ and f satisfies $(f1) - (f2)$. Then for all $\lambda > 0$ and $\beta < \lambda_1$ the problem (3) has solutions.*

Proof: By Theorem 2.2 there exists $u \in C^2(\Omega) \cap C(\bar{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta u = \lambda \dfrac{|\nabla u|^p}{f(u)} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Obviously $\underline{u} := u$ is a sub-solution of (3). Now, we consider the problem

$$(4)\qquad \begin{cases} -\Delta \omega = \beta\omega + \alpha & \text{in } \Omega \\ \omega > 0 & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega \end{cases}$$

where we fix $0 < \beta < \lambda_1$ and $\alpha \geq 2$. Since $\beta < \lambda_1$ we observe that the conditions of the Theorem 1.2.5 from [7] are fulfilled, therefore there exists a solution $\omega \in C^2(\bar{\Omega})$ of (4).

Using the fact $0 < p < 1$, and (f2) we can choose $C_\lambda > 0$ large enough such that

$$C_\lambda > \lambda |C_\lambda \nabla\omega|^p = \lambda |\nabla C_\lambda \omega|^p$$

$$1 < f(C_\lambda \omega), \quad C_\lambda > \beta \sup u(x) \quad \text{in } \Omega$$

From (4) we obtain

$$-\Delta C_\lambda \omega = \beta C_\lambda \omega + \alpha C_\lambda \geq C_\lambda \beta\omega + 2C_\lambda \geq$$
$$\geq C_\lambda \beta \omega + \lambda |\nabla(C_\lambda\omega)|^p + \beta \sup_\Omega u(x) \geq \lambda |\nabla(C_\lambda\omega)|^p + \beta(C_\lambda\omega) +$$
$$\beta u = \lambda |\nabla(C_\lambda\omega)|^p + \beta(u + C_\lambda\omega).$$

Now, we claim that $\overline{u} = u + C_\lambda\omega$ is a super-solution of (3). Indeed, we have

$$-\Delta\overline{u} = -\Delta(u + C_\lambda\omega) = -\Delta u - \Delta C_\lambda \omega \geq \lambda |\nabla(C_\lambda\omega)|^p +$$
$$\beta(u + C_\lambda\omega) + \lambda \frac{|\nabla u|^p}{f(u)}$$
$$\geq \lambda \frac{|\nabla(C_\lambda\omega)|^p}{f(C_\lambda\omega)} + \beta(u + C_\lambda\omega) + \lambda \frac{|\nabla u|^p}{f(u)}.$$

From $\overline{u} = u + C_\lambda\omega > \max(u, C_\lambda\omega)$ we find

$$\beta(u + C_\lambda\omega) + \frac{\lambda}{f(\overline{u})}(|\nabla(C_\lambda\omega)|^p + |\nabla u|^p) \geq \beta(u + C_\lambda\omega) + \frac{\lambda}{f(\overline{u})}(|\nabla(C_\lambda\omega)| + |\nabla u|)^p \geq$$
$$\beta(u + C_\lambda\omega) + \frac{\lambda}{f(\overline{u})}(|\nabla(u + C_\lambda\omega)|)^p = \lambda \frac{|\nabla\overline{u}|^p}{f(\overline{u})} + \beta\overline{u} \quad \text{in } \Omega.$$

Therefore we obtained $-\Delta\overline{u} \geq \lambda\frac{|\nabla\overline{u}|^p}{f(\overline{u})} + \beta\overline{u}$ in Ω.

Hence, $(\underline{u}, \overline{u})$ is a pair of sub and super-solution of (3), obviously $\underline{u} < \overline{u}$ and thus the problem (3) has a classical solution u provided $\lambda > 0$ and $\beta \in (0, \lambda_1)$.

□

Let us consider the problem

$$(5) \qquad \begin{cases} -\Delta\omega = \omega^{-\alpha} + \lambda & \text{in } \Omega \\ \omega > 0 & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega \end{cases}$$

By Theorem 1.2.5 from [7] the above problem (for $0 < \lambda < \lambda_1$) has solution. We can choose $m > 1$ large enough such that $1 < f(m\omega)$ from where $m > \frac{|\nabla m\omega|^p}{f(m\omega)}$. From (5) we find

$$-\Delta(m\omega) = m\omega^{-\alpha} + \lambda m \geq (m\omega)^{-\alpha} + \lambda\frac{|\nabla(m\omega)|^p}{f(m\omega)}$$

Let $\overline{u} := m\omega$, we find a super-solution for the below problem, that is $-\Delta\overline{u} \geq \overline{u}^{-\alpha} + \lambda\frac{|\nabla\overline{u}|^p}{f(\overline{u})}$.

$$(6) \qquad \begin{cases} -\Delta u = \lambda\dfrac{|\nabla u|^p}{f(u)} + u^{-\alpha} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Obviously, $\underline{u} = v$ is a sub-solution of the problem (6), where v is the unique solution for the problem

$$\begin{cases} -\Delta v = v^{-\alpha} & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

and $c_1\delta(x) \leq v(x) \geq c_2\delta(x)$ with $c_1, c_2 > 0$. Hence, for all $\lambda \in (0, \lambda_1)$ there exist a solution u for the problem (6).

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