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On the Darboux Property in the Multivalued Case

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Abstract. In this paper we study the concepts of atom, pseudo-atom, non-atomicity and the Darboux property for multivalued set functions. We establish the relationships between the Darboux property and non-atomicity and we point out the differences which appear here from the case of set functions. We prove that the range of a multimeasure having the Darboux property is a convex set.

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1. Introduction

Darboux property, non-atomicity, regularity are important problems in the measure theory. In the last years, the non-additive case received a special attention because of its applications in mathematical economics, statistics or theory of games.

In this paper we present, for the multivalued case, the relationships between atoms and pseudo-atoms, between non-atomicity and the Darboux property and we prove that the range of a multimeasure with the Darboux property is a convex set. Also, as we shall see, if for set functions, non-atomicity and the Darboux property are equivalent under some conditions, these results are not valid for the multivalued case.

We recall now some definitions and results used in this paper.

Let T be an abstract nonvoid set and $\mathcal{C}$ a ring of subsets of T.

By $\mathcal{P}(T)$ we mean the family of all subsets of T.

By $i = \overline{1, n}$ we mean $i \in \{1, \ldots, n\}$, for all $n \in \mathbb{N}^*$, where $\mathbb{N}$ is the set of all naturals and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We also denote $\mathbb{R}_+ = [0, +\infty)$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

Definition 1.1. *A set function $\nu : \mathcal{C} \to \overline{\mathbb{R}}_+$, with $\nu(\emptyset) = 0$, is said to be:*

(i) a submeasure (in the sense of Drewnowski [4]) if ν is monotone and $\nu(A \cup B) \leq \nu(A) + \nu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$;

(ii) o-continuous if $\lim_{n\to\infty} \nu(A_n) = 0$, for every $(A_n)_{n\in\mathbb{N}^*} \subseteq \mathcal{C}$, with $A_n \supseteq A_{n+1}$, for every $n \in \mathbb{N}^*$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$;

(iii) a Dobrakov submeasure (Dobrakov [3]) if ν is a submeasure and it is also o-continuous.

Definition 1.2. *Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be a set function, with $\nu(\emptyset) = 0$.*

(i) One says that ν has the Darboux property (DP) if for every $A \in \mathcal{C}$ and every $p \in (0, 1)$, there exists a set $B \in \mathcal{C}$ such that $B \subseteq A$ and $\nu(B) = p\nu(A)$.

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(ii) ν is said to be non-atomic (NA) if for every $A \in \mathcal{C}$ with $\nu(A) > 0$, there exists $B \in \mathcal{C}, B \subseteq A$, such that $\nu(B) > 0$ and $\nu(A \backslash B) > 0$.

(iii) One says that ν has the Saks property (SP) if for every $A \in \mathcal{C}$ and every $\varepsilon > 0$, there exists a $\mathcal{C}$-partition $\{B_i\}_{i=1}^n$ of A such that $\nu(B_i) < \varepsilon$, for every $i \in \{1, \ldots, n\}$.

Remark 1.1. I) In what concerns non-atomicity and the Darboux property, we remarkind now the relationships established in literature for set functions. Suppose $\nu : \mathcal{C} \to \mathbb{R}_+$.

(i) If $\mathcal{C}$ is a δ-ring and ν is a measure, then $NA \Leftrightarrow DP$ (Dinculeanu [2]).

(ii) If $\mathcal{C}$ is a σ-algebra and ν is a Dobrakov submeasure, then $NA \Leftrightarrow DP \Leftrightarrow SP$ (Klimkin and Svistula [11]).

(iii) If $\mathcal{C}$ is an algebra and ν is bounded, finitely additive, then $DP \Rightarrow SP \Rightarrow NA$ (Rao and Rao [12]).

(iv) If $\mathcal{C}$ is a σ-algebra and ν is finitely additive, then $DP \Leftrightarrow SP$ (Klimkin and Svistula [11]).

(v) If $\mathcal{C}$ is a σ-algebra and ν is a submeasure, then $SP \Rightarrow DP$ (Klimkin and Svistula [11]).

II) Concerning the range of a non-atomic set function, the following results are known:

(i) If $\mathcal{C}$ is a σ-algebra and $\nu : \mathcal{C} \to \mathbb{R}_+$ is a non-atomic measure, then the range of ν is $R(\nu) = [0, \nu(T)]$ (Dinculeanu [2]).

(ii) If $\mathcal{C}$ is a σ-algebra and $\nu : \mathcal{C} \to \mathbb{R}_+$ is finitely additive with SP, then $R(\nu) = [0, \nu(T)]$ (Klimkin and Svistula [11]).

(iii) If $\mathcal{C}$ is a ring and $\nu : \mathcal{C} \to X$ is a vector measure with DP, then $R(\nu)$ is convex (Bandyopadhyay [1]).

In the sequel $(X, \|\cdot\|)$ will be a real normed space, with the distance d induced by its norm. $\mathcal{P}_0(X)$ the family of all non-empty subsets of X, $\mathcal{P}_f(X)$ the family of non-empty closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of non-empty bounded closed subsets of X, $\mathcal{P}_{bfc}(X)$ the family of non-empty bounded closed convex subsets of X and h defined by:

$$h(M, N) = \max\{e(M, N), e(N, M)\},$$

where $e(M, N) = \sup_{x \in M} d(x, N)$ is the excess of M over N, for every $M, N \in \mathcal{P}_0(X)$ and $d(x, N)$ is the distance from x to N.
It is known that h is an extended metric on $\mathcal{P}_f(X)$ (i.e. a metric which can also take the value $+\infty$) and it becomes a metric on $\mathcal{P}_{bf}(X)$, named the Hausdorff metric (Hu and Papageorgiou [10]).

We denote $|M| = h(M, \{0\})$, for every $M \in \mathcal{P}_0(X)$, where 0 is the origin of X. Obviously, $|M| = \sup_{x \in M} \|x\|$, for every $M \in \mathcal{P}_0(X)$.

On $\mathcal{P}_0(X)$ we introduce the Minkowski addition " $\stackrel{\bullet}{+}$ ", defined by:

$$M \stackrel{\bullet}{+} N = \overline{M + N}, \text{ for every } M, N \in \mathcal{P}_0(X).$$

where $M + N = \{x + y | \ x \in M, y \in N\}$ and $\overline{M + N}$ is the closure of $M + N$.

Definition 1.3. *(Gavriluţ [5], [6]) If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a multivalued set function, then μ is said to be:*

(i) monotone if $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$;

(ii) a multisubmeasure if it is monotone, $\mu(\emptyset) = \{0\}$ and

$\mu(A \cup B) \subseteq \mu(A) + \mu(B)$, *for every* $A, B \in \mathcal{C}$, *with* $A \cap B = \emptyset$ *(or, equivalently, for every* $A, B \in \mathcal{C}$*);*

(iii) a multimeasure if $\mu(\emptyset) = \{0\}$ *and* $\mu(A \cup B) = \mu(A) + \mu(B)$, *for every* $A, B \in \mathcal{C}$, *with* $A \cap B = \emptyset$;

(iv) exhaustive (with respect to h) if $\lim_{n\to\infty} |\mu(A_n)| = 0$, *for every sequence of mutual disjoint sets* $(A_n)_{n\in\mathbb{N}^*} \subseteq \mathcal{C}$*;*

(v) order continuous (shortly, o-continuous) (with respect to h) if $\lim_{n\to\infty} |\mu(A_n)| = 0$, *for every* $(A_n)_{n\in\mathbb{N}^*} \subseteq \mathcal{C}$, *with* $A_n \searrow \emptyset$ *(i.e.* $A_{n+1} \subseteq A_n$, *for every* $n \in \mathbb{N}^*$ *and* $\bigcap_{n=1}^{\infty} A_n = \emptyset$*).*

Remark 1.2. If μ is $\mathcal{P}_f(X)$-valued, then in Definition 1.3-(ii) and (iii) it usually appears "$\dot{+}$" instead of "$+$", because the sum of two closed sets is not always closed.

Example 1.1. Let $\nu : \mathcal{C} \to \mathbb{R}_+$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. If ν is a submeasure (finitely additive, respectively), then μ is a multisubmeasure (monotone multimeasure, respectively).

μ is called *the multisubmeasure* (*multimeasure,* respectively) *induced by* ν.

Definition 1.4. *For a multivalued set function* $\mu : \mathcal{C} \to \mathcal{P}_0(X)$, *with* $\mu(\emptyset) = \{0\}$, *we consider the following set functions:*

(i) $\overline{\mu} : \mathcal{P}(T) \to \overline{\mathbb{R}}_+$, *called the variation of* μ, *defined for every* $A \in \mathcal{P}(T)$ *by:*

$\overline{\mu}(A) = \sup\left\{\sum_{i=1}^{n} |\mu(A_i)|; A_i \subseteq A,\ A_i \in \mathcal{C}, \forall i = \overline{1,n}, A_i \cap A_j = \emptyset,\ \textit{for } i \neq j\right\}$ *and*

(ii) $|\mu| : \mathcal{C} \to \overline{\mathbb{R}}_+$, *defined by* $|\mu|(A) = |\mu(A)|$, *for every* $A \in \mathcal{C}$.

Remark 1.3. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a multivalued set function, with $\mu(\emptyset) = \{0\}$. Then:

I) $|\mu(A)| \leq \overline{\mu}(A)$, for every $A \in \mathcal{C}$. So, $\overline{\mu}(A) = 0$ implies $|\mu(A)| = 0$.

II) $\overline{\mu}$ is a monotone set function.

III) If μ is monotone, then $|\mu|$ is also monotone.

IV) If μ is a multisubmeasure, then $\overline{\mu}$ is finitely additive and $|\mu|$ is a submeasure.

V) Let $A \in \mathcal{C}$. Then:

(a) $\mu(A) = \{0\}$ if and only if $|\mu(A)| = 0$.

(b) If μ is monotone, then $\mu(A) = \{0\}$ if and only if $\overline{\mu}(A) = 0$.

2. Atoms and pseudo-atoms for multivalued set functions

In this section we establish some properties of atoms, pseudo-atoms and non-atomicity for multivalued set functions.

Definition 2.1. *(Gavriluţ [7], Gavriluţ and Croitoru [8]) Let* $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ *be a multivalued set function, with* $\mu(\emptyset) = \{0\}$.

(i) A set $A \in \mathcal{C}$ *is said to be an atom of* μ *if* $\mu(A) \supsetneq \{0\}$ *and for every* $B \in \mathcal{C}$, *with* $B \subseteq A$, *we have* $\mu(B) = \{0\}$ *or* $\mu(A \backslash B) = \{0\}$.

(ii) If μ *is monotone, then* μ *is said to be non-atomic (NA) if it has no atoms (that is, for every* $A \in \mathcal{C}$, *with* $\mu(A) \supsetneq \{0\}$, *there exists* $B \in \mathcal{C}$, *with* $B \subseteq A$, $\mu(B) \supsetneq \{0\}$ *and* $\mu(A \backslash B) \supsetneq \{0\}$*).*

(iii) A set $A \in \mathcal{C}$ *is called a pseudo-atom of* μ *if* $\mu(A) \supsetneq \{0\}$ *and for every* $B \in \mathcal{C}$, *with* $B \subseteq A$, *we have* $\mu(B) = \{0\}$ *or* $\mu(B) = \mu(A)$.

(iv) If μ is monotone, then μ is said to be non-pseudo-atomic (NPA) if it has no pseudo-atoms (that is, for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, there exists $B \in \mathcal{C}$, with $B \subseteq A$, $\mu(B) \supsetneq \{0\}$ and $\mu(B) \subsetneq \mu(A)$).

Remark 2.1. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$ and let $A \in \mathcal{C}$. Then the following statements hold:

I) A is an atom of μ if and only if A is an atom of $|\mu|$.

II) If μ is monotone, then A is an atom of μ if and only if A is an atom of $\overline{\mu}$.

III) If A is a pseudo-atom of μ, then A is a pseudo-atom of $|\mu|$. The converse is not always valid. For example, let $T = \{x, y, z\}$, $\mathcal{C} = \mathcal{P}(T)$ and $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ defined by

$$\mu(A) = \begin{cases} [0,3], & A = T \\ \{1,2,3\}, & A \in \mathcal{P}(T) \backslash \{T, \emptyset\} \\ \{0\}, & A = \emptyset \end{cases}, \text{ for every } A \in \mathcal{C}.$$

Then $|\mu|(A) = \begin{cases} 3, & A \neq \emptyset \\ 0, & A = \emptyset \end{cases}$, for every $A \in \mathcal{C}$. It follows that T is a pseudo-atom of $|\mu|$, but T is not a pseudo-atom of μ.

IV) Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be a set function and let $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ defined by $\mu(E) = [0, \nu(E)]$, for every $E \in \mathcal{C}$. If $A \in \mathcal{C}$ is a pseudo-atom of $|\mu|$, then A is also a pseudo-atom of μ. Moreover, if ν is finitely additive, then A is a pseudo-atom of μ if and only if it is a pseudo-atom of $\overline{\mu}$.

Remark 2.2. I) Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multisubmeasure and let $A, B \in \mathcal{C}$, with $B \subseteq A$. Then $\mu(A \backslash B) = \{0\}$ implies $\mu(B) = \mu(A)$. Indeed, we have $\mu(B) \subset \mu(A) \subset \mu(B) \overset{\bullet}{+} \mu(A \backslash B) = \mu(B)$. So $\mu(B) = \mu(A)$. It follows that every atom of μ is a pseudo-atom of μ. As we shall see in Example 2.1-I), the converse is not valid. Moreover, if μ is NPA, then μ is NA.

II) If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a multimeasure and $A, B \in \mathcal{C}$ are so that $B \subseteq A$, then $\mu(A \backslash B) = \{0\}$ implies $\mu(B) = \mu(A)$. Indeed, since μ is a multimeasure, it results $\mu(A) = \mu((A \backslash B) \cup B) = \mu(A \backslash B) + \mu(B) = \mu(B)$. It also follows that every atom of μ is a pseudo-atom of μ.

III) Let $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ be a multimeasure and let $A, B \in \mathcal{C}$, with $B \subseteq A$. Then $\mu(B) = \mu(A)$ implies $\mu(A \backslash B) = \{0\}$. Indeed, by the cancellation law on $\mathcal{P}_{bfc}(X)$ (Godet-Thobie [9]), since $\mu(B) = \mu(A) = \mu(B \cup (A \backslash B)) = \mu(B) \overset{\bullet}{+} \mu(A \backslash B)$, it results $\mu(A \backslash B) = \{0\}$. It follows that every pseudo-atom of μ is an atom of μ. Consequently, $A \in \mathcal{C}$ is an atom of a multimeasure $\mu : \mathcal{C} \to \mathcal{P}_{bfc}(X)$ if and only if A is a pseudo-atom of μ. So, μ is NPA if and only if μ is NA.

Example 2.1. I) Let $T = \{x, y, z\}$, $\mathcal{C} = \mathcal{P}(T)$ and for every $A \in \mathcal{C}$, let

$$\mu(A) = \begin{cases} [0,1], & \text{if } A \neq \emptyset, \\ \{0\}, & \text{if } A = \emptyset. \end{cases}$$

One can easily check that $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R})$ is a multisubmeasure and it is not a multimeasure.

Let $A = \{x, y\}$. There is $B = \{x\} \subseteq A$, such that $\mu(B) \supsetneq \{0\}$ and $\mu(A \backslash B) = \mu(\{y\}) = [0,1] \supsetneq \{0\}$.

So A is not an atom of μ.

Now, for every $C \in \mathcal{C}$, with $C \subseteq A$, we have $\mu(C) = \{0\}$, for $C = \emptyset$ or $\mu(C) = [0,1] = \mu(A)$, for $C \neq \emptyset$, which shows that A is a pseudo-atom of μ. So, there are pseudo-atoms of a multisubmeasure, which are not atoms.

II) Let $T=\{x,y\}$ $(x\neq y)$, $\mathcal{C}=\mathcal{P}(T)$ and for every $A\in\mathcal{C}$, let

$$\mu(A)=\begin{cases}[0,2], & \text{if } A=\{x,y\}\\ [0,1], & \text{if } A=\{y\}\\ \{0\}, & \text{if } A=\emptyset \text{ or } A=\{x\}\end{cases}.$$

Evidently, μ is not a multisubmeasure.

Let $A=\{x,y\}$. There exists $B=\{y\}\subseteq A$, such that $\mu(B)=[0,1]\supsetneq\{0\}$ and $\mu(B)\neq\mu(A)$. So A is not a pseudo-atom of μ.

Now, for every $C\in\mathcal{C}$, we have:

(i) if $C=\emptyset$, then $\mu(C)=\{0\}$;

(ii) if $C=\{x\}$, then $\mu(C)=\{0\}$;

(iii) if $C=\{y\}$, then $\mu(C)=[0,1]\supsetneq\{0\}$ and $\mu(A\backslash C)=\mu(\{x\})=\{0\}$;

(iv) if $C=\{x,y\}$, then $\mu(A\backslash C)=\mu(\emptyset)=\{0\}$.

Consequently, A is an atom of μ.

So, if μ is not a multisubmeasure, then there are atoms of μ which are not pseudo-atoms of μ.

III) Let $\mathcal{C}=\{\emptyset,\{1\},\{2\},\{1,2\}\}$ and $\nu:\mathcal{C}\to\mathbb{R}_+$, defined for every $A\in\mathcal{C}$, by

$$\nu(A)=\begin{cases}0, & \text{if } A=\emptyset\\ 1, & \text{if } A=\{1\} \text{ or } A=\{2\}\\ \frac{3}{2}, & \text{if } A=\{1,2\}.\end{cases}$$

Then $\{1\}$ and $\{2\}$ are atoms for the multisubmeasure μ induced by the submeasure ν: $\mu(A)=[0,\nu(A)]$, for every $A\in\mathcal{C}$.

IV) Suppose T is a countable set. Let $\mathcal{C}=\{A; A\subseteq T,\ A \text{ is finite or } cA \text{ is finite}\}$ and the multisubmeasure $\mu:\mathcal{C}\to\mathcal{P}_f(\mathbb{R})$, defined for every $A\in\mathcal{C}$, by

$$\mu(A)=\begin{cases}\{0\}, & \text{if } A \text{ is finite}\\ \{0,1\}, & \text{if } cA \text{ is finite.}\end{cases}$$

Then every $A\in\mathcal{C}$, such that cA is finite, is an atom of μ.

V) Let $\mathcal{C}=\mathcal{P}(\mathbb{N})$ and $\mu:\mathcal{C}\to\mathcal{P}_f(\mathbb{R})$ defined for every $A\in\mathcal{C}$ by

$$\mu(A)=\begin{cases}\{0\}, & \text{if } A=\emptyset\\ (A+1)\cup\{0\}, & \text{if } A\neq\emptyset\end{cases},$$

where $A+1=\{x+1|\ x\in A\}$.

One can prove that μ is a multisubmeasure. Then every $A\in\mathcal{C}$, with $\operatorname{card} A=1$ is an atom of μ (and a pseudo-atom of μ too, according to Remark 2.2-I) and every $A\in\mathcal{C}$, with $\operatorname{card} A\geq 2$ is not a pseudo-atom of μ (and not an atom of μ, according to Remak 2.2-I).

VI) Let $T=2\mathbb{N}=\{0,2,4,\ldots\}$, $\mathcal{C}=\mathcal{P}(T)$ and $\mu:\mathcal{C}\to\mathcal{P}_f(\mathbb{R})$ defined for every $A\in\mathcal{C}$ by

$$\mu(A)=\begin{cases}\{0\}, & \text{if } A=\emptyset\\ \frac{1}{2}A\cup\{0\}, & \text{if } A\neq\emptyset\end{cases}$$

where $\frac{1}{2}A=\{\frac{x}{2}|\ x\in A\}$. One can prove that μ is a multisubmeasure.

If $A\in\mathcal{C}$, with $\operatorname{card} A=1$ and $A\neq\{0\}$ or $A\in\mathcal{C}, A=\{0,2n\}$, $n\in\mathbb{N}^*$, then A is an atom of μ (and a pseudo-atom of μ too, according to Remark 2.2-I).

If $A \in \mathcal{C}$, with card $A \geq 2$ and there exist $a, b \in A$ such that $a \neq b$ and $ab \neq 0$, then A is not a pseudo-atom of μ (and not an atom of μ, according to Remark 2.2-I).

VII) Let $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ defined for every $A \in \mathcal{C}$ by

$$\mu(A) = \begin{cases} \{0\}, & \text{if } A \text{ is finite} \\ \{0\} \cup [n_A, +\infty), & \text{if } A \text{ is infinite and } n_A = \min A. \end{cases}$$

Then μ is monotone and NPA.

Remark 2.3. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction with $\mu(\emptyset) = \{0\}$ and let $A \in \mathcal{C}$. By the definitions, we obtain:

I) If A is a pseudo-atom of μ and $B \in \mathcal{C}, B \subseteq A$ so that $\mu(B) \supsetneq \{0\}$, then B is a pseudo-atom of μ and $\mu(B) = \mu(A)$.

II) Suppose μ is monotone. If A is an atom of μ and $B \in \mathcal{C}, B \subseteq A$ so that $\mu(B) \supsetneq \{0\}$, then B is an atom of μ and $\mu(A \backslash B) = \{0\}$.

Remark 2.4. Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be monotone, with $\mu(\emptyset) = \{0\}$. According to Remark 1.3-I, we have $|\mu(A)| \leq \overline{\mu}(A)$, for every $A \in \mathcal{C}$.

Moreover, if $A \in \mathcal{C}$ is an atom of μ, then $\overline{\mu}(A) \leq |\mu(A)|$. Indeed, let $\{B_i\}_{i=1}^n$ be a $\mathcal{C}$-partition of A. Then there is at most one $i_0 \in \{1, \dots, n\}$ such that $\mu(B_{i_0}) \supsetneq \{0\}$ and $\mu(B_i) = \{0\}$, for every $i \in \{1, \dots, n\}, i \neq i_0$. Since μ is monotone, we have $\sum\limits_{i=1}^{n} |\mu(B_i)| \leq |\mu(A)|$, which implies $\overline{\mu}(A) \leq |\mu(A)|$, so $\overline{\mu}(A) = |\mu(A)|$.

Theorem 2.1. *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be monotone, with $\mu(\emptyset) = \{0\}$ and let $A, B \in \mathcal{C}$ be two pseudo-atoms of μ.*

I) If $\mu(A) \neq \mu(B)$, then $\mu(A \cap B) = \{0\}$.

II) Suppose, moreover, that μ is a multisubmeasure. If $\mu(A \cap B) = \{0\}$, then $A \backslash B$ and $B \backslash A$ are pseudo-atoms of μ and $\mu(A \backslash B) = \mu(A), \mu(B \backslash A) = \mu(B)$.

Proof. I. If $\mu(A \cap B) \supsetneq \{0\}$, it results from Remark 2.3-I that $A \cap B$ is a pseudo-atom of μ and $\mu(A \cap B) = \mu(A) = \mu(B)$, which is false.

II. We prove that $\mu(A \backslash B) \supsetneq \{0\}$. Suppose on the contrary that $\mu(A \backslash B) = \{0\}$. Since μ is a multisubmeasure, we get that

$$\mu(A) = \mu((A \backslash B) \cup (A \cap B)) \subseteq \mu(A \backslash B) \stackrel{\bullet}{+} \mu(A \cap B) = \{0\},$$

which is false because A is a pseudo-atom of μ. So, $\mu(A \backslash B) \supsetneq \{0\}$. By Remark 2.3-I, it follows that $A \backslash B$ is a pseudo-atom of μ and $\mu(A \backslash B) = \mu(A)$.

Analogously, $B \backslash A$ is a pseudo-atom of μ and $\mu(B \backslash A) = \mu(B)$. □

Theorem 2.2. *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a multisubmeasure and let $A, B \in \mathcal{C}$ be pseudo-atoms of μ. Then there exist mutual disjoint sets $C_1, C_2, C_3 \in \mathcal{C}$, with $A \cup B = C_1 \cup C_2 \cup C_3$, such that, for every $i \in \{1, 2, 3\}$, either C_i is a pseudo-atom of μ, or $\mu(C_i) = \{0\}$.*

Proof. Let $C_1 = A \cap B, C_2 = A \backslash B, C_3 = B \backslash A$. We have the following cases:

(i) $\mu(C_1) = \{0\}$. According to Theoremark 2.1-II, C_2 and C_3 are pseudo-atoms of μ and $\mu(C_2) = \mu(A), \mu(C_3) = \mu(B)$.

(ii) $\mu(C_1) \supsetneq \{0\}, \mu(C_2) \supsetneq \{0\}, \mu(C_3) \supsetneq \{0\}$. According to Remark 2.3-I, C_1 is a pseudo-atom of μ and $\mu(C_1) = \mu(A) = \mu(B)$. Now we prove that C_2 is a pseudo-atom.

Let $D \in \mathcal{C}$, $D \subseteq C_2$, with $\mu(D) \supsetneq \{0\}$. Then $D \subseteq C_2 \subseteq A$ and, since A is a pseudo-atom of μ, we have $\mu(D) = \mu(C_2) = \mu(A)$. So C_2 is a pseudo-atom of μ. Analogously, C_3 is a pseudo-atom of μ, too.

(iii) $\mu(C_1) \supsetneq \{0\}, \mu(C_2) = \{0\}, \mu(C_3) \supsetneq \{0\}$. From Remark 2.3-I, it results that C_1 is a pseudo-atom of μ and $\mu(C_1) = \mu(A) = \mu(B)$. As in (ii), we obtain that C_3 is a pseudo-atom of μ and $\mu(C_3) = \mu(B)$.

The last two cases are similar to (iii).

(iv) $\mu(C_1) \supsetneq \{0\}, \mu(C_2) \supsetneq \{0\}, \mu(C_3) = \{0\}$.

(v) $\mu(C_1) \supsetneq \{0\}, \mu(C_2) = \mu(C_3) = \{0\}$. □

Corollary 2.1. *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a multisubmeasure and let $A, B \in \mathcal{C}$ be pseudo-atoms of μ. If $\mu(A \cap B) \supsetneq \{0\}, \mu(A \backslash B) = \mu(B \backslash A) = \{0\}$, then $A \cap B$ is a pseudo-atom of μ and $\mu(A \triangle B) = \{0\}$.*

3. Darboux property for set multifunctions

In this section we present some relationships between non-atomicity, the Darboux property and the Saks property for $\mu, |\mu|$ and $\overline{\mu}$ and prove that the range of a multimeasure having the Darboux property is a convex set.

Definition 3.1. *Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a multivalued set function, with $\mu(\emptyset) = \{0\}$. We say that μ has the Darboux property (DP) if for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$ and every $p \in (0, 1)$, there exists a set $B \in \mathcal{C}$ such that $B \subseteq A$ and $\mu(B) = p\ \mu(A)$.*

Remark 3.1. Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be a submeasure and μ be the multisubmeasure induced by ν. Then μ has the Darboux property if and only if ν has the Darboux property.

Proposition 3.1. *If $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ is a multisubmeasure and if $|\mu| : \mathcal{C} \to \mathbb{R}_+$ has the Saks property, then μ is bounded (that is, there exists $M > 0$ so that $|\mu(A)| \leq M$, for every $A \in \mathcal{C}$).*

Proof. By the Saks property, for $\varepsilon = 1$, there exists a $\mathcal{C}$-partition $(B_i)_{i=1}^N$ of T such that $|\mu(B_i)| < 1$, for every $i \in \{1, \ldots, N\}$. Then, for every $A \in \mathcal{C}$, we have:

$$|\mu(A)| \leq |\mu(T)| = |\mu(\bigcup_{i=1}^{N} B_i)| \leq \sum_{i=1}^{N} |\mu(B_i)| < N,$$

so μ is bounded. □

In what follows, we establish that the range of a multimeasure with the Darboux property, is a convex set:

Theorem 3.1. *If a multimeasure $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ has the Darboux property, then the following properties hold:*

(i) for every $Z_1, Z_2 \in \mathcal{R}(\mu) = \{\mu(A) | A \in \mathcal{C}\}$ and every $\alpha \in (0, 1)$, we have $\alpha Z_1 + (1 - \alpha) Z_2 \in \mathcal{R}(\mu)$;

(ii) $R(\mu) = \bigcup_{A \in \mathcal{A}} \mu(A)$ is convex.

Proof. (i) Let $Z_1 = \mu(A)$ and $Z_2 = \mu(B)$, where $A, B \in \mathcal{C}$. Since μ has the Darboux property, there exist $C, D \in \mathcal{C}$ such that $C \subseteq A \backslash B$, $D \subseteq B \backslash A$ and $\mu(C) = \alpha \mu(A \backslash B), \mu(D) = (1 - \alpha) \mu(B \backslash A)$. Then, for $E = C \cup D \cup (A \cap B) \in \mathcal{A}$, it results $\alpha Z_1 + (1 - \alpha) Z_2 = \mu(E) \in \mathcal{R}(\mu)$.

(ii) Let $x_1, x_2 \in R(\mu)$ and $\alpha \in (0, 1)$. Suppose $x_1 \in \mu(A), x_2 \in \mu(B)$, with $A, B \in \mathcal{C}$. From (i) it follows there is $E \in \mathcal{C}$ such that $\alpha \mu(A) + (1 - \alpha) \mu(B) = \mu(E)$. So, $\alpha x_1 + (1 - \alpha) x_2 \in R(\mu)$. □

Further we investigate some relationships between non-atomicity and the Darboux property for $\mu, |\mu|$ and $\overline{\mu}$.

Theorem 3.2. *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a multivalued set function, with $\mu(\emptyset) = \{0\}$ and $|\mu| : \mathcal{C} \to \mathbb{R}_+$.*

(i) If μ is monotone, then the following statements are equivalent:

(a) μ is NA.

(b) $|\mu|$ is NA.

(c) $\overline{\mu}$ is NA.

(ii) If μ is a multisubmeasure with the Darboux property, then μ is non-atomic.

(iii) If $\mathcal{C}$ is a σ-algebra and if μ is a multisubmeasure such that $|\mu|$ has the Saks property, then μ is non-atomic.

(iv) If μ has DP, then $|\mu|$ has DP.

(v) If $\mathcal{C}$ is an algebra, μ is monotone and if $\overline{\mu}$ is bounded, finitely additive and has the Darboux property, then μ is non-atomic.

Proof. (i) follows immediately.

(ii) Suppose, on the contrary, that μ has an atom $A_0 \in \mathcal{C}$. Let $p \in (0,1)$. According to the Darboux property, there is $B \in \mathcal{C}, B \subseteq A_0$ such that $\mu(B) = p\mu(A_0)$. Since A_0 is an atom, it follows $\mu(B) = \{0\}$ or $\mu(A_0 \backslash B) = \{0\}$. If $\mu(B) = \{0\}$, then $\mu(A_0) = \{0\}$, which is false.

If $\mu(A_0 \backslash B) = \{0\}$, then $\mu(A_0) = \mu(B) = p\mu(A_0)$. It results $|\mu(A_0)| = p|\mu(A_0)|$ and so, $\mu(A_0) = \{0\}$, which is false. Consequently, μ is non-atomic.

(iii) Since $|\mu|$ is a submeasure with the Saks property on a σ-algebra, then it has the Darboux property, so we get that $|\mu|$ is non-atomic. But this is equivalent to the non-atomicity of μ.

(iv) We apply the definitions.

(v) Since $\overline{\mu}$ is bounded, finitely additive on an algebra and has the Darboux property, then it is non-atomic. But this is equivalent to the non-atomicity of μ. □

Remark 3.2. The converses of Theoremark 3.3-(ii) and (iv) are not true.

Indeed, let $\mathcal{C}$ be an algebra of subsets of an abstract set T, $m : \mathcal{C} \to \mathbb{R}_+$ a bounded finitely additive set function with DP and $\mu : \mathcal{C} \to \mathcal{P}_{bf}(\mathbb{R})$ the multivalued set function, defined for every $A \in \mathcal{C}$, by

$$\mu(A) = \begin{cases} [-m(A), m(A)], & \text{if } m(A) \leq 1 \\ [-m(A), 1], & \text{if } m(A) > 1. \end{cases}$$

Then μ is a multisubmeasure and $|\mu| = \overline{\mu} = m$.

Let us prove that μ has not the Darboux property. Suppose, on the contrary, that for every $A \in \mathcal{C}$, with $\mu(A) \supsetneq \{0\}$, and every $p \in (0,1)$, there exists $B \in \mathcal{C}, B \subseteq A$ such that $\mu(B) = p\mu(A)$. Considering $m(A) > 1$, we have $p\mu(A) = [-pm(A), p]$.

(i) If $m(B) \leq 1$, then $\mu(B) = [-m(B), m(B)] = [-pm(A), p]$. It follows $m(B) = p$ and $-p = -pm(A)$. So, $m(A) = 1$. False.

(ii) If $m(B) > 1$, then we have $\mu(B) = [-m(B), 1] = [-pm(A), p]$. So $1 = p$, which is false.

So, μ has not the Darboux property, although $|\mu|$ and $\overline{\mu}$ have it.

Since $|\mu|$ has DP, then $|\mu|$ is non-atomic, hence μ is non-atomic.

Open problems

1. What happens with NA and NPA without monotonicity?

2. In what general hypothesis we have: $\mu\ NA \Rightarrow |\mu|\ SP$?
3. Establishing relationships between NPA and DP.

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