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Path-based Reasoning in Semantic Schemas

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Abstract. The concept of **semantic schema** was introduced in [4] in order to extend that of **semantic network**. A semantic schema is a tuple of abstract entities, each one specifying some feature of the representation process.

In the present paper we define a new kind of path for a semantic schema, named the *deductive path*. This new concept does not change dramatically the reasoning mechanism of the semantic schema as this paper proves. But, based on the deductive paths we can link two or more semantic schemas in a new structure. The resulted structure, also a semantic schema, generates a reasoning environment for distributed knowledge representation and reasoning by analogy.

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1. Prerequistes

As we mentioned, semantic schemas are generalizations of semantic networks. A semantic network is a *labeled directed graph* that represents connections (relationships) between concepts in some specific domain of knowledge.

There are different kinds of relationships that can be represented in a semantic network. The most common relationships are **A-KIND-OF**, **IS-A** and **HAS**.

In order to obtain a semantic schema from a semantic network we have to replace all the represented objects and the relationships with abstract symbols.

Using proper interpretations for its abstract symbols, we can obtain various pieces of knowledge that respect the structure of the semantic schema.

An interpretation for a semantic schema defines the domains of its component entities as it happens in mathematical logic, where an interpretation establishes a logic value for some formula. If $\mathcal{S}$ is a semantic schema and $\mathcal{I}$ an interpretation for $\mathcal{S}$ then the pair $(\mathcal{S}, \mathcal{I})$ defines an environment for the reasoning process. Various interpretations for the same semantic schema can be considered. Thus the pairs $(\mathcal{S}, \mathcal{I}_1)$, ..., $(\mathcal{S}, \mathcal{I}_n)$ can represent n knowledge pieces KP_1, ..., KP_n if these knowledge pieces have the same abstract structure given by $\mathcal{S}$ (see Figure 2).

But, a semantic schema is not just an abstract semantic network. It comprises two aspects:

- **A formal aspect** by means of which some formal computations in a Peano-algebra are obtained; this aspect deals with the syntactical representations of the semantic schema

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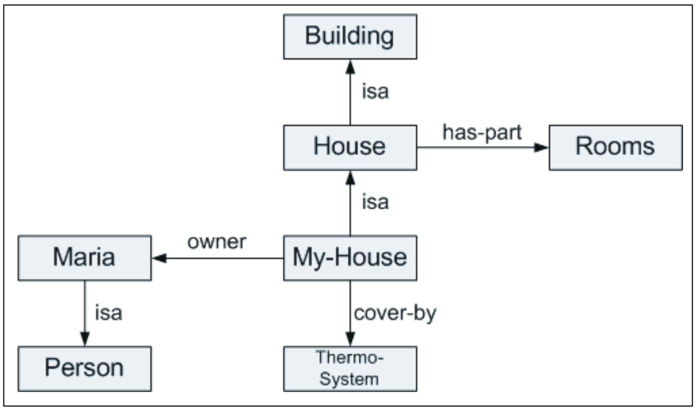


FIGURE 1. Example of a semantic network

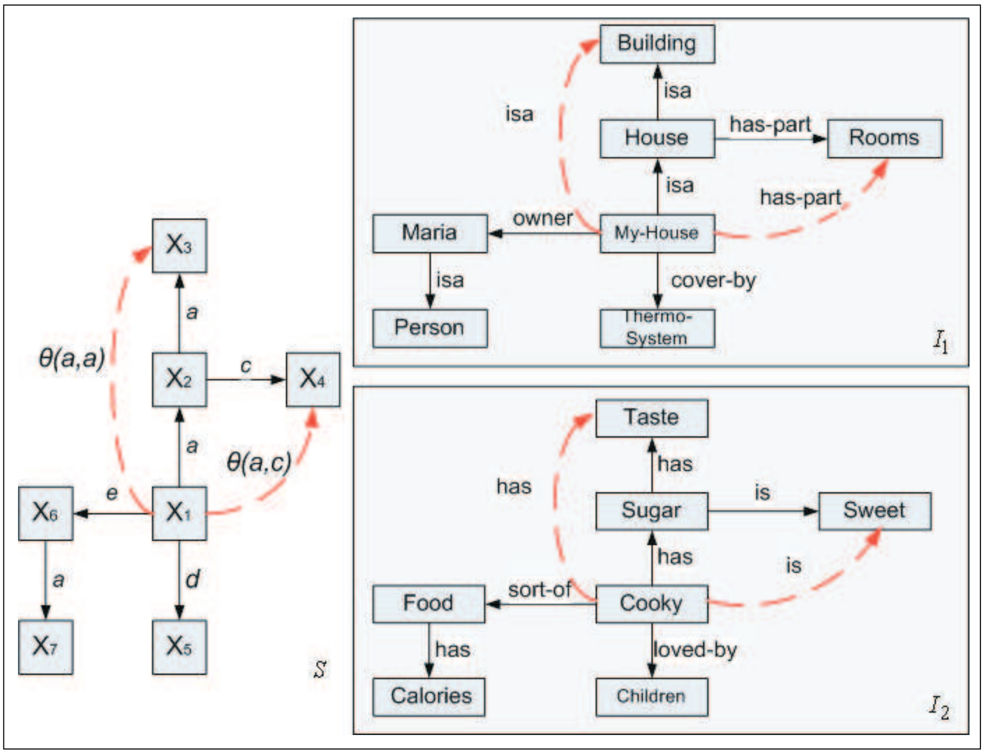


FIGURE 2. A single semantic schema $\mathcal{S}$ and two distinct interpretations of it, noted with $\mathcal{I}_1$, $\mathcal{I}_2$

- **An evaluation aspect** is described in the context of an interpretation by means of which the abstract entities defined in the previous step get values from a space named **output space**

1.1. Semantic schemas. Syntactical aspects. Consider a symbol θ of arity 2 and a finite non-empty set A_0. We denote by $\overline{A_0}$ the **Peano θ-algebra** generated by

A_0, therefore $\overline{A_0} = \bigcup_{n\geq 0} A_n$, where A_n are defined recursively as follows ([8]):

$$A_{n+1} = A_n \cup \{\theta(u,v) \mid u, v \in A_n\}$$

For every $\alpha \in \overline{A_0}$ we define $trace(\alpha)$ as follows:

(1) if $\alpha \in A_0$ then $trace(\alpha) =< \alpha >$

(2) if $\alpha = \theta(u,v)$ then $trace(\alpha) =< p, q >$, where $trace(u) =< p >$ and $trace(v) =< q >$

If $E \subseteq A_1 \times \ldots \times A_n$ and $i \in \{1, \ldots, n\}$ then we denote:

$$pr_i E = \{x \in A_i \mid \exists (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in E\}$$

Definition 1. *([4]) A* **semantic θ-schema** *is a system $\mathcal{S} = (X, A_0, A, R)$ where:*

- *X is a finite non-empty set of symbols and its elements are named* **object symbols**
- *A_0 is a finite non-empty set of elements named* **label symbols**
- *$A_0 \subseteq A \subseteq \overline{A_0}$, where $\overline{A_0}$ is the Peano θ-algebra generated by A_0*
- *$R \subseteq X \times A \times X$ is a non-empty set which fulfills the following conditions:*

$$(x, \theta(u,v), y) \in R \Rightarrow \exists z \in X : (x,u,z) \in R, (z,v,y) \in R \tag{1}$$

$$\theta(u,v) \in A, (x,u,z) \in R, (z,v,y) \in R \Rightarrow (x, \theta(u,v), y) \in R \tag{2}$$

$$pr_2 R = A \tag{3}$$

In the remainder of this work we say shortly θ-schema instead of semantic θ-schema. We denote:

$$R_0 = R \cap (X \times A_0 \times X) \tag{4}$$

Let $\mathcal{S} = (X, A_0, A, R)$ be a semantic schema. We consider a symbol h of arity 1, a symbol σ of arity 2 and take the set:

$$M = \{h(x,a,y) \mid (x,a,y) \in R_0\}$$

We denote by $\mathcal{H}$ the Peano σ-algebra generated by M. It follows that $\mathcal{H} = \bigcup_{n\geq 0} M_n$, where M_n are defined recursively as follows:

$$\begin{cases} M_0 = M \\ M_{n+1} = M_n \cup \{\sigma(u,v) \mid u, v \in M_n\}, n \geq 0 \end{cases} \tag{5}$$

We denote by Z the alphabet which includes the symbol σ, the elements of X, the elements of A, the left and the right parentheses, the symbol h and comma. We denote by Z^* the set of all the worlds over Z. As in the case of a rewriting system we define two rewriting rules as in the next definition.

Definition 2. *([4]) Let be $w_1, w_2 \in Z^*$. We define the binary relation $\Rightarrow$ as follows:*

- *If $(x,a,y) \in R_0$ then $w_1(x,a,y)w_2 \Rightarrow w_1 h(x,a,y)w_2$*
- *Let be $(x, \theta(u,v), y) \in R$. By the Property 1 we know that there is an element $z \in X$ such that: $(x,u,z) \in R$ and $(z,v,y) \in R$. Thus, we have:*

$$w_1(x, \theta(u,v), y) \Rightarrow w_1 \sigma((x,u,z),(z,v,y))w_2$$

The relation $\Rightarrow$ is named the **direct derivation** *relation over Z^*. We denote by $\Rightarrow^*$ and $\Rightarrow^+$ the reflexive and respectively the transitive closure of the relation $\Rightarrow$. The relation $\Rightarrow^*$ will be called simply the* **derivation** *relation over Z^*.*

Definition 3. *([4]) For each $w \in Z^*$ where $w = w_1 \ldots w_n$ with $w_i \in Z$, $i \in \{1, \ldots, n\}$, $n \geq 1$, we denote:*

$$first(w) = w_1 \text{ and } last(w) = w_n$$

Definition 4. *(*[4]*) The* **mapping generated** *by $\mathcal{S}$ is the mapping:*

$$\mathcal{G}_{\mathcal{S}} : R \rightarrow 2^{\mathcal{H}}$$

defined as follows:

- $\mathcal{G}_{\mathcal{S}}(x, a, y) = \{h(x, a, y)\}$*, for* $a \in A_0$
- $\mathcal{G}_{\mathcal{S}}(x, \theta(u, v), y) = \{w \in \mathcal{H} \mid (x, \theta(u, v), y) \Rightarrow^* w\}$

The set $\mathcal{H}$ is an infinite one. We extract from $\mathcal{H}$ those elements which can be derived from R and we denote this set by $\mathcal{F}_{comp}(\mathcal{S})$. In other words:

$$\mathcal{F}_{comp}(\mathcal{S}) = \{w \in \mathcal{H} \mid \exists(x, u, y) \in R : (x, u, y) \Rightarrow^* w\}$$

Obviously, we have:

$$\mathcal{F}_{comp}(\mathcal{S}) = \bigcup_{(x,u,y)\in R} \mathcal{G}_{\mathcal{S}}(x, u, y)$$

Definition 5. *(*[2]*) If $w \in \mathcal{F}_{comp}(\mathcal{S})$ then the element $u \in A$ such that $(x, u, y) \in R$ and $(x, u, y) \Rightarrow^* w$ is named* **the sort** *of w and we denote this property by $sort(w) = u$.*

In [2] it is proved that for every element $w \in \mathcal{F}_{comp}(\mathcal{S})$, $sort(w)$ is uniquely determined.

1.2. Semantic schema. Semantical aspects. As we have said, by means of an appropriate interpretation the abstract entities of a semantic schema receives values in an **output space**. We define the interpretation of a semantic schema as a system endowed with a set of algorithms, which partition the output space in classes organized hierarchically.

The classes of the output space are defined as follows:

Definition 6. *We define recursively:*

- *The object $o = Alg_a(ob(x), ob(y))$ for $a \in A_0$ and $x, y \in X$ is a* **complex object of class** *a and we note this property by $cls(o) = a$*
- *If $cls(o_1) = u$, $cls(o_2) = v$ and $\theta(u, v) \in A$ then $o = Alg_{\theta(u,v)}(o_1, o_2)$ is a* **complex object of class** *$\theta(u, v)$ and $cls(o) = \theta(u, v)$.*

Definition 7. *(*[4], [2]*) Let be $\mathcal{S} = (X, A_0, A, R)$ a semantic schema. An interpretation $\mathcal{I}$ of $\mathcal{S}$ is a system $\mathcal{I} = (Ob, ob, Y, \{Alg_u\}_{u\in A})$ where:*

- *Ob is a finite set of elements which are called* **objects** *of the interpretation*
- *$ob : X \rightarrow Ob$ is a bijective function*
- *Y is a nonempty set of elements which are called the* **output elements** *of the interpretation:*

$$Y = \bigcup_{u\in A} Y_u \tag{6}$$

where:

$$Y_a = \{Alg_a(ob(x), ob(y)) \mid (x, a, y) \in R_0\}, a \in A_0$$
$$Y_{\theta(u,v)} = \{Alg_{\theta(u,v)}(o_1, o_2) \mid o_1 \in Y_u, o_2 \in Y_v\}, \theta(u, v) \in A \setminus A_0$$

As it can be seen in (6) the output space Y is broken into layers. A layer is a set Y_u for some $u \in A$. We observe that each element Y_u has the class u.

The mapping defined by this kind of interpretation is given in the next definition:

Definition 8. *(*[4]*) We define recursively the valuation mapping:*

$$Val_{\mathcal{I}} : \mathcal{F}_{comp}(\mathcal{S}) \rightarrow Y$$

as follows:

- $Val_{\mathcal{I}}(h(x, a, y)) = Alg_a(ob(x), ob(y))$
- $Val_{\mathcal{I}}(\sigma(\alpha, \beta)) = Alg_{\theta(u,v)}(Val_{\mathcal{I}}(\alpha), Val_{\mathcal{I}}(\beta))$ *if* $sort(\sigma(\alpha, \beta)) = \theta(u, v)$.

We remark that the elements of A are viewed as **sorts** for the elements of $\mathcal{F}_{comp}(\mathcal{S})$ and classes for objects.

$$sort(\sigma(\alpha, \beta)) = cls(Val_{\mathcal{I}}(\sigma(\alpha, \beta)))$$

The output mapping of a semantic schema generated by an interpretation computes for each pair of nodes $(x, y) \in X \times X$ all the meanings assigned in the output space Y.

Definition 9. *([4]) If $\mathcal{I}$ is an interpretation of a semantic schema $\mathcal{S}$ then we can define the output mapping:*

$$Out_{\mathcal{I}} : X \times X \to 2^Y$$

as follows:

$$Out_{\mathcal{I}}(x, y) = \bigcup_{(x,u,y)\in R} \bigcup_{w\in\mathcal{G}_{\mathcal{S}}(x,u,y)} \{Val_{\mathcal{I}}(w)\}$$

2. Deductive Paths in Semantic Schemas

In the subsequent we will consider a θ-schema $\mathcal{S}$ such that $\mathcal{S} = (X, A_0, A, R)$.

Definition 10. *We denote by $ORD(\mathcal{S})$ the least set of pairs $([x_1, \ldots, x_r], [a_1, \ldots, a_n])_{r>1,n\geq 1}$ satisfying the following properties:*
- *If $(x, a, y) \in R_0$ then $([x, y], a) \in ORD(\mathcal{S})$.*
- *If $([x_i, \ldots, x_k], b_1) \in ORD(\mathcal{S})$ and $([x_k, \ldots, x_r], b_2) \in ORD(\mathcal{S})$, where $i < k < r$, then $([x_i, \ldots, x_r], [b_1, b_2]) \in ORD(\mathcal{S})$.*

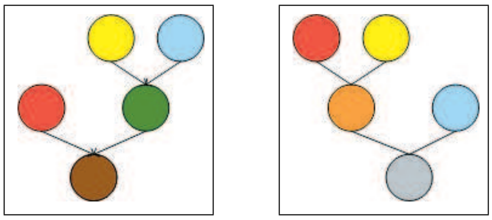
*An element of $ORD(\mathcal{S})$ is an **ordered path** of $\mathcal{S}$.*

Remark 11. *The sentence "the least set" in Definition 10 shows that for every $([x_i, \ldots, x_r], u) \in ORD(\mathcal{S})$ there are $([x_i, \ldots, x_k], b_1) \in ORD(\mathcal{S})$ and $([x_k, \ldots, x_r], b_2) \in ORD(\mathcal{S})$ such that $i < k < r$ and $u = [b_1, b_2]$.*

Let us consider the θ-schema $\mathcal{S}$ represented in Figure 5. We exemplify the following ordered paths of $\mathcal{S}$:
- $([x_1, x_2], a) \in ORD(\mathcal{S})$, $([x_2, x_3], b) \in ORD(\mathcal{S})$.
- $([x_1, x_2, x_3], [a, b]) \in ORD(\mathcal{S})$, $([x_2, x_3, x_4], [b, a]) \in ORD(\mathcal{S})$
- $([x_1, x_2, x_3, x_4], [[a, b], a]) \in ORD(\mathcal{S})$, $([x_1, x_2, x_3, x_4], [a, [b, a]]) \in ORD(\mathcal{S})$.

Remark 12. *The name of "ordered" path comes from the fact that some order is introduced between the arcs of the path.*

The combination order of two consecutive entities of an application can give distinct final results. For example, consider the following game: we consider n colored balls b_1,. . .,b_n; two consecutive balls can be replaced by another ball by applying some rule; the game finishes when we obtain only one ball. In order to exemplify this game we take a sequence $(yellow, blue, orange, green, red, brown, grey)$ of five balls of color yellow, blue, orange, green red, brown and respectively grey. The replacing rules are the following:
- $(yellow, blue) \to green$
- $(red, green) \to brown$
- $(red, yellow) \to orange$
- $(orange, blue) \to grey$.



(a) (b)

FIGURE 3. 3(a) the ball corresponding to the sequence $(red, (yellow, blue))$ and 3(b) corresponds to the sequence $((red, yellow), blue)$

If we apply the sequence $(red, (yellow, blue))$ then we obtain a brown ball (see Figure3(a)). Changing the order, the sequence $((red, yellow), blue)$ gives a grey ball (see Figure3(b)).

We denote $L(\mathcal{S}) = pr_2 ORD(\mathcal{S})$. We define the following entities for an element of $L(\mathcal{S})$:

- $length(a) = 1$, $\triangle(a) =< a >$ for $a \in A_0$
- $length([u, v]) = length(u) + length(v)$
- $\triangle([u, v]) =< a_1, \ldots, a_s, b_1, \ldots, b_p >$, for $\triangle(u) =< a_1, \ldots, a_s >$, $\triangle(v) =< b_1, \ldots, b_p >$

We define also the mapping $\omega : L(\mathcal{S}) \longrightarrow \overline{A_0}$ by:
(1) $\omega(a) = a$ for $a \in A_0$;
(2) $\omega([u, v]) = \theta(\omega(u), \omega(v))$
We observe that the mapping ω works as a "morphism" if we consider that $L(\mathcal{S})$ is a partial algebra. For example,

$$\omega([[a, b], [b, a]]) = \theta(\omega([a, b]), \omega([b, a])) = \theta(\theta(a, b), \theta(b, a))$$

Proposition 13. *If $u \in L(\mathcal{S}) \setminus A_0$ then there is $u_1, u_2 \in L(\mathcal{S})$ such that $u = [u_1, u_2]$.*

Proof. We have $L(\mathcal{S}) = pr_2 ORD(\mathcal{S})$, therefore there is a sequence $[x_1, \ldots, x_{n+1}]$ of nodes such that $([x_1, \ldots, x_{n+1}], u) \in ORD(\mathcal{S})$. By Remark 11 we find $([x_1, \ldots, x_r], b_1) \in ORD(\mathcal{S})$, $([x_r, \ldots, x_{n+1}], b_2) \in ORD(\mathcal{S})$ such that $u = [b_1, b_2]$. It follows that $b_1 \in L(\mathcal{S})$, $b_2 \in L(\mathcal{S})$ and the proposition is proved. □

Proposition 14. *The mapping $\omega : L(\mathcal{S}) \longrightarrow \overline{A_0}$ is injective.*

Proof. We prove by induction on n the following property: $\forall u, v \in L(\mathcal{S})$, if $\omega(u) = \omega(v)$, for $trace(\omega(u)) =< a_1, \ldots, a_n >$, then $u = v$.

For $n = 1$ we have $u = a_1 = v$ and the property is verified.

Suppose the property is true for every $n \leq p$ and take $u, v \in L(\mathcal{S})$ such that $\omega(u) = \omega(v)$, $trace(\omega(u)) =< a_1, \ldots, a_{p+1} >$. By Proposition 13 we find $u_1, u_2, v_1, v_2 \in L(\mathcal{S})$ such that $u = [u_1, u_2]$, $v = [v_1, v_2]$. Obviously $trace(\omega(u_i)) \leq p$ and $trace(\omega(v_i)) \leq p$ for $i \in \{1, 2\}$. But $\omega(u) = \theta(\omega(u_1), \omega(u_2))$, $\omega(v) = \theta(\omega(v_1), \omega(v_2))$ and $\omega(u) = \omega(v)$. The set $\overline{A_0}$ is a Peano algebra therefore $\omega(u_1) = \omega(v_1)$ and $\omega(u_2) = \omega(v_2)$. Applying the inductive assumption we obtain $u_1 = v_1$ and $u_2 = v_2$, therefore $u = v$. □

Proposition 15. *If $w = [u, v] = [\alpha, \beta] \in L(\mathcal{S})$, $u, v, \alpha, \beta \in L(\mathcal{S})$ then $u = \alpha$ and $v = \beta$.*

Proof. We have $\omega(w) = \theta(\omega(u), \omega(v)) = \theta(\omega(\alpha), \omega(\beta)) \in \overline{A_0}$. But $\overline{A_0}$ is a Peano algebra, therefore $\omega(u) = \omega(\alpha)$ and $\omega(v) = \omega(\beta)$. By Proposition 14 we have $u = \alpha$ and $v = \beta$. The Proposition is proved. □

Corollary 16. *If $w \in L(\mathcal{S})$ then there are $u \in L(\mathcal{S})$ and $v \in L(\mathcal{S})$, uniquely determined, such that $w = [u, v]$.*

Proof. Immediate by Proposition 13 and Proposition 15. □

Proposition 17. *If $([x_1, \ldots, x_{n+1}], w) \in ORD(\mathcal{S})$ then $length(w) = n$.*

Proof. We apply Definition 10. We proceed by induction on n. If $n = 1$ then $(x_1, w, x_2) \in R_0$, therefore $w \in A_0$. Thus the property is true in this case. Suppose that the property is true for every $n \leq p$ and consider $([x_1, \ldots, x_{p+2}], w) \in ORD(\mathcal{S})$. Because $ORD(\mathcal{S})$ is the *least set* satisfying Definition 10 we deduce that there are w_1, w_2 and a natural number k such that $2 \leq k \leq p + 1$ such that $w = [w_1, w_2]$, $([x_1, \ldots, x_k], w_1) \in ORD(\mathcal{S})$ and $([x_k, \ldots, x_{p+2}], w_2) \in ORD(\mathcal{S})$. Applying the inductive assumption we have $length(w_1) = k - 1$, $length(w_2) = p + 2 - k$. But $length(w) = length(w_1) + length(w_2) = k - 1 + p + 2 - k = p + 1$ and the proposition is proved. □

Definition 18. *An ordered path $([x_1, \ldots, x_k], w) \in ORD(\mathcal{S})$ is a **deductive path** if $\omega(w) \in A$. We denote by $Ded(\mathcal{S})$ the set of all deductive paths of $\mathcal{S}$.*

Based on this definition we can use the notation $([x_1, \ldots, x_k], \omega(w))$ for a deductive path. This notation can be explained by the fact that the entity $\omega(w)$ defines all the properties of the corresponding path.

Proposition 19. *If $([x_1, \ldots, x_{n+1}], \omega(w)) \in Ded(\mathcal{S})$ and $trace(\omega(w)) =< a_1, \ldots, a_k >$ then $k = n$ and $\triangle(w) =< a_1, \ldots, a_k >$.*

Proof. Immediate by Proposition 17 and the morphism property of ω. □

Remark 20. *We relieve the following remarks:*

- *In Figure 4 we have two deductive paths:*
  $([x_1, x_2, x_3, x_4], \theta(\theta(a, b), a)) \in Ded(\mathcal{S})$
  $([x_1, x_2, x_3, x_4], \theta(a, \theta(b, a)) \in Ded(\mathcal{S})$
  The order is given by the square brackets $[[a, b], a]$ and $[a, [b, a]]$ respectively. This order is obtained by a "splitting" property of the deductive path.
- *For Figure 5 we relieve the following property. We have $(x_1, \theta(\theta(a, b), a), x_4) \in R$. The components of this tuple specify the initial node x_1 and the final node x_4 for a deductive path corresponding to the order $[[a, b], a]$. We observe that there are two deductive paths defined by $(x_1, \theta(\theta(a, b), a), x_4) \in R$:*
  $$([x_1, x_2, x_3, x_4], \theta(\theta(a, b), a)) \in Ded(\mathcal{S})$$
  $$([x_1, y_1, y_2, x_4], \theta(\theta(a, b), a)) \in Ded(\mathcal{S})$$

3. Properties of the Deductive Paths

In what follows we develop the properties specified in the previous section. The deductive paths $([x, y], a) \in Ded(\mathcal{S})$ for $(x, a, y) \in R_0$ can be considered as "atomic" entities and only the non-atomic paths can be decomposed into a sequence of two deductive paths. This property is stated in the next proposition.

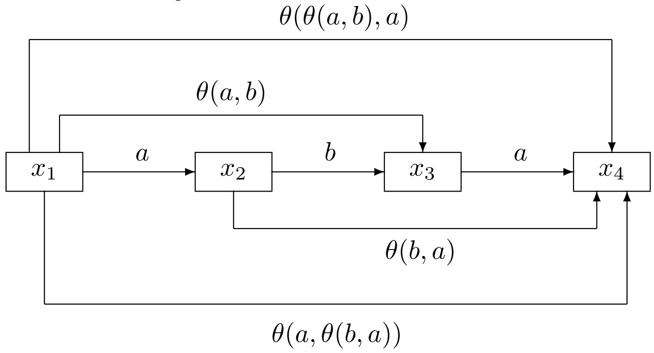


FIGURE 4. Deductive paths

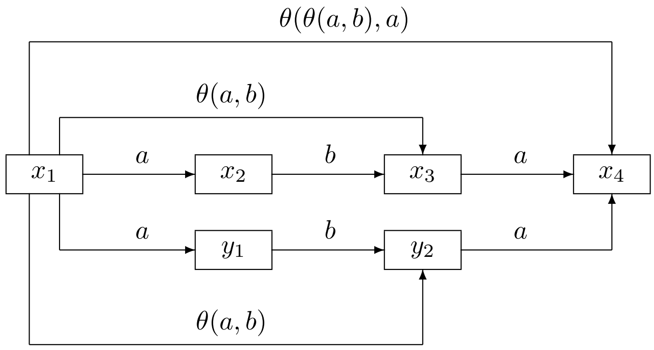


FIGURE 5. Deductive paths

Proposition 21. *If* $([x_1,\ldots,x_{n+1}],\theta(u,v)) \in Ded(\mathcal{S})$*, where* $n \geq 2$*,* $u,v \in A$*, then there is* $k \in \{1,\ldots,n{-}1\}$*, uniquely determined, such that* $([x_1,\ldots,x_{k+1}],u) \in Ded(\mathcal{S})$ *and* $([x_{k+1},\ldots,x_{n+1}],\, v) \in Ded(\mathcal{S})$*.*

Proof. We proceed by induction on n. If $n = 2$ then we have the deductive path $([x_1,x_2,x_3],\, \theta(a,b))$, where $\theta(a,b) \in A$, $a,b \in A_0$. In this case the property is verified for $k = 1$ because $([x_1,x_2],a) \in Ded(\mathcal{S})$ and $([x_2,x_3],b) \in Ded(\mathcal{S})$. Suppose the property is true for every $n \leq p$ and take a path $([x_1,\ldots,x_{p+2}],\theta(u,v)) \in Ded(\mathcal{S})$. There is $w \in L(\mathcal{S})$ such that $([x_1,\ldots,x_{p+2}],w) \in ORD(\mathcal{S})$ and $\omega(w) = \theta(u,v)$. By Corollary 16 we find $\alpha \in L(\mathcal{S})$ and $\beta \in L(\mathcal{S})$, uniquely determined, such that $w = [\alpha,\beta]$. By the definition of the mapping ω we have $\omega(w) = \theta(\omega(\alpha),\omega(\beta))$. But $\omega(w) = \theta(u,v)$, therefore using the properties of the Peano algebra $\overline{A_0}$ we have $\omega(\alpha) = u$ and $\omega(\beta) = v$. By Remark 11 there are $([x_1,\ldots,x_k],b_1) \in ORD(\mathcal{S})$ and $([x_k,\ldots,x_{p+2}],b_2) \in ORD(\mathcal{S})$ such that $1 < k < p+2$ and $w = [b_1,b_2]$. By the definition of $L(\mathcal{S})$ we deduce that $b_1 \in L(\mathcal{S})$ and $b_2 \in L(\mathcal{S})$. We have $\omega(w) = \theta(\omega(b_1),\omega(b_2))$, therefore $\omega(\alpha) = \omega(b_1)$ and $\omega(\beta) = \omega(b_2)$. Applying Proposition 14 we obtain $\alpha = b_1$ and $\beta = b_2$. It follows that $([x_1,\ldots,x_k],\omega(\alpha)) \in Ded(\mathcal{S})$ and $([x_k,\ldots,x_{p+2}],\omega(\beta)) \in Ded(\mathcal{S})$. In other words $([x_1,\ldots,x_k],u) \in Ded(\mathcal{S})$ and $([x_k,\ldots,x_{p+2}],v) \in Ded(\mathcal{S})$ and the proposition is proved. □

Proposition 22. *If* $([x_1,\ldots,x_{n+1}],u) \in Ded(\mathcal{S})$ *then* $(x_1,u,x_{n+1}) \in R$*.*

Proof. We prove this property by induction on n. For $n = 1$ the proposition is true because if $([x_1,x_2],u) \in Ded(\mathcal{S})$ then $(x_1,u,x_2) \in R_0 \subseteq R$. Consider a natural number $p \geq 1$. We suppose the proposition is true for every $n \leq p$ and consider $([x_1, \ldots, x_{p+2}], u) \in Ded(\mathcal{S})$. Using the properties of the set A we deduce that there are $\alpha \in A$ and $\beta \in A$ such that $u = \theta(\alpha, \beta)$. By Proposition 21 we find a natural number $k \in \{1, \ldots, p\}$, uniquely determined, such that $([x_1, \ldots, x_{k+1}], \alpha) \in Ded(\mathcal{S})$ and $([x_{k+1}, \ldots, x_{p+2}], \beta) \in Ded(\mathcal{S})$. By the inductive assumption we have $(x_1, \alpha, x_{k+1}) \in R$ and $(x_{k+1}, \beta, x_{p+2}) \in R$. But $\theta(\alpha, \beta) \in A$ and by the properties of the set A we obtain $(x_1, \theta(\alpha, \beta), x_{p+2}) \in R$. Thus $(x_1, u, x_{p+2}) \in R$ and the proposition is proved. □

Proposition 23. *If* $([x_1, \ldots, x_{k+1}], u) \in Ded(\mathcal{S})$, $([x_{k+1}, \ldots, x_{n+1}], v) \in Ded(\mathcal{S})$ *and* $\theta(u, v) \in A$ *then* $([x_1, \ldots, x_{n+1}], \theta(u, v)) \in Ded(\mathcal{S})$.

Proof. There are $w_1 \in L(\mathcal{S})$, $w_2 \in L(\mathcal{S})$ such that $([x_1, \ldots, x_{k+1}], w_1) \in ORD(\mathcal{S})$, $([x_{k+1}, \ldots, x_{n+1}], w_2) \in ORD(\mathcal{S})$, $\omega(w_1) = u$ and $\omega(w_2) = v$. By Definition 10 we have $([x_1, \ldots, x_{n+1}], [w_1, w_2]) \in ORD(\mathcal{S})$. But $\omega([w_1, w_2]) = \theta(\omega(w_1), \omega(w_2)) = \theta(u, v)$, therefore $([x_1, \ldots, x_{n+1}], \theta(u, v)) \in Ded(\mathcal{S})$. □

Proposition 24. *If* $(x_1, u, x_{n+1}) \in R$ *then there is* $([x_1, \ldots, x_{n+1}], u) \in Ded(\mathcal{S})$.

Proof. We proceed by induction on k, where $trace(u) =< a_1, \ldots, a_k >$.

For $k = 1$ we have $u = a \in A_0$ and therefore $(x_1, u, x_{n+1}) \in R_0$. Based on Definition 10 we have $([x_1, x_{n+1}], a) \in ORD(\mathcal{S})$. But $\omega(a) = a \in A_0$ and $A_0 \subseteq A$, therefore $\omega(a) \in A$. Thus $([x_1, x_{n+1}], u) \in Ded(\mathcal{S})$ and the property is verified for $k = 1$.

Suppose the property is true for every $k < p$ and consider an element $(x_1, u, x_{p+1}) \in R$ such that $trace(u) =< a_1, \ldots, a_p >$. Based on the properties of a Peano algebra and the properties of $\mathcal{S}$ we can write $u = \theta(\alpha, \beta)$, where $\alpha \in A$, $\beta \in A$ and there is $r < p$ such that $(x_1, \alpha, x_{r+1}) \in R$, $(x_{r+1}, \beta, x_{p+1}) \in R$. Applying the inductive assumption we deduce that there are $([x_1, \ldots, x_{r+1}], \alpha) \in Ded(\mathcal{S})$ and $([x_{r+1}, \ldots, x_{p+1}], \beta) \in Ded(\mathcal{S})$. By Proposition 23 we have $([x_1, \ldots, x_{p+1}], \theta(\alpha, \beta)) \in Ded(\mathcal{S})$. □

4. Path-based Reasoning in Semantic Schemas

In this section we present a new reasoning mechanism for a semantic schema based on deductive paths. In comparison with the usual formalism we will consider a path-driven mechanism and as a consequence, we obtain a slight modification of the valuation mapping.

We consider a θ-schema $\mathcal{S} = (X, A_0, A, R)$. Let us denote by h a symbol of arity 1 and take the set:

$$M = \{h([x, y], a) \quad | \quad (x, a, y) \in R_0\}$$

where $R_0 = (X \times A_0 \times X) \cap R$ and we used the notation $h([x, y], a)$ instead of $h(([x, y], a))$. We consider a symbol σ of arity 2 and denote by $\mathcal{H}_{\mathcal{S}}$ *the Peano* σ-*algebra generated by* M.

We denote by Z the alphabet including the symbol σ, the elements of X and of A, the left and right parentheses, the square brackets [and], the symbol h and comma. We denote by Z^* the set of all words over Z.

Let be $w_1, w_2 \in Z^*$. We define the following binary relation on Z^*, denoted by $\Rightarrow_{\mathcal{S}}$:

- If $(x, a, y) \in R_0$ then $w_1([x, y], a)w_2 \Rightarrow_{\mathcal{S}} w_1 h([x, y], a)w_2$
- Let be $([x_1, \ldots, x_{n+1}], \theta(u, v)) \in Ded(\mathcal{S})$, for $u, v \in A$. If $([x_1, \ldots, x_{k+1}], u) \in Ded(\mathcal{S})$ and $([x_{k+1}, \ldots, x_{n+1}], v) \in Ded(\mathcal{S})$ then

$$w_1([x_1, \ldots, x_{n+1}], \theta(u, v))w_2 \Rightarrow_{\mathcal{S}} w_1 \sigma(([x_1, \ldots, x_{k+1}], u), ([x_{k+1}, \ldots, x_{n+1}], v))w_2$$

We denote by $\Rightarrow_{\mathcal{S}}^*$ the reflexive and transitive closure of the relation $\Rightarrow_{\mathcal{S}}$.

The set $\mathcal{H}_{\mathcal{S}}$ is an infinite one. We extract from $\mathcal{H}_{\mathcal{S}}$ those elements which can be derived from $Ded(\mathcal{S})$ and we denote this set by $\mathcal{F}(\mathcal{S})$. In other words,

$$\mathcal{F}(\mathcal{S}) = \{w \in \mathcal{H}_{\mathcal{S}} \mid \exists d \in Ded(\mathcal{S}) : d \Rightarrow_{\mathcal{S}}^* w\}$$

Proposition 25. *If $w \in \mathcal{F}(\mathcal{S})$ then there is a deductive path $d \in Ded(\mathcal{S})$, uniquely determined, such that $d \Rightarrow_{\mathcal{S}}^* w$.*

Proof. We define the following mapping $\Omega : \mathcal{F}(\mathcal{S}) \longrightarrow A$ as follows:

$\Omega(h([x,y],a)) = a$ for $a \in A_0$

$\Omega(\sigma(w_1, w_2)) = \theta(\Omega(w_1), \Omega(w_2))$

We denote by $NodeList(\mathcal{S}) = \{[x_1, \ldots, x_n] \mid n \geq 1, x_i \in X, i = 1, \ldots, n\}$ the set of all lists with elements from X. We define the partial mapping $Reun : NodeList(\mathcal{S}) \times NodeList(\mathcal{S}) \rightarrow NodeList(\mathcal{S})$

$Reun([x_1, \ldots, x_n], [x_n, x_{n+1} \ldots, x_k]) = [x_1, \ldots, x_n, x_{n+1}, \ldots, x_k]$

If we introduce the mapping $\delta : \mathcal{F}(\mathcal{S}) \rightarrow NodeList(\mathcal{S})$ by

$\delta(h([x,y],a)) = [x,y]$

$\delta(\sigma(w_1, w_2)) = Reun(\delta(w_1), \delta(w_2))$

then for every $w \in \mathcal{F}(\mathcal{S})$, $(\exists!)d = (\delta(w), \Omega(w)) \in Ded(\mathcal{S})$ such that d satisfies the property: $d \Rightarrow_{\mathcal{S}}^* w$.

We prove by induction on the number m of σ existing in the word w. Because $w \in \mathcal{F}(\mathcal{S})$ we have two possibilities for w:

- $w = h([x,y],a)$, $(x,a,y) \in R_0$, $m = 0$
- $w = \sigma(w_1, w_2)$, for $w_1, w_2 \in \mathcal{F}(\mathcal{S})$, $m \geq 1$

For $m = 0$ we have that there is an unique $d \in Ded(\mathcal{S})$ such that $d \Rightarrow_{\mathcal{S}}^* w$, $d = ([x,y],a) = (\delta(w), \Omega(w))$.

If $m = 1$ then $w = \sigma(w_1, w_2)$, $w_1 = h([x,y],a), w_2 = h([y,z],b)$, $(x,a,y),(y,b,z) \in R_0$. According to the derivation rules on Z^* results $(\exists!)d \in Ded(\mathcal{S})$:

$$d = ([x,y,z], \theta(a,b)) \Rightarrow_{\mathcal{S}}^* \sigma(h([x,y],a), h([y,z],b)) = \sigma(w_1, w_2) = w$$

We obtain $d = (Reun(\delta(w_1), \delta(w_2)), \theta(\Omega(w_1), \Omega(w_2)))$ that is $d = (\delta(w), \Omega(w))$.

Suppose that the property is true for all words w such that $m < p$ and consider $w = \sigma(w_1, w_2)$ with $m = p$. Obviously, the words w_1 and w_2 contain a smaller number of σ then w and thus:

$(\exists!)d_1 = (\delta(w_1), \Omega(w_1)) \in Ded(\mathcal{S}) : d_1 \Rightarrow_{\mathcal{S}}^* w_1$

$(\exists!)d_2 = (\delta(w_2), \Omega(w_2)) \in Ded(\mathcal{S}) : d_2 \Rightarrow_{\mathcal{S}}^* w_2$

According to the derivation rules we have:

$$d = (Reun(\delta(w_1), \delta(w_2)), \theta(\Omega(w_1), \Omega(w_2))) \Rightarrow_{\mathcal{S}}^* \sigma(w_1, w_2)$$

From the way the mappings δ and Ω are defined we obtain $d = (\delta(w), \Omega(w)) \Rightarrow_{\mathcal{S}}^* w$ and the property s proved. □

Remark 26. *If $w \in \mathcal{F}(\mathcal{S})$ and $d \Rightarrow_{\mathcal{S}}^* w$ then $pr_2 d$ is named the **sort** of w and we denote $sort(w) = pr_2 d$.*

The set $\mathcal{F}(\mathcal{S})$ is the result of the formal computations defined by the schema $\mathcal{S}$.

We consider the interpretation $\mathcal{I} = (Ob, ob, \{Alg_u\}_{u \in A})$ of $\mathcal{S}$. Based on $\mathcal{I}$ we define the **valuation mapping** based on the deductive paths as follows:

$$Val_{\mathcal{I}} : \mathcal{F}(\mathcal{S}) \longrightarrow Y$$

as follows:

- $Val_{\mathcal{I}}(h([x,y],a)) = Alg_a(ob(x), ob(y))$

- $Val_{\mathcal{I}}(\sigma(\alpha,\beta)) = Alg_{\theta(u,v)}(Val_{\mathcal{I}}(\alpha), Val_{\mathcal{I}}(\beta))$ if $sort(\sigma(\alpha,\beta)) = \theta(u,v)$.

5. Future work

We intend to continue our work by defining a new structure over two different semantic schemas based on their deductive paths. We will name this structure as the *hyper-schema*.

An hyper-schema relieves a special kind of cooperation between the schemas over which is defined. As a mathematical structure, an hyper-schema is an aggregation of the two schemas. But an hyper-schema also benefits of a knowledge transfer from the schemas. This transfer is described by means of the deductive paths that can be interconnected between the two schemas.

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