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The Multiple Zeta Function and the Computation of Some Integrals in Compact Form

Radu-Octavian Vîlceanu

Abstract. Based on the Euler-Zagier multiple zeta function ζ and the extended zeta function $\tilde{\zeta}$, we compute integrals of the form

$$I^{\pm}_{k,l,r,m} = \int_0^1 \frac{(-\log(1 \pm x))^k}{(1 \pm x)^m} \cdot x^r (-\log x)^l \, dx,$$
$$J^{\pm}_{l,r,m,k} = \int_0^1 \frac{x^{kr+k-1}}{(1 \pm x^k)^m} \cdot (-\log x)^l \, dx$$

and

$$I_{l,r,m_1,m_2} = \int_0^1 \frac{x^r (-\log x)^l}{(1-x)^{m_1}(1+x)^{m_2}} dx$$

where k, l, r, m, m_1, m_2 are integers.

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1. Introduction

In recent years much attention was paid to the computation in compact form of the integrals involving special functions. The aim of this paper is to consider the integrals of the type

$$I^{\pm}_{k,\ell,r,m} = \int_0^1 \frac{(-\log(1 \pm x))^k}{(1 \pm x)^m} \cdot x^r (-\log x)^\ell \, dx, \tag{1.1}$$

and

$$J^{\pm}_{\ell,r,m,k} = \int_0^1 \frac{x^{kr+k-1}}{(1 \pm x^k)^m} \cdot (-\log x)^\ell \, dx, \tag{1.2}$$

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where k, ℓ, r, m are integers. Some particular cases can be found in handbooks like that by I. S. Gradzhteyn and I. M. Ryzhyk [11]. For example,

$$\int_0^1 \frac{1}{1+x} \cdot \log x dx = -\frac{\pi^2}{12} \quad \text{([11], formula 4.231.1)} \tag{1.3}$$

$$\int_0^1 \frac{1}{1-x} \cdot \log x dx = -\frac{\pi^2}{6} \quad \text{([11], formula 4.231.2)} \tag{1.4}$$

$$\int_0^1 \frac{1}{1-x} \cdot \log^3 x dx = -\frac{\pi^4}{15} \quad \text{([11], formula 4.262.2)} \tag{1.5}$$

$$\int_0^1 \frac{x}{1-x} \cdot \log x dx = 1-\frac{\pi^2}{6} \quad \text{([11], formula 4.231.3)} \tag{1.6}$$

$$\int_0^1 \log(1-x) \cdot \log x dx = 2-\frac{\pi^2}{6} \quad \text{([11], formula 4.221.1)} \tag{1.7}$$

$$\int_0^1 \log(1+x) \cdot \log x dx = 2-2\log 2-\frac{\pi^2}{12} \quad \text{([11], formula 4.221.2)} \tag{1.8}$$

All the examples above were known to L. Euler (in connection with his work on the zeta function). The computation of the integrals $I^{\pm}_{k,\ell,r,m}$ and $J^{\pm}_{\ell,r,m,k}$ is a bit more involving since it needs the *Euler-Zagier multiple zeta function,*

$$\zeta_\ell(s_1, s_2, ..., s_\ell) = \sum_{n_1>n_2>n_3>\cdots>n_\ell\geq 1} \frac{1}{n_1^{s_1} \cdot n_2^{s_2} \cdots n_\ell^{s_\ell}}, \tag{1.9}$$

where the summation runs over the integer values of $n_1 > n_2 > n_3 > \cdots > n_\ell$. Here $\ell \in \mathbb{N}^*$. For $\ell = 1$, we retrieve the usual zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(also called Riemann's zeta function).

The Euler-Zagier function ζ_ℓ is absolutely convergent in the region

$$\operatorname{Re}(s_\ell) > 1, \ \sum\nolimits_{j=1}^{\ell} \operatorname{Re}(s_j) > \ell. \tag{1.10}$$

See [18] for details.

A remarkable result due to L. Euler asserts that $\zeta(2n)$ can be computed in compact form for all positive integers n :

Lemma 1.1. (L. Euler; see [15], p. 22 and p. 48). *For every $n \in \mathbb{N}^*$,*

$$\zeta(2n) = \frac{(-1)^{n+1} 2^{2n-1} \pi^{2n}}{(2n)!} B_{2n}, \tag{1.11}$$

where the B_n's are the Bernoulli numbers.

The Bernoulli numbers are usually defined as the coefficients of the Maclaurin expansion

$$\frac{t}{e^t-1} = \frac{1}{\sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!}.$$

Thus

$$\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}...\right)\left(B_0+B_1 t+\frac{B_2 t^2}{2!}+\frac{B_3 t^3}{3!}...\right) = 1.$$

A useful remark is the following recursion formula:

Lemma 1.2. (see [1]). *If* $n \geq 2$, *then*

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{1.12}$$

The discussion above yields

$$B_0 = 1, \quad B_1 = -1/2, \quad B_2 = 1/6 \quad B_4 = -1/30, \quad B_6 = 1/42, \ldots$$

and

$$B_3 = B_5 = B_7 = \ldots = 0.$$

Thus

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945} \text{ etc.}$$

The computation of the multiple zeta function needs some additional remarks (all available in [4]):

- *Euler's reduction formula,*

$$\zeta_2(s,1) = \frac{s}{2}\zeta(s+1) - \frac{1}{2}\sum_{k=1}^{s-2}\zeta(k+1)\zeta(s-k),\ \forall s \in \mathbb{Z},\ s > 1; \tag{1.13}$$

- *Euler's reflexive formula,*

$$\zeta_2(s_1,s_2) + \zeta_2(s_2,s_1) = \zeta(s_1)\zeta(s_2) - \zeta(s_1+s_2),\ \forall s_1,s_2 \in \mathbb{Z},\ s_1,s_2 > 1; \tag{1.14}$$

- *the summation formula,*

$$\sum_{\substack{a_1+a_2+\cdots+a_r=n \\ a_1\geq 0,\ldots,a_r\geq 0}} \zeta_r(a_1+2, a_2+1, \ldots, a_r+1) = \zeta(n+r+1),\ \forall n,r \in \mathbb{N}. \tag{1.15}$$

Therefore

$$\begin{aligned}
\zeta_2(2,1) &= \zeta(3); \\
\zeta_2(3,1) &= \frac{3}{2}\zeta(4) - \frac{1}{2}\zeta^2(2) = \frac{\pi^4}{360}; \\
\zeta_2(4,1) &= 2\zeta(5) - \zeta(2)\zeta(3); \\
\zeta_2(5,1) &= \frac{5}{2}\zeta(6) - \zeta(2)\zeta(4) - \frac{1}{2}\zeta^2(3); \\
\zeta_2(6,1) &= 3\zeta(7) - \zeta(5)\zeta(2) - \zeta(4)\zeta(3).
\end{aligned}$$

Markett and Broadhurst (see [8]) have proved the following formulae:

$$\begin{aligned}
\zeta_3(2,1,1) &= \zeta(4); \\
\zeta_3(3,1,1) &= 2\zeta(5) - \zeta(3)\zeta(2); \\
\zeta_3(4,1,1) &= \frac{23}{16}\zeta(6) - \zeta^2(3); \\
\zeta_3(5,1,1) &= -\frac{5}{4}\zeta(3)\zeta(4) + 5\zeta(7) - 2\zeta(5)\zeta(2).
\end{aligned}$$

The *multiple extended zeta function* $\tilde{\zeta}$ is the complex function

$$\tilde{\zeta}_\ell(s_1, s_2, \ldots, s_\ell; \sigma_1, \sigma_2, \ldots, \sigma_\ell) = \sum_{n_1>n_2>n_3>\cdots>n_\ell\geq 1} \frac{\sigma_1^{n_1}\cdot\sigma_2^{n_2}\cdots\sigma_\ell^{n_\ell}}{n_1^{s_1}\cdot n_2^{s_2}\cdots n_\ell^{s_\ell}}, \tag{1.16}$$

where $\ell \in \mathbb{N}^*$, $s_1, s_2, ..., s_\ell \in \mathbb{Z}^*$, and $\sigma_i = \pm 1$. See [5].

For $\sigma_i = \operatorname{sgn}(s_i)$ $(i \in \{1, ..., \ell\})$ we put (see [4]),

$$\tilde{\zeta}_\ell(s_1, s_2, ..., s_\ell) = \sum_{n_1 > n_2 > n_3 > ... > n_\ell \geq 1} \frac{\sigma_1^{n_1} \cdot \sigma_2^{n_2} \cdots \sigma_\ell^{n_\ell}}{n_1^{|s_1|} \cdot n_2^{|s_2|} \cdots n_\ell^{|s_\ell|}}, \tag{1.17}$$

Thus

$$\tilde{\zeta}_\ell(s_1, s_2, ..., s_\ell) = \zeta_\ell(s_1, s_2, ..., s_\ell),$$

provided that $s_1, s_2, ..., s_\ell \in \mathbb{N}^*$.

Lemma 1.3. (see [16], p. 125). *If $\tilde{\zeta}_1$ is the function defined above for $\ell = 1$, and ζ is the Riemann zeta function, then:*

$$\tilde{\zeta}_1(-p) = \begin{cases} -\ln(2) & \text{if } \; p = 1 \\ \left(2^{1-p} - 1\right)\zeta(p) & \text{if } \; p > 1 \end{cases}.$$

Therefore

$$\begin{aligned}
\tilde{\zeta}_1(-1) &= -\ln(2); \\
\tilde{\zeta}_1(-2) &= \left(2^{1-2} - 1\right)\zeta(2) = -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}; \\
\tilde{\zeta}_1(-3) &= \left(2^{1-3} - 1\right)\zeta(3) = -\frac{3}{4}\zeta(3); \\
\tilde{\zeta}_1(-4) &= \left(2^{1-4} - 1\right)\zeta(4) = -\frac{7}{8}\zeta(4) = -\frac{7\pi^4}{720}; \\
\tilde{\zeta}_1(-5) &= \left(2^{1-5} - 1\right)\zeta(5) = -\frac{15}{16}\zeta(5); \\
\tilde{\zeta}_1(-6) &= \left(2^{1-6} - 1\right)\zeta(6) = -\frac{31}{32}\zeta(6) = -\frac{31\pi^6}{30240}.
\end{aligned}$$

According to J. M. Borwein, D. M. Bradley and D. J. Broadhurst [5] and M. Bigotte, G. Jacob, N. E. Oussous and M. Petitot [10], we have

$$\begin{aligned}
\zeta_4(2,1,1,1) &= \zeta(5); \\
\zeta_4(3,1,1,1) &= \zeta_2(5,1); \\
\zeta_4(4,1,1,1) &= \zeta_3(5,1,1); \\
\tilde{\zeta}_{n+1}(-1, \{1\}_n) &= (-1)^{n+1}\frac{1}{(n+1)!}\log^{n+1}(2); \\
\tilde{\zeta}_2(-2,1) &= \frac{1}{8}\zeta(3); \\
\tilde{\zeta}_3(-2,1,1) &= -\frac{1}{40}\zeta^2(2) + \frac{1}{2}\tilde{\zeta}_2(-3,1);
\end{aligned}$$

where $\{a\}_n$ means

$$\underbrace{a, a, ..., a}_{n \text{ times}}.$$

The *Hurwitz multiple ζ function* is the complex function

$$\zeta_\ell(s_1, s_2, ..., s_\ell; \theta_1, \theta_2, ..., \theta_\ell) = \sum_{n_1 > n_2 > n_3 > \cdots > n_\ell \geq 1} \frac{1}{(n_1 + \theta_1)^{s_1} \cdot (n_2 + \theta_2)^{s_2} \cdots (n_\ell + \theta_\ell)^{s_\ell}}, \tag{1.18}$$

where $\ell \in \mathbb{N}^*$, $s_1, s_2, ..., s_\ell \in \mathbb{Z}^*$, and $\theta_k \in [0, 1)$ for $k \in \{1, 2, ..., \ell\}$. See [14]

When $\theta_k = 0$ for all $k \in \{1, 2, ..., \ell\}$), the Hurwitz multiple ζ_ℓ function coincides with the Euler-Zagier multiple ζ_ℓ function.

Lemma 1.4. (see [14]) *The Hurwitz multiple ζ_ℓ function is analytic in the region*

$$\operatorname{Re}(s_k + \cdots + s_\ell) > \ell - k + 1, \text{ for } k \in \{1, 2, ..., \ell\}.$$

Lemma 1.5. (see [14]) *The Hurwitz multiple ζ_ℓ function has a meromorphic extension to $\mathbb{C}^\ell$ having as possible poles:*

a) one simple pole in the hyperplane $s_\ell = 1$;

b) one simple pole in the hyperplane $s_k + ... + s_\ell - \ell + k - 1 = n$ for all the integers $n \leq 0$ and $k \in \{1, 2, ..., \ell - 1\}$.

2. The main results

Lemma 2.1. *If k, r and ℓ are nonnegative integers and $\ell \geq 1$, then:*

$$I^{\pm}_{k,\ell,r,1} = \int_0^1 \frac{(-\log(1 \pm x))^k}{1 \pm x} \cdot x^r (-\log x)^\ell \, dx$$
$$= \mp k!\ell! \sum_{n_1 > n_2 > n_3 > \cdots > n_{k+1} \geq 1} \frac{(\mp 1)^{n_1}}{(n_1 + r)^{\ell+1} \cdot n_2 \cdot n_3 \cdots n_{k+1}}.$$

Proof. For $k = 0$ we have the Maclaurin expansion

$$\frac{1}{1 \pm x} = \sum_{n_1=0}^{\infty} (\mp x)^{n_1},$$

which yields

$$I^{\pm}_{0,\ell,r,1} = \sum_{n_1=0}^{\infty} (\pm 1)^{n_1} \int_0^1 x^{n_1+r} (-\log x)^\ell \, dx = \mp \ell! \sum_{n_1=1}^{\infty} \frac{(\mp 1)^{n_1}}{(n_1 + r)^{\ell+1}}.$$

For $k \geq 1$ we have

$$\frac{(-\log(1 \pm x))^k}{1 \pm x} = k! \sum_{n_1 \geq n_2 > n_3 > \cdots > n_{k+1} \geq 1} \frac{(\mp x)^{n_1}}{n_2 \cdot n_3 \cdots n_{k+1}}.$$

Then

$$I^{\pm}_{k,\ell,r,1} = k! \sum_{n_1 \geq n_2 > n_3 > \cdots > n_{k+1} \geq 1} \frac{(\mp 1)^{n_1}}{n_2 \cdot n_3 \cdot \cdots \cdot n_{k+1}} \int_0^1 x^{n_1+r} (-\log x)^\ell \, dx$$
$$= k! \sum_{n_1 \geq n_2 > n_3 > \cdots > n_{k+1} \geq 1} \frac{(\mp 1)^{n_1}}{n_2 \cdot n_3 \cdots n_{k+1}} \cdot \frac{\ell!}{(n_1 + r + 1)^{\ell+1}}$$
$$= \mp k!\ell! \sum_{n_1 > n_2 > n_3 > \cdots > n_{k+1} \geq 1} \frac{(\mp 1)^{n_1}}{(n_1 + r)^{\ell+1} \cdot n_2 \cdot n_3 \cdots n_{k+1}}.$$

∎

Lemma 2.2. *For $r=-1$, $k\geq 1$, $\ell\geq 1$*

$$\begin{aligned} I^{\pm}_{k,\ell,-1,1} &= \int_0^1 \frac{\left(-\log\left(1\pm x\right)\right)^k}{1\pm x}\cdot x^{-1}\left(-\log x\right)^{\ell}dx \\ &= k!\ell! \sum_{n_1\geq n_2>n_3>\ldots>n_{k+1}\geq 1} \frac{\left(\mp 1\right)^{n_1}}{n_1^{\ell+1}\cdot n_2\cdot n_3\cdots n_{k+1}} \\ &= k!\ell! \sum_{n_1>n_2>n_3>\cdots>n_{k+1}\geq 1} \frac{\left(\mp 1\right)^{n_1}}{n_1^{\ell+1}\cdot n_2\cdot n_3\cdots n_{k+1}} \\ &+ k!\ell! \sum_{n_1=n_2>n_3>\cdots>n_{k+1}\geq 1} \frac{\left(\mp 1\right)^{n_1}}{n_1^{\ell+1}\cdot n_2\cdot n_3\cdots n_{k+1}} \\ &= k!\ell!\left(\tilde{\zeta}_{k+1}\left(\mp\left(\ell+1\right),\{1\}_k\right)+\tilde{\zeta}_k\left(\mp\left(\ell+2\right),\{1\}_{k-1}\right)\right), \end{aligned}$$

and

$$\begin{aligned} I^{+}_{k,0,-1,1} &= \int_0^1 \frac{\left(-\log\left(1+x\right)\right)^k}{1+x}\cdot x^{-1}dx \\ &= k! \sum_{n_1\geq n_2>n_3>\ldots>n_{k+1}\geq 1} \frac{\left(-1\right)^{n_1}}{n_1\cdot n_2\cdot n_3\cdot\ldots\cdot n_{k+1}} \\ &= k!\left(\tilde{\zeta}_{k+1}\left(-1,\{1\}_k\right)+\tilde{\zeta}_k\left(-2,\{1\}_{k-1}\right)\right) \end{aligned}$$

and

$$\left|I^{-}_{k,\ell,r,1}\right| = \infty,\ \textit{for } r<-1,\ r\in\mathbb{Z} \textit{ or } r=-1,\ k=0 \textit{ or } r=-1,\ \ell=0.$$

Corollary 2.1. (See also [19]). *Suppose that k and ℓ are integers with $k\geq 0$, $\ell\geq 1$. Then:*

$$\begin{aligned} I^{\pm}_{k,\ell,0,1} &= \int_0^1 \frac{\left(-\log\left(1\pm x\right)\right)^k}{1\pm x}\cdot\left(-\log x\right)^{\ell}dx \\ &= \mp k!\ell!\tilde{\zeta}_{k+1}\left(\mp\left(\ell+1\right),\{1\}_k\right); \end{aligned}$$

and

$$\begin{aligned} I^{+}_{k,0,0,1} &:= \int_0^1 \frac{\left(-\log\left(1+x\right)\right)^k}{1+x}dx \\ &= -k!\tilde{\zeta}_{k+1}\left(-1,\{1\}_k\right). \end{aligned}$$

Corollary 2.2. *Suppose that k and ℓ are integers with $k\geq 0$, $\ell\geq 1$. Then:*

$$\begin{aligned} I^{\pm}_{k,\ell,1,1} &:= \int_0^1 \frac{\left(-\log\left(1\pm x\right)\right)^k}{1\pm x}\cdot x\left(-\log x\right)^{\ell}dx \\ &= k!\ell!\left(\tilde{\zeta}_{k+1}\left(\mp\left(\ell+1\right),\{1\}_k\right)\right. \\ &\left.\pm \sum_{n_1=n_2>n_3>\ldots>n_{k+1}\geq 1} \frac{\left(\mp 1\right)^{n_1}}{\left(n_2+1\right)^{\ell+1}\cdot n_2\cdot n_3\cdot\ldots\cdot n_{k+1}}\right), \end{aligned}$$

and

$$I^{+}_{k,0,1,1}=\int_0^1\frac{\left(-\log\left(1+x\right)\right)^k}{1+x}\cdot xdx$$
$$=k!\left(\tilde{\zeta}_{k+1}\left(-1,\{1\}_k\right)+\sum_{n_1=n_2>n_3>\cdots>n_{k+1}\geq 1}\frac{\left(-1\right)^{n_1}}{\left(n_2+1\right)\cdot n_2\cdot n_3\cdots n_{k+1}}\right).$$

In particular,

$$\begin{aligned}
I^{\pm}_{0,\ell,1,1} &= \int_0^1\frac{1}{1\pm x}\cdot x\left(-\log x\right)^\ell dx=\ell!\left(\tilde{\zeta}\left(\mp\left(\ell+1\right)\right)\pm 1\right), \quad \textit{for } \ell\in\mathbb{N}^*\\
I^{+}_{0,0,1,1} &= \int_0^1\frac{1}{1+x}\cdot xdx=\tilde{\zeta}\left(-1\right)+1=1-\log\left(2\right),\\
I^{-}_{1,\ell,1,1} &= -\int_0^1\frac{\log\left(1-x\right)}{1-x}\cdot x\left(-\log x\right)^\ell dx\\
&= -\ell!\left(\zeta_2\left(\ell+1,1\right)-\ell-1-\sum_{i=2}^{\ell+1}\zeta\left(i\right)\right), \ \textit{for } \ell\in\mathbb{N}^*\\
I^{+}_{1,\ell,1,1} &= -\int_0^1\frac{\log\left(1+x\right)}{1+x}\cdot x\left(-\log x\right)^\ell dx\\
&= -\ell!\left(\tilde{\zeta}_2\left(-\ell-1,1\right)+\ell+1+2\tilde{\zeta}\left(-1\right)+\sum_{i=2}^{\ell+1}\tilde{\zeta}\left(-i\right)\right), \ \textit{for } \ell\in\mathbb{N}.
\end{aligned}$$

Proof. We start with the integral $I^{-}_{1,\ell,1,1}$. According to the Corollary 2.2,

$$I^{-}_{1,\ell,1,1}=-\int_0^1\frac{\log\left(1-x\right)}{1-x}\cdot x\left(-\log x\right)^\ell dx$$
$$=-\ell!\left(\zeta_2\left(\ell+1,1\right)-\sum_{n_2=1}^{\infty}\frac{1}{\left(n_2+1\right)^{\ell+1}\cdot n_2}\right).$$

From the relation

$$\frac{1}{\left(n_2+1\right)^{\ell+1}\cdot n_2}=\frac{1}{n_2}-\sum_{i=1}^{\ell+1}\frac{1}{\left(n_2+1\right)^i}$$

we infer

$$\sum_{n_2=1}^{\infty}\frac{1}{\left(n_2+1\right)^{\ell+1}\cdot n_2}=\ell+1-\sum_{i=2}^{\ell+1}\zeta\left(i\right)$$

and thus

$$I^{-}_{1,\ell,1,1}=-\ell!\left(\zeta_2\left(\ell+1,1\right)-\ell-1+\sum_{i=2}^{\ell+1}\zeta\left(i\right)\right).$$

We pass now to the integral $I^{+}_{1,\ell,1,1}$. Using the Corollary 2.2 we have

$$I^{+}_{1,\ell,1,1}=-\int_0^1\frac{\log\left(1+x\right)}{1+x}\cdot x\left(-\log x\right)^\ell dx$$
$$=-\left(\tilde{\zeta}_2\left(-\ell-1,1\right)+\sum_{n_1=1}^{\infty}\frac{\left(-1\right)^{n_1}}{\left(n_1+1\right)^{\ell+1}\cdot n_1}\right).$$

The relation

$$\frac{(-1)^{n_1}}{(n_1+1)^{\ell+1}\cdot n_1}=(-1)^{n_1}\left(\frac{1}{n_1}-\sum_{i=1}^{\ell+1}\frac{1}{(n_1+1)^i}\right),$$

leads us to

$$\sum_{n_1=1}^{\infty}\frac{(-1)^{n_1}}{(n_1+1)^{\ell+1}\cdot n_1}=\ell+1+\sum_{i=2}^{\ell+1}\tilde{\zeta}(-i),$$

whence

$$I_{1,\ell,1,1}^{+}=-\ell!\left(\tilde{\zeta}_2(-\ell-1,1)+\ell+1+2\tilde{\zeta}(-1)+\sum_{i=2}^{\ell+1}\tilde{\zeta}(-i)\right).$$

∎

Corollary 2.3. *Let ℓ and r be integers such that $\ell\geq 1$, $r\geq 0$. Then:*

$$I_{0,\ell,r,1}^{\pm}=\int_0^1\frac{x^r}{1\pm x}\cdot(-\log x)^{\ell}\,dx=\ell!\,(\mp 1)^{r+1}\left(\tilde{\zeta}(\mp(\ell+1))-\sum_{n_1=1}^{r}\frac{(\mp 1)^{n_1}}{n_1^{\ell+1}}\right). \quad (2.1)$$

and

$$I_{0,0,r,1}^{+}=\int_0^1\frac{x^r}{1+x}dx=(-1)^{r+1}\left(\tilde{\zeta}(-1)-\sum_{n_1=1}^{r}\frac{(-1)^{n_1}}{n_1}\right).$$

Theorem 2.1. *Let k, r and ℓ be integers such that $k,r\geq 0$, $\ell\geq 1$. Then:*

$$\begin{aligned}I_{k,l,r,0}^{\pm}&=\int_0^1(-\log(1\pm x))^k\cdot x^r(-\log x)^{\ell}\,dx\\&=k!\ell!\sum_{n_1>n_2>n_3>\cdots>n_k\geq 1}\frac{(-1)^{n_1}}{(n_1+r+1)^{\ell+1}\cdot n_1\cdot n_2\cdot n_3\cdots n_k},\end{aligned}$$

and

$$\begin{aligned}I_{k,0,r,0}^{+}&=\int_0^1(-\log(1+x))^k\cdot x^r dx\\&=k!\sum_{n_1>n_2>n_3>\cdots>n_k\geq 1}\frac{(-1)^{n_1}}{(n_1+r+1)\cdot n_1\cdot n_2\cdot n_3\cdots n_k}.\end{aligned}$$

In particular,

$$\begin{aligned}I_{1,0,r,0}^{+}=-\int_0^1\log(1+x)\cdot x^r dx&=\sum_{n_1=1}^{\infty}\frac{(-1)^{n_1}}{(n_1+r+1)\cdot n_1}\\&=\begin{cases}\frac{2}{r+1}\log(2)+S_r & \text{if} \quad r=2p\\ -S_r & \text{if} \quad r=2p+1\end{cases},\end{aligned}$$

where $S_r=\frac{1}{r+1}\sum_{i=1}^{r+1}\frac{(-1)^i}{i}$ and $p\in\mathbb{Z}$.

Proof. The basic remark is

$$(-\log(1\pm x))^k=k!\sum_{n_1>n_2>n_3>\cdots>n_k\geq 1}\frac{(\mp x)^{n_1}}{n_1\cdot n_2\cdot n_3\cdots n_k},$$

from where we infer that

$$\begin{aligned}I^{\pm}_{k,\ell,r,0} &= k! \sum_{n_1>n_2>n_3>\cdots>n_k\geq 1} \frac{(\mp 1)^{n_1}}{n_1\cdot n_2\cdot n_3\cdots n_k}\int_0^1 x^{n_1+r}\left(-\log x\right)^{\ell} dx \\ &= k! \sum_{n_1>n_2>n_3>\cdots>n_k\geq 1} \frac{(\mp 1)^{n_1}}{n_1\cdot n_2\cdot n_3\cdots n_k}\cdot\frac{\ell!}{(n_1+r+1)^{\ell+1}} \\ &= k!\ell! \sum_{n_1>n_2>n_3>\cdots>n_k\geq 1} \frac{(\mp 1)^{n_1}}{(n_1+r+1)^{\ell+1}\cdot n_1\cdot n_2\cdot n_3\cdots n_k}.\end{aligned}$$

∎

Remark 2.1. *For $r=-1$, $k\geq 1$, $\ell\geq 0$ we have*

$$\begin{aligned}I^{\pm}_{k,\ell,-1,0} &= \int_0^1 \frac{\left(-\log(1\pm x)\right)^k\left(-\log x\right)^{\ell}}{x}dx \\ &= k!\ell! \sum_{n_1>n_2>n_3>\cdots>n_k\geq 1} \frac{(\mp 1)^{n_1}}{n_1^{\ell+2}\cdot n_2\cdot n_3\cdots n_k} \\ &= k!\ell!\tilde{\zeta}_k\left(\mp(\ell+2),\{1\}_{k-1}\right),\end{aligned}$$

and

$$I^{-}_{k,\ell,r,0}=\infty,\ \textit{for } r<-1, r\in\mathbb{Z}.$$

Theorem 2.2. *Let k,r,ℓ and m be integers with $k,r\geq 0$ and m, $\ell\geq 1$. Then:*

$$I^{-}_{k,\ell,r,m}=\int_0^1 \frac{\left(-\log(1-x)\right)^k}{(1-x)^m}\cdot x^r\left(-\log x\right)^{\ell}dx=\sum_{i=0}^{r}(-1)^i\binom{r}{i}I^{-}_{\ell,k,i-m,0}.$$

Proof. Performing the variable $t=1-x$, we get

$$\begin{aligned}I^{-}_{k,\ell,r,m} &= \int_0^1 \frac{(-\log t)^k}{t^m}\cdot(1-t)^r\left(-\log(1-t)\right)^{\ell}dx \\ &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}\int_0^1(-\log t)^k\cdot t^{i-m}\left(-\log(1-t)\right)^{\ell}dx \\ &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}I^{-}_{\ell,k,i-m,0}.\end{aligned}$$

∎

Corollary 2.4. *If $\ell\geq 1$ is an integer, then*

$$\begin{aligned}I^{-}_{1,\ell,0,0} &= -\int_0^1\log(1-x)\cdot(-\log x)^{\ell}dx=-\ell!\sum_{n_1=1}^{\infty}\frac{1}{(n_1+1)^{\ell+1}\cdot n_1} \\ &= -\ell!\left(\ell+1-\sum_{i=2}^{\ell+1}\zeta(i)\right),\end{aligned}$$

while for $\ell \geq 0$:

$$I_{1,\ell,0,0}^{+} = -\int_{0}^{1} \log(1+x) \cdot (-\log x)^{\ell} \, dx = -\ell! \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1}}{(n_1+1)^{\ell+1} \cdot n_1}$$

$$= \ell! \left(\ell + 1 + \tilde{\zeta}(-1) + \sum_{i=1}^{\ell+1} \tilde{\zeta}(-i) \right).$$

Theorem 2.3. *Let* ℓ, m, r, k *be integers such that* $\ell, m, r \geq 1, k \geq 2$. *Then:*

$$J_{\ell,r,m,k}^{\pm} = \int_{0}^{1} \frac{x^{kr+k-1}}{(1 \pm x^{k})^{m}} \cdot (-\log x)^{\ell} \, dx = \frac{1}{k^{\ell+1}} I_{0,\ell,r,m.}^{\pm}$$

Proof. This results immediately via the change of variable

$$x^{k} = t.$$

∎

Corollary 2.5. *If* $\ell \geq 1$ *is an integer, then*

$$J_{\ell,0,1,2}^{\pm} = \int_{0}^{1} \frac{x}{1 \pm x^{2}} \cdot (-\log x)^{\ell} \, dx = \frac{1}{2^{\ell+1}} I_{0,\ell,0,1}^{\pm} = \mp \frac{1}{2^{\ell+1}} \ell! \tilde{\zeta}(\mp(\ell+1)).$$

Theorem 2.4. *Let* r *,*ℓ*,* m_1 *and* m_2 *be integers with* $\ell, m_1, m_2 \geq 0$. *Then*

$$I_{\ell,r,m_1,m_2} = \int_{0}^{1} \frac{x^{r}(-\log x)^{\ell}}{(1-x)^{m_1}(1+x)^{m_2}} dx = \sum_{i=1}^{m_1} A_i I_{0,\ell,r,i}^{-} + \sum_{j=1}^{m_2} B_j I_{0,\ell,r,j}^{+} \ ,$$

where $A_i, B_j \in \mathbb{R}$, $i \in \{1, 2, ..., m_1\}$, $j \in \{1, 2, ..., m_2\}$.

Proof. Because of the decomposition

$$\frac{1}{(1-x)^{m_1}(1+x)^{m_2}} = \sum_{i=1}^{m_1} \frac{A_i}{(1-x)^{i}} + \sum_{j=1}^{m_2} \frac{B_j}{(1+x)^{j}},$$

(where $A_i, B_j \in \mathbb{R}$ are suitable constants), we have

$$I_{\ell,r,m_1,m_2} = \int_{0}^{1} \left[\sum_{i=1}^{m_1} \frac{A_i}{(1-x)^{i}} + \sum_{j=1}^{m_2} \frac{B_j}{(1+x)^{j}} \right] \cdot x^{r}(-\log x)^{\ell} \, dx$$

$$= \sum_{i=1}^{m_1} A_i \int_{0}^{1} \frac{1}{(1-x)^{i}} \cdot x^{r}(-\log x)^{\ell} \, dx + \sum_{j=1}^{m_2} B_j \int_{0}^{1} \frac{1}{(1+x)^{j}} \cdot x^{r}(-\log x)^{\ell} \, dx$$

$$= \sum_{i=1}^{m_1} A_i I_{0,\ell,r,i}^{-} + \sum_{j=1}^{m_2} B_j I_{0,\ell,r,j}^{+}.$$

∎

Corollary 2.6. *If* $\ell \geq 1$ *an integer, then*

$$I_{\ell,0,1,1} = \int_{0}^{1} \frac{1}{1-x^{2}} \cdot (-\log x)^{\ell} \, dx = \frac{1}{2} \left(I_{0,\ell,0,1}^{-} + I_{0,\ell,0,1}^{+} \right)$$

$$= \frac{1}{2} \ell! \left(\zeta(\ell+1) - \tilde{\zeta}(-\ell-1) \right)$$

$$= \ell! \left(1 - \frac{1}{2^{\ell+1}} \right) \zeta(\ell+1)$$

and

$$I_{\ell,1,1,1} = \int_0^1 \frac{x}{1-x^2} \cdot (-\log x)^{\ell}\, dx = \frac{1}{2}\left(I^-_{0,\ell,1,1} + I^+_{0,\ell,1,1}\right)$$
$$= \frac{1}{2}\ell!\left(\zeta(\ell+1) + \tilde{\zeta}(-\ell-1)\right) = \frac{\ell!}{2^{\ell+1}}\zeta(\ell+1).$$

3. Applications

We illustrate here how the results above work.

1. Our first example is

$$-I^-_{2,3,-1,1} = \int_0^1 \frac{\log^2(1-x)}{x(1-x)} \log^3 x dx = -\frac{1}{36}\pi^6 + 18\zeta^2(3). \tag{3.1}$$

According to the remark 2.2,

$$-I^-_{2,3,-1,1} = \int_0^1 \frac{\log^2(1-x)}{x(1-x)} \log^3 x dx = -2!3!\left(\zeta_3(4,1,1) + \zeta_2(5,1)\right).$$

Because

$$\zeta_3(4,1,1) = \frac{23}{16}\zeta(6) - \zeta^2(3)$$

and

$$\zeta_2(5,1) = \frac{5}{2}\zeta(6) - \zeta(2)\zeta(4) - \frac{1}{2}\zeta^2(3)$$

we obtain

$$-I^-_{2,3,-1,1} = -12\left(\frac{63}{16}\zeta(6) - \zeta(2)\zeta(4) - \frac{3}{2}\zeta^2(3)\right).$$

But

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945}$$

so that

$$-I^-_{2,3,-1,1} = -\frac{1}{36}\pi^6 + 18\zeta^2(3). \quad \blacksquare$$

2. A second example:

$$I^+_{2,0,-1,1} = \int_0^1 \frac{\log^2(1+x)}{x(1+x)} dx = \frac{1}{4}\zeta(3) - \frac{1}{3}\log^3(2). \tag{3.2}$$

From the remark 2.2, it results:

$$I^+_{2,0,-1,1} = 2!\left(\tilde{\zeta}_3(-1,1,1) + \tilde{\zeta}_2(-2,1)\right).$$

Because

$$\tilde{\zeta}_3(-1,1,1) = -\frac{1}{6}\log^3(2)$$

and

$$\tilde{\zeta}_2(-2,1) = \frac{1}{8}\zeta(3)$$

we conclude that

$$I^+_{2,0,-1,1} = \frac{1}{4}\zeta(3) - \frac{1}{3}\log^3(2). \quad \blacksquare$$

3. The third example:

$$I^-_{2,2,0,1} = \int_0^1 \frac{\log^2(1-x)}{1-x} \cdot \log^2 x dx = 8\zeta(5) - \frac{2}{3}\pi^2\zeta(3) \tag{3.3}$$

For the demonstration we use corollary 2.1:

$$I^{-}_{2,2,0,1}=\int_0^1 \frac{\log^2(1-x)}{1-x}\cdot\log^2 x dx = 2!2!\zeta_3(3,1,1).$$

Because

$$\zeta_3(3,1,1)=\zeta_2(4,1)$$

and

$$\zeta_2(4,1)=2\zeta(5)-\zeta(2)\zeta(3)=2\zeta(5)-\frac{\pi^2}{6}\zeta(3)$$

we conclude that

$$I^{-}_{2,2,0,1}=8\zeta(5)-\frac{2}{3}\pi^2\zeta(3). \quad \blacksquare$$

4. The fourth example:

$$-I^{+}_{3,0,0,1}=\int_0^1 \frac{\log^3(1+x)}{1+x}dx=\frac{1}{4}\log^4(2). \tag{3.4}$$

In fact, by Corollary 2.1 we infer that

$$\begin{aligned} I^{+}_{3,0,0,1} &= -3!\tilde{\zeta}_4(-1,1,1,1)\\ &= (-3!)\times\frac{1}{4!}\log^4(2)\\ &= \frac{1}{4}\log^4(2). \quad \blacksquare \end{aligned}$$

5. The fifth example:

$$I^{-}_{1,3,1,1}=\int_0^1 \frac{\log(1-x)}{1-x}\cdot x\cdot\log^3 x dx = 12\zeta(5)-\pi^2\zeta(3)-24+\pi^2+6\zeta(3)+\frac{\pi^4}{15} \tag{3.5}$$

By Corollary 2.2:

$$\begin{aligned} I^{-}_{1,3,1,1} &= \int_0^1 \frac{\log(1-x)}{1-x}\cdot x\cdot\log^3 x dx = (-1)^4\, 3!\left(\zeta_2(4,1)-4+\sum_{i=2}^{4}\zeta(i)\right)\\ &= 6\zeta_2(4,1)-24+6\zeta(2)+6\zeta(3)+6\zeta(4). \end{aligned}$$

Taking into account that

$$\zeta_2(4,1)=2\zeta(5)-\frac{\pi^2}{6}\zeta(3),$$

and

$$\zeta(2)=\frac{\pi^2}{6} \text{ and } \zeta(4)=\frac{\pi^4}{90},$$

we conclude that

$$I^{-}_{1,3,1,1}=12\zeta(5)-\pi^2\zeta(3)-24+\pi^2+6\zeta(3)+\frac{\pi^4}{15}.$$

6. The sixth example:

$$I^{+}_{1,1,1,1}=\int_0^1 \frac{\log(1+x)}{1+x}\cdot x\cdot\log x dx = 2-2\log(2)-\frac{\pi^2}{12}+\frac{1}{8}\zeta(3). \tag{3.6}$$

By Corollary 2.2,

$$I^{+}_{1,1,1,1}=\int_0^1 \frac{\log(1+x)}{1+x}\cdot x\cdot\log x dx = \tilde{\zeta}_2(-2,1)+2+2\tilde{\zeta}(-1)+\tilde{\zeta}(-2),$$

and it remains to take into account that

$$\begin{aligned}\tilde{\zeta}_2(-2,1) &= \frac{1}{8}\zeta(3);\\ \tilde{\zeta}(-1) &= -\log(2);\\ \tilde{\zeta}(-2) &= -\frac{\pi^2}{12}. \quad \blacksquare\end{aligned}$$

7. The seventh example is:

$$I^{-}_{1,3,0,0} = \int_0^1 \log(1-x)\cdot\log^3 x dx = 24 - \pi^2 - 6\zeta(3) - \frac{\pi^4}{15}. \tag{3.7}$$

In fact, according to Corollary 2.4:

$$\begin{aligned}I^{-}_{1,3,0,0} &= \int_0^1 \log(1-x)\cdot\log^3 x dx = 3!\left(4 - \sum_{i=2}^{4}\zeta(i)\right)\\ &= 6(4-\zeta(2)-\zeta(3)-\zeta(4)) = 24 - \pi^2 - 6\zeta(3) - \frac{\pi^4}{15}. \quad \blacksquare\end{aligned}$$

8. The eighth example:

$$J^{-}_{2,0,1,2} = I_{2,1,1,1} = \int_0^1 \frac{x}{1-x^2}\log^2 x dx = \frac{1}{4}\zeta(3). \tag{3.8}$$

See Corollary 2.5 or 2.6.

4. The use of computer

The *polylogarithm multiple function* in one single complex variable is defined by the formula (see [17]):

$$\underset{(s_1,s_2,\ldots,s_\ell)}{\mathrm{Li}}(z_1) = \sum_{n_1>n_2>n_3>\cdots>n_\ell\geq 1} \frac{z_1^{n_1}}{n_1^{s_1}\cdot n_2^{s_2}\cdots n_\ell^{s_\ell}},\ \ (s_i\geq 1, 1\leq i\leq \ell) \tag{4.1}$$

defined for $|z_1|<1$, or thus for $|z_1|=1$ if $s_1\geq 2$.

We call *polylogarithm multiple function* Li *multivariable* the complex function:

$$\underset{(s_1,s_2,\ldots,s_\ell)}{\mathrm{Li}}(z_1,\ldots,z_\ell) = \sum_{n_1>n_2>n_3>\cdots>n_\ell\geq 1} \frac{z_1^{n_1}\cdot z_2^{n_2}\cdots z_\ell^{n_\ell}}{n_1^{s_1}\cdot n_2^{s_2}\cdots n_\ell^{s_\ell}},\ \ (s_i\geq 1, 1\leq i\leq \ell)$$

defined for $|z_1|<1$ and $|z_i|\leq 1$ $(2\leq i\leq \ell)$, so thus for $|z_i|\leq 1$ $(1\leq i\leq \ell)$ if $s_1\geq 2$.

Lemma 4.1. (see [17])

$$\begin{aligned}&\underset{(s_1,s_2,\ldots,s_\ell)}{\mathrm{Li}}(1) = \zeta_\ell(s_1,s_2,\ldots,s_\ell),\ \textit{for } s_1,s_2,\ldots,s_\ell\in\mathbb{N}^*;\\ &\underset{(s_1,s_2,\ldots,s_\ell)}{\mathrm{Li}}(0) = 0;\\ &\frac{d}{dx}\underset{(s_1,s_2,\ldots,s_\ell)}{\mathrm{Li}}(x) = \frac{1}{x}\underset{(s_1-1,s_2,\ldots,s_\ell)}{\mathrm{Li}}(x),\ \textit{for } s_1>1;\\ &\frac{d}{dx}\underset{(1,s_2,\ldots,s_\ell)}{\mathrm{Li}}(x) = \frac{1}{1-x}\underset{(s_2,\ldots,s_\ell)}{\mathrm{Li}}(x),\ \textit{for } s_1=1.\end{aligned}$$

This lemma helps us to compute with high accuracy the values in positive round variables of the multiple ζ_ℓ function with the help of a computer.

For example, by calculating $\zeta_4(6,1,1,1)$, it can by solved in *Maple* the differential system

```
> dsys6:=
> diff(y6111(x),x)=y5111(x)/x,
> diff(y5111(x),x)=y4111(x)/x,
> diff(y4111(x),x)=y3111(x)/x,
> diff(y3111(x),x)=y2111(x)/x,
> diff(y2111(x),x)=y1111(x)/x,
> diff(y1111(x),x)=y111(x)/(1-x),
> diff(y111(x),x)=y11(x)/(1-x),
> diff(y11(x),x)=y1(x)/(1-x),
> diff(y1(x),x)=1/(1-x);
>init6:=y6111(0)=0,y5111(0)=0,y4111(0)=0,y3111(0)=0,
>         y2111(0)=0,y1111(0)=0,y111(0)=0,y11(0)=0,y1(0)=0;
```

and we obtain

$$\zeta_4(6,1,1,1) \approx .00010609000327002678114903988058412790975005495541176.$$

Using the program, made in C, that is found at the web address:

`http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi`,

we can compute the sum:

$$z(s_1, s_2, ..., s_k) = \sum_{n_1 > n_2 > n_3 > \cdots > n_\ell \geq 1} \frac{\sigma_1^{n_1} \cdot \sigma_2^{n_2} \cdots \sigma_\ell^{n_\ell}}{n_1^{|s_1|} \cdot n_2^{|s_2|} \cdots n_\ell^{|s_\ell|}},$$

for the non-null integers $s_1, s_2, ..., s_k$ and $\sigma_j = signum(s_j)$, that are the values of $\tilde{\zeta}_k$, and

$$zp(p, s_1, s_2, ..., s_k) = \sum_{n_1 > n_2 > n_3 > \cdots > n_\ell \geq 1} \frac{p^{-n_1}}{n_1^{s_1} \cdot n_2^{s_2} \cdots n_\ell^{s_\ell}}$$

for the positive integers $s_1, s_2, ..., s_k$ and real number $p \geq 1$, that are the values $\mathrm{Li}_{(s_1, s_2, ..., s_\ell)}(1/p)$. For $p = 1$ there can be calculated the positive values of ζ_k.

Using this program we obtain:

```
z(6,1,1,1)= .10609022891021752051405595491455175898810503330 08e-3
z(-6,1,1,1)= .23721805349405091926870095513763279997383803809 23e-4
```

With these results and an algorithm to find some constants calculated with a great number of decimals (that is based on PSLQ, in trained by Ferguson in 1991), or even the algorithm PSLQ, we can use the computer in order to find various formulas with the integrals we study in this article.

For example, in the event we use Maple we have

$$L_1 = \int_0^1 \frac{\log^2(1-x)}{x(1-x)} \log^3 x dx$$
$$= -.69632100528637370705976423848588296811 55...$$

Applying the algorithm PSLQ to the vector $\begin{bmatrix}\pi^6 & \zeta^2(3) & -L_1\end{bmatrix}$ we find

$$\begin{bmatrix}1 & -648 & -36\end{bmatrix},$$

that is,

$$\pi^6 - 648\zeta^2(3) + 36L_1 = 0. \tag{4.2}$$

It results

$$L_1 = -\frac{1}{36}\pi^6 + 18\zeta^2(3),$$

rigorously demonstrated at (3.1).

Computing with 40 decimals,

$$L_2 = \int_0^1 \frac{\log^2(1-x)}{1-x} \cdot \log^2 x dx$$
$$= .3862046399577749378625821257157710561280\ldots .$$

Using the *tryconsts* procedure, which uses the PSLQ algorithm (see [12]),

$$tryconsts(.3862046399577749378625821257157710561280);$$

we find

$$L_2 = 8\zeta(5) - \frac{2}{3}\pi^2\zeta(3),$$

which is formula (3.3) above.

5. Some final comments

The theorems and corollaries obtained in this article offer the possibility to compute integrals of the type (1.1) and (1.2) (a few of them being mentioned in the book of I. S. Gradzhteyn and I. M. Ryzhyk [11]. Sergey Zlobin [19] used the integrals

$$I^{-}_{k,\ell,0,1} = \int_0^1 \frac{(-\log(1-x))^k}{1-x} \cdot (-\log x)^\ell dx$$

in order to compute a type of double logarithmic integrals.

In solving Problem 11275 published in February 2007 in *American Mathematical Monthly,* (see [3]), D. H. Bailey and J. M. Borwein reached at the following integral

$$I = \int_1^\infty \frac{\log u}{u^2-1} du \tag{5.1}$$

By changing the variable $u = 1/v$ we obtain

$$I = \int_0^1 \frac{\log v}{v^2-1} dv = \frac{1}{2}\left(\int_0^1 \frac{\log v}{v-1} dv - \int_0^1 \frac{\log v}{v+1} dv\right).$$

We already noticed that

$$-I^{-}_{0,1,0,1} = \int_0^1 \frac{\log v}{v-1} dv = \frac{\pi^2}{6} \tag{5.2}$$

and

$$-I^{+}_{0,1,0,1} = \int_0^1 \frac{\log v}{v+1} dv = -\frac{\pi^2}{12}. \tag{5.3}$$

Consequently

$$I = \int_0^1 \frac{\log v}{v^2-1} dv = \frac{\pi^2}{8}. \tag{5.4}$$

The integral I is actually an integral of type I_{ℓ,r,m_1,m_2} (precisely $I = -I_{1,0,1,1}$). It may be found in I. S. Gradzhteyn and I. M. Ryzhyk book (see [11] , formulae 4.231.13).

Also in this book one can find the formula

$$\int_0^1 \frac{1-x}{1+x} \cdot \log x dx = 1 - \frac{\pi^2}{6}, \quad \text{([11], formula 4.231.4)} \tag{5.5}$$

Its proof is simple since

$$\int_0^1 \frac{1-x}{1+x} \cdot \log x dx = \int_0^1 \frac{1}{1+x} \log x dx - \int_0^1 \frac{x}{1+x} \log x dx.$$

By Corollary 2.1,

$$-I^{+}_{0,1,0,1} = \int_0^1 \frac{1}{1+x} \log x dx = \tilde{\zeta}\left(-2\right) = -\frac{\pi^2}{12}.$$

while Corollary 2.2 leads us to

$$I^{+}_{0,1,1,1} = -\int_0^1 \frac{x}{1+x} \log x dx = \tilde{\zeta}\left(-2\right) + 1 = -\frac{\pi^2}{12} + 1.$$

Consequently

$$\int_0^1 \frac{1-x}{1+x} \cdot \log x dx = -I^{+}_{0,1,0,1} + I^{+}_{0,1,1,1} = 1 - \frac{\pi^2}{6}.$$

Corollary 2.1 gives us the possibility to compute certain values of the functions ζ and $\tilde{\zeta}$ by using a definite integral:

$$\begin{aligned}\zeta_{k+1}\left(\ell+1, \{1\}_k\right) &= \sum_{n_1>n_2>n_3>\cdots>n_\ell\geq 1} \frac{1}{n_1^{\ell+1}\cdot n_2\cdots n_\ell} \\ &= \frac{(-1)^{k+\ell}}{\Gamma\left(k+1\right)\Gamma(\ell+1)} \int_0^1 \frac{\log^k\left(1-x\right)}{1-x} \cdot \log^\ell x dx,\end{aligned}$$

where ℓ, k is integer $k \geq 0, \ell \geq 1$,

$$\begin{aligned}\tilde{\zeta}_{k+1}\left(-\ell-1, \{1\}_k\right) &= \sum_{n_1>n_2>n_3>\cdots>n_\ell\geq 1} \frac{(-1)^{n_1}}{n_1^{\ell+1}\cdot n_2\cdots n_\ell} \\ &= \frac{(-1)^{k+\ell+1}}{\Gamma\left(k+1\right)\Gamma(\ell+1)} \int_0^1 \frac{\log^k\left(1+x\right)}{1+x} \cdot \log^\ell x dx,\end{aligned}$$

where ℓ, k is integer k,$\ell \geq 0$.

For example, with the help of Maple

$$\begin{aligned}\zeta_4\left(6,1,1,1\right) &= \frac{(-1)^{3+5}}{\Gamma\left(4\right)\Gamma(6)} \int_0^1 \frac{\log^3\left(1-x\right)}{1-x} \cdot \log^5 x dx \\ &\approx .0001060902289102175205140559549145517589881...\end{aligned}$$

and

$$\begin{aligned}\tilde{\zeta}_4\left(-6,1,1,1\right) &= \frac{(-1)^{3+5+1}}{\Gamma\left(4\right)\Gamma(6)} \int_0^1 \frac{\log^3\left(1+x\right)}{1+x} \cdot \log^5 x dx \\ &\approx .0000237218053494050919268700955137632799973 83...\end{aligned}$$

We observe that these values coincide with the values of z(6,1,1,1) and z(-6,1,1,1) calculated using the computer available at the web address

http://oldweb.cecm.sfu.ca/cgi-bin/EZFace/zetaform.cgi .

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(Radu-Octavian Vîlceanu) Department of Mathematics, University of Craiova,
Al.I. Cuza Street, No. 13, Craiova RO-200585, Romania, Tel. & Fax: 40-251412673
E-mail address: radu.vilceanu@yahoo.com