On the lattice of deductive systems of a residuated lattice

DANA PICIU, ANTOANETA JEFLEA AND RALUCA CREȚAN

Abstract. In any residuated lattice $A$ the set $Ds(A)$ of all deductive systems of $A$ forms a pseudo-complemented distributive lattice and we denote by $D^\circ$ the pseudocomplement of $D$ in this lattice (it is proved that $D^\circ = \{ a \in A : a \lor x = 1, \text{ for every } x \in D \}$). In this paper we give a characterization for regular deductive systems and we study the lattice $Ds^p(A)$ of deductive systems of the form $[a]^\circ$. If $A$ is a hyperarchimedean residuated lattice, then $Ds^p(A)$ is a Boolean algebra. Also, for $X \subseteq A$ we denote by $X^* = \{ a \in A : a \to x = x, \text{ for any } x \in X \}$ which is a deductive system and we show that the set $R^*_s(Ds(A)) = \{ D \in Ds(A) : D = D^{**} \}$ does a Boolean algebra.

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1. Introduction

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([13]), Dilworth ([7]), Ward and Dilworth ([20]), Ward ([19]), Balbes and Dwinger ([1]) and Pavelka ([16]).

In [10], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: BCK- latices in [9], full BCK- algebras in [13], FLew- algebras in [14], and integral, residuated, commutative l-monoids in [3].

Residuated lattices have been studied extensively and include important classes of algebras such as BL-algebras, introduced by Hájek as the algebraic counterpart of his Basic Logic, and MV-algebras, the algebraic setting for Łukasiewicz propositional logic.

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [2], [4], [7], [12], [15], [19], [20]).

In order to simplify the notation a residuated lattice $(A, \land, \lor, \circ, \to, 0, 1)$ will be referred by its support set $A$.

By $B(A)$ we denote the Boolean algebra of all complemented elements in the lattice $L(A) = (A, \land, \lor, 0, 1)$.

In any residuated lattice $A$ the set $Ds(A)$ of all deductive systems of $A$ forms a pseudo-complemented distributive lattice and we denote by $D^\circ$ the pseudocomplement of $D$ in this lattice (it is proved that $D^\circ = \{ a \in A : a \lor x = 1, \text{ for every } x \in D \}$). In this paper we give a characterization for regular deductive systems denoted by $R_s(Ds(A)) = \{ D \in Ds(A) : D = D^{**} \}$. Also, for $X \subseteq A$ we denote by $X^* = \{ a \in A : a \to x = x, \text{ for any } x \in X \}$ which is a deductive system and we show that the set $R^*_s(Ds(A)) = \{ D \in Ds(A) : D = D^{**} \}$ does a Boolean algebra. We prove that $R_s(Ds(A)) \subseteq R^*_s(Ds(A))$ and $D \in R^*_s(Ds(A))$ iff $D = |e|$, with $e \in B(A)$.

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Finally, we study the lattice $Ds^\circ_p(A)$ of deductive systems of the form $[a]^n$ with $a \in A$.

If $A$ is a hyperarchimedean residuated lattice, then $Ds^\circ_p(A)$ is a Boolean algebra.

## 2. Preliminaries

**Definition 2.1.** A residuated lattice [2], [18]) is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ equipped with an order $\leq$ satisfying the following:

1. $(LR_1) \ (A, \wedge, \vee, 0, 1)$ is a bounded lattice;
2. $(LR_2) \ (A, \odot, 1)$ is a commutative ordered monoid;
3. $(LR_3) \ \odot$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$.

The relations between the pair of operations $\odot$ and $\rightarrow$ expressed by $(LR_3)$, is a particular case of the law of residuation [2]). Lukasiewicz structure, Gödel structure, Products structure are residuated lattices (see [18]).

**Example 2.1.** If $(A, \vee, \wedge, 0, 1)$ is a Boolean algebra and we define for every $x, y \in A, x \odot y = x \wedge y, x \rightarrow y = x' \vee y$, then $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ become a residuated lattice.

**Remark 2.1.** [18] A residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an MV-algebra if it satisfies the additional condition: $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for any $x, y \in A$.

We give an example of finite residuated lattice:

**Example 2.2.** [11] Let $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but $a, b$ are incomparable. $A$ become a residuated lattice relative to the following operations:

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We refer the reader to [4], [12], [18] for basic results in the theory of residuated lattices. In the following, we only present the material needed in the reminder of the paper.

In what follows by $A$ we denote a residuated lattice; for $x \in A$ and a natural number $n$, we define $x^* = x \rightarrow 0, (x^*)^* = x^{**}, x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

**Theorem 2.1.** [4], [12], [18] Let $x, x_1, x_2, y, y_1, y_2, z \in A$. Then we have the following rules of calculus:

1. $(c_1) \ x \rightarrow x = x, x \rightarrow x = 1, y \leq x \rightarrow y, x \rightarrow 1 = 0, x \rightarrow 0 = 1$;
2. $(c_2) \ x \odot 0 = 0, x \odot y \leq x, y, x \odot 0 = 0$ and $(x \vee y = 1$ implies $x \odot y = x \wedge y)$;
3. $(c_3) \ (x \leq y$ iff $x \rightarrow y = 1)$ and $(x \rightarrow y = y \rightarrow x = 1$ iff $x = y)$;
4. $(c_4) \ x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow z = y \rightarrow (x \rightarrow z)$;
5. $(c_5) \ x \rightarrow y \leq y \rightarrow x$;
6. $(c_6) \ x \odot x^* = 0$ and $x \odot y = 0$ iff $x \leq y^*$;
7. $(c_7) \ x \leq x^{**}, x^{**} \leq x^* \rightarrow x, 1^* = 0, 0^* = 1$;
8. $(c_8) \ x \rightarrow y \leq y^* \rightarrow x^*, x^{**} \leq x^*, (x \odot y)^* = x \rightarrow y^* = y \rightarrow x^* = x^{**} \rightarrow y^*$;
9. $(c_9) \ x \odot (y_1 \vee y_2) = (x \odot y_1) \vee (x \odot y_2), (y_1 \vee y_2) \rightarrow x = (y_1 \rightarrow x) \wedge (y_2 \rightarrow x)$ and $x \rightarrow (y_1 \vee y_2) \geq (x \rightarrow y_1) \vee (x \rightarrow y_2)$;
10. $(c_{10}) \ x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

**Corollary 2.1.** [12] Let $a_1, \ldots, a_n \in A$. 

(c_{11}) If \( a_1 \lor \ldots \lor a_n = 1 \), then \( a_1^k \lor \ldots \lor a_n^k = 1 \), for every natural number \( k \).

**Proposition 2.1.** If \( A \) is a residuated lattice and \( a, b, x \in A \), then
\[
(c_{12}): x \lor (a \rightarrow b) \leq (x \lor a) \rightarrow (x \lor b).
\]

**Proof.** We have
\[
(x \lor a) \rightarrow (x \lor b) \equiv (x \lor b) \land (a \rightarrow (x \lor b)) = 1 \land (a \rightarrow (x \lor b)) = a \rightarrow (x \lor b) \geq x \lor (a \rightarrow b) \].

**Proposition 2.2.** ([4]) For \( e \in A \) the following are equivalent:
(i) \( e \in B(A) \);
(ii) \( e \lor e^* = 1 \).

**Lemma 2.1.** ([4], [12]) If \( e \in B(A) \), then
\[
(c_{13}) e \land x = e \land x, \text{ for every } x \in A;
\]
\[
(c_{14}) e \land (x \lor y) = (e \land x) \lor (e \land y), \text{ for every } x, y \in A.
\]

3. The regular deductive systems of a residuated lattice

**Definition 3.1.** ([12], [18]) A nonempty subset \( D \subseteq A \) is called a deductive system of \( A \) if the following conditions are satisfied:
(\( Ds_1 \)) \( 1 \in D \);
(\( Ds_2 \)) If \( x, x \rightarrow y \in D \), then \( y \in D \).

**Remark 3.1.** ([12], [18]) A nonempty subset \( D \subseteq A \) is a deductive system of \( A \) if for all \( x, y \in A \):
(\( Ds'_1 \)) If \( x, y \in D \), then \( x \lor y \in D \);
(\( Ds'_2 \)) If \( x \in D, y \in A, x \leq y \), then \( y \in D \).

Every deductive system of \( A \) is a filter for \( L(A) \), but a filter of \( L(A) \) is not, in general, a deductive system of \( A \) (see [18]).

We denote by \( Ds(A) \) the set of all deductive systems of \( A \).

For a nonempty subset \( S \subseteq A \), the smallest deductive system of \( A \) which contains \( S \), i.e. \( S \cap \{ D \in Ds(A) : S \subseteq D \} \), is said to be the deductive system of \( A \) generated by \( S \) and will be denoted by \( [S] \).

If \( S = \{ a \} \), with \( a \in A \), we denote by \( [a] \) the deductive system generated by \( \{ a \} \) (\( [a] \) is called principal).

For \( D \in Ds(A) \) and \( a \in A \), we denote by \( D(a) = [D \cup \{ a \}] \) (clearly, if \( a \in D \), then \( D(a) = D \)).

**Proposition 3.1.** ([12], [18]) Let \( S \subseteq A \) a nonempty subset of \( A, a \in A, D, D_1, D_2 \in Ds(A) \). Then
(i) If \( S \) is a deductive system, then \( [S] = S \);
(ii) \( [S] = \{ x \in A : s_1 \ldots \lor s_n \leq x, \text{ for some } n \geq 1 \text{ and } s_1, \ldots, s_n \in S \} \). In particular, \( [a] = \{ x \in A : x \geq a^n, \text{ for some } n \geq 1 \} \);
(iii) \( D(a) = \{ x \in A : x \geq d \lor a^n, \text{ with } d \in D \text{ and } n \geq 1 \} \);
(iv) \( D_1 \cup D_2 = \{ x \in A : x \geq d_1 \lor d_2 \text{ for some } d_1 \in D_1 \text{ and } d_2 \in D_2 \} \).

**Proposition 3.2.** Let \( D \in Ds(A) \) and \( a, b \in A \). Then \( D(a) \cap D(b) = D(a \lor b) \).

**Proof.** Let \( x \in D(a) \cap D(b) \). Then there are \( d_1, d_2 \in D \) and \( m, n \geq 1 \) such that \( x \geq d_1 \land a^m \land d_2 \land b^n \). Then \( x \geq (d_1 \land a^m) \lor (d_2 \land b^n) \lor (d_1 \lor b^n) \lor (d_2 \lor a^m) \lor (a \lor b)^{mn} \), hence by Proposition 3.1, \( x \in D(a \lor b) \), since \( d_1 \lor d_2, d_1 \lor b^n, d_2 \lor a^m \in D \). We deduce that \( D(a) \cap D(b) \subseteq D(a \lor b) \).
Conversely, let \( x \in D(a \lor b) \), there is \( d \in D \) and \( m \geq 1 \) such that \( x \geq d \lor (a \lor b)^m \geq d \lor a^m, d \lor b^m \), that is, \( D(a \lor b) \subseteq D(a) \cap D(b) \), so we obtain the desired equality. ■

**Corollary 3.1.** Let \( D \in Ds(A) \) and \( a_1, \ldots, a_n \in A \). Then \( D(a_1) \cap \ldots \cap D(a_n) = D(a_1 \lor \ldots \lor a_n) \).

**Corollary 3.2.** Let \( D \in Ds(A) \) and \( a_1, \ldots, a_n \in A \) such that \( a_1 \lor \ldots \lor a_n \in D \). Then \( D(a_1) \cap \ldots \cap D(a_n) = D \).

The lattice \( (Ds(A), \subseteq) \) is a complete Brouwerian lattice (hence distributive), where for a family \( \mathcal{F} = (D_i)_{i \in I} \) of deductive systems, \( \inf(\mathcal{F}) = \bigcap D_i \) and \( \sup(\mathcal{F}) = \bigcup D_i \).

Clearly, in this lattice \( 1 \) and \( 0 = \{1\} \).

**Proposition 3.3.** ([17]) If \( a, b \in A \), then
(i) \( \{a\} = \{x \in A : a \leq x \} \iff a \lor a = a \);
(ii) \( a \leq b \) implies \( \{b\} \subseteq \{a\} \);
(iii) \( \{a\} \cap \{b\} = \{a \land b\} \);
(iv) \( \{a\} \lor \{b\} = \{a \lor b\} \);
(v) \( \{a\} = 1 \iff a = 1 \).

For \( D_1, D_2, D \in Ds(A) \) we denote
\[
D_1 \hookrightarrow D_2 = \{a \in A : D_1 \cap \{a\} \subseteq D_2\} \text{ and } D^\circ = D \hookrightarrow 0 = D \hookrightarrow \{1\}.
\]

**Lemma 3.1.** ([6]) If \( D_1, D_2 \in Ds(A) \) then
(i) \( D_1 \hookrightarrow D_2 \in Ds(A) \);
(ii) If \( D \in Ds(A) \), then \( D_1 \cap D \subseteq D_2 \iff D \subseteq D_1 \hookrightarrow D_2 \), that is,
\[
D_1 \hookrightarrow D_2 = \sup\{D \in Ds(A) : D_1 \cap D \subseteq D_2\};
\]
(iii) \( D_1 \hookrightarrow D_2 = \{a \in A : a \lor y \in D_2, \text{ for all } y \in D_1\} \).

**Corollary 3.3.** \((Ds(A), \lor, \cap, \hookrightarrow, \{1\}, A)\) is a Heyting algebra, where for \( D \in Ds(A) \),
\[
D^\circ = \{x \in A : x \lor y = 1, \text{ for every } y \in D\},
\]
**Remark 3.2.** From Lemma 3.1, (ii), we deduce that if \( D_1, D_2 \in Ds(A) \) and \( x \in A \) such that \( x \in D_1 \) and \( x \in D_1 \hookrightarrow D_2 \), then \( x \in D_2 \). Also, if \( D \in Ds(A) \) then \( D \hookrightarrow D = A \) and \( D \subseteq D^\circ \).

**Proposition 3.4.** \( D^\circ = \{a \in A : a \rightarrow x = x \land x \rightarrow a = a, \text{ for every } x \in D\} \).

**Proof.** Let \( a \in D^\circ \). Since \( 1 = a \lor x \leq [(a \rightarrow x) \rightarrow x] \land [(x \rightarrow a) \rightarrow a] \) for every \( x \in D \) we deduce that \( (a \rightarrow x) \rightarrow x = (x \rightarrow a) \rightarrow a = 1 \), hence \( a \rightarrow x = x \) and \( x \rightarrow a = a \), for every \( x \in D \). ■

For \( X \subseteq A \) we denote by \( X^* = \{a \in A : a \rightarrow x = x, \text{ for any } x \in X\} \).

**Proposition 3.5.** \( X^* \in Ds(A) \), for every set \( X \subseteq A \).

**Proof.** Obvious \( 1 \in X^* \) since by \( c_1, 1 \rightarrow x = x, \text{ for any } x \in X \). Let \( a, b \in X^* \). Then \( a \rightarrow x = x \) and \( b \rightarrow x = x \), for any \( x \in X \). By \( c_3 \), we have \( (a \lor b) \rightarrow x = a \lor (b \rightarrow x) = a \rightarrow x, \text{ hence } a \lor b \in X^* \). If \( a \leq b \) and \( a \in X^* \) then \( a \rightarrow x = x, \text{ for any } x \in X \). By \( c_4 \), \( 1 = a \lor b \leq (b \rightarrow x) \rightarrow (a \rightarrow x), \text{ so } (b \rightarrow x) \rightarrow (a \rightarrow x) = 1 \). Using \( c_1 \), \( x \leq b \rightarrow x \leq a \rightarrow x = x, \text{ for every } x \in X, \text{ so } b \rightarrow x = x \). We deduce \( b \in X^* \). ■
Proposition 3.6. If \( D \in Ds(A) \), then \( D^\circ \subseteq D^* \).

Proof. Let \( a \in D^\circ \) and \( x \in D \). Then \( a \lor x = 1 \Rightarrow (a \lor x) \rightarrow x = 1 \rightarrow x = x \lor x \Rightarrow (a \rightarrow x) \land (x \rightarrow x) = x \Rightarrow (a \rightarrow x) \land 1 = x \Rightarrow a \rightarrow x \Rightarrow a \in D^* \Rightarrow D^\circ \subseteq D^* . \]

Remark 3.3. By Remark 2.1, if the residuated lattice \( A \) is a MV-\( \Delta \)-algebra then \( D^\circ = D^* \).

Proposition 3.7. For every subset \( X \subseteq A \), we have \( X \cap X^* = \emptyset \) or \( X \cap X^* = \{1\} \).

Proof. If \( 1 \in X \), since \( X^* \in Ds(A) \) we deduce that \( 1 \in X \cap X^* \). Let \( x \in X \cap X^* \). Then \( x \rightarrow x = x \), so \( x = 1 \) and \( X \cap X^* = \{1\} \).

If \( 1 \notin X \) we prove that \( X \cap X^* = \emptyset \). Suppose that exists \( x \in X \cap X^* \), obvious, \( x \neq 1 \). Then \( x \rightarrow x = x \), so \( x = 1 \), a contradiction.\( \blacksquare \)

Corollary 3.4. If \( D \in Ds(A) \), then \( D \cap D^* = \{1\} \).

Lemma 3.2. Let \( X, Y \) two subsets of \( A \). If \( X \subseteq Y \) then \( Y^* \subseteq X^* \).

Proof. Let \( y \in Y^* \). Then \( y \rightarrow z = z \), for every \( z \in Y \). Since \( X \subseteq Y \) we deduce that \( y \rightarrow z = z \), for every \( z \in X \), so \( y \in X^* \), that is, \( Y^* \subseteq X^* . \)

Proposition 3.8. Let \( D_1, D_2 \in Ds(A) \). Then \( D_1 \cap D_2 = \{1\} \) iff \( D_1 \subseteq D_2^* \).

Proof. Suppose that \( D_1 \cap D_2 = \{1\} \). Let \( d_1 \in D_1 \). For any \( d_2 \in D_2 \), \( d_1 \leq (d_1 \rightarrow d_2) \rightarrow d_2 \) so \( (d_1 \rightarrow d_2) \rightarrow d_2 \in D_1 \cap D_2 = \{1\} \). We obtain \( d_1 \rightarrow d_2 = d_2 \), hence \( d_1 \in D_2^* \).

Conversely, we assume that \( D_1 \subseteq D_2^* \). Since \( D_1, D_2 \in Ds(A) \), \( 1 \in D_1 \cap D_2 \subseteq D_2^* \cap D_2 = \{1\} \), by Remark 3.4, that is, \( D_1 \cap D_2 = \{1\} \).\( \blacksquare \)

Lemma 3.3. If \( D \in Ds(A) \) then \( D \subseteq D^{**} \).

Proof. Let \( d \in D \). For any \( x \in D^* \), since \( D, D^* \) are deductive systems and \( x, d \leq (d \rightarrow x) \rightarrow x \), we deduce that \( (d \rightarrow x) \rightarrow x \in D \cap D^* = \{1\} \), so, \( d \rightarrow x = x \), hence \( D \subseteq D^{**} . \)

Remark 3.4. The set of deductive systems \( Ds(A) \) forms two pseudocomplemented lattices (with \(* \) and with \( \circ \)). By Remark 3.3, if the residuated lattice \( A \) is a MV-algebra, then the two pseudocomplemented lattices coincide.

Remark 3.5. It follows from Glivenko’s theorem that the sets \( R_*(Ds(A)) = \{D \in Ds(A) : D = D^{***}\} \) and \( R_*(Ds(A)) = \{D \in Ds(A) : D = D^{\infty}\} \) are Boolean algebras. For \( D_1, D_2 \in Ds(A) \), \( (D_1^* \cap D_2^*)^* \) (respectively, \( (D_1^* \cap D_2^*)^\circ \)) is the least deductive system including \( D_1, D_2 \). Hence for \( D_1, D_2 \in Ds(A) \), we have \( sup\{D_1, D_2\} \) in \( R_*(Ds(A)) \) (respectively, \( R_*(Ds(A)) \) ) is \( (D_1^* \cap D_2^*)^* \) (respectively, \( (D_1^* \cap D_2^*)^\circ \)).

Remark 3.6. If \( D \in Ds(A) \) then \( (D = D^{**} \iff D \lor D^* = A) \) and \( (D = D^{\infty} \iff D \lor D^\circ = A) \).

Theorem 3.1. \( R_*(Ds(A)) \subseteq R_*(Ds(A)) \).

Proof. By Proposition 3.6, we have \( D^\circ \subseteq D^* \). Let \( D \in R_*(Ds(A)) \). Then \( D \lor D^\circ = A \). But \( A = D \lor D^\circ \subseteq D \lor D^* \), so \( D \lor D^* = A \), hence \( D \in R_*(Ds(A)) \).\( \blacksquare \)

Proposition 3.9. The following assertions are equivalent:

(i) \( e \in B(A) \);
(ii) \( |e|^\circ = |e| \);
(iii) \( |e|^{\infty} = |e| \).
Proof. (i) ⇒ (ii). Let \( e \in B(A) \). Since \( e \vee e^* = 1 \) and \( [e]^\circ = \{ x \in A : e \vee x = 1 \} \), we deduce that \( e^* \in [e]^\circ \), so \( [e] \hookrightarrow [e]^\circ \). If \( x \in [e]^\circ \), since \( e \vee x = 1 \), we have \( e^* = e^* \wedge 1 = e^* \wedge (e \vee x) \overset{\text{iii}}{=} e^* \circ (e \vee x) \overset{\text{iii}}{=} (e^* \circ e) \vee (e^* \circ x) \overset{\text{iii}}{=} 0 \vee (e^* \wedge x) = e^* \wedge x \), so \( e^* \leq x \). It follow that \( x \in [e^*] \) and we deduce \( [e]^\circ = [e^*] \).

(ii) ⇒ (i). Using Proposition 2.2, \( [e]^\circ = [e^*] \Rightarrow e^* \in [e]^\circ \Rightarrow e \vee e^* = 1 \Rightarrow e \in B(A) \).

Proof. (i) ⇒ (ii). Let \( D \in R_\circ(Ds(A)) \); since \( D \vee D^\circ = A \), there exist \( e \in D \), \( a \in D^\circ \) such that \( e \circ a = 0 \).

Since \( a \in D^\circ \), we have \( a \vee e = 1 \). Using \( c_2 \) we deduce that \( a \wedge e = a \circ e = 0 \), that is, \( e \in B(A) \).

For every \( x \in D \), \( a \vee x = 1 \). We have \( e \wedge x = 0 \vee (e \wedge x) = (e \wedge a) \vee (e \wedge x) \overset{\text{iii}}{=} e \wedge (a \vee x) = e \wedge 1 = e \), so \( e \leq x \), that is, \( D = [e] \).

(ii) ⇒ (i). By Proposition 3.9, (iii). 

We say that the inverse image of an deductive system under a morphism of residuated lattices is also a deductive system. Hence we have the following results:

Theorem 3.3. Let \( A, B \) two residuated lattices and \( f : A \to B \) a morphism of residuated lattice. If \( Y \) is a nonempty subset of \( B \), then \( f^{-1}(Y^*) \) is a deductive system of \( A \) containing \( f^{-1}(Y)^* \). Moreover, if \( D \) is deductive system of \( B \), then \( f^{-1}(D)^\circ \) is a deductive system of \( A \) containing \( [f^{-1}(D)]^\circ \).

Theorem 3.4. Let \( A, B \) two residuated lattices, \( f : A \to B \) a morphism of residuated lattice and \( X \subseteq A \) a nonempty subset of \( A \). Then \( f(X^*) \subseteq [f(X)]^* \).

Proof. Let \( b \in f(X^*) \) and \( y \in f(X) \). Then there exist \( a \in X^* \) and \( x \in X \) such that \( f(a) = b \) and \( f(x) = y \). Since \( a \in X^* \) and \( x \in X \) we deduce that \( a \rightarrow x = x \). It follows that \( b \rightarrow y = f(a) \rightarrow f(x) = f(a \rightarrow x) = f(x) = y \), so \( b \in [f(X)]^* \). We deduce that \( f(X^*) \subseteq [f(X)]^* \).

Theorem 3.5. Let \( A, B \) two residuated lattices, \( f : A \to B \) a surjective morphism of residuated lattice and \( D \in Ds(A) \). Then

(i): \( f(D^\circ), f(D^\circ) \in Ds(B) \);

(ii): \( f(D^\circ) \subseteq [f(D)]^\circ \) and \( f(D^\circ) \subseteq [f(D)]^* \);

(iii): If \( D^\circ \) (respectively \( D^\circ \)) is a maximal deductive system of \( A \) such that \( f(D^\circ) \) (respectively \( f(D^\circ) \)) is a proper, then \( f(D^\circ) \) (respectively \( f(D^\circ) \)) is a maximal deductive system of \( B \).

Proof. (i). Obviously, \( 1 = f(1) \in f(D^\circ) \). Let \( x, y \in f(D^\circ) \), that is there are \( a, b \in D^\circ \) such that \( f(a) = x \) and \( f(b) = y \). Since \( D^\circ \in Ds(A) \), we deduce that \( a \circ b \in D^\circ \) and \( x \circ y = f(a \circ b) = f(a \circ b) \in f(D^\circ) \). Let \( x, y \in B \) such that \( x \leq y \) and \( x \in f(D^\circ) \). Hence, there is \( a \in D^\circ \) such that \( f(a) = x \) and since \( f \) is surjective, there exists \( b \in A \) such that \( f(b) = y \). Then \( y = x \circ y = f(a) \circ f(b) = f(a \circ b) \) and \( a \circ b \geq a \in D^\circ \), so \( a \circ b \in D^\circ \) and \( y \in f(D^\circ) \). We obtain that \( f(D^\circ) \in Ds(B) \). Similarly for \( f(D^\circ) \in Ds(B) \).
Let $D'$ be a proper deductive system of $B$ such that $f(D') \subseteq D'$. We have that $D^* \subseteq f^{-1}(f(D^*)) \subseteq f^{-1}(D')$ and since $f^{-1}(D')$ is a proper deductive system of $A$, we must have $D^* = f^{-1}(D')$. We deduce that $f(D^*) = f(f^{-1}(D')) = D'$, since $f$ is a surjective morphism. Similarly for $f(D^*)$.\[\square\]

**Remark 3.8.** For $D \in Ds(A)$, if $D^*$ is a maximal deductive system of $A$, by Remark 3.6 we deduce that $D^* = D^*$, and by Theorem 3.5 if $f : A \to B$ is a surjective morphism of residuated lattice, then $f(D^*) = f(D^*)$ is a maximal deductive system of $B$.

With any deductive system $D$ of $A$ we can (see [12], [18]) associate a congruence $\theta_D$ on $A$ by defining $(a, b) \in \theta_D$ iff $a \to b, b \to a \in D$ iff $(a \to b) \cup (b \to a) \in D$. Conversely, for $\theta \in Con(A)$, the subset $D_{\theta}$ of $A$ defined by $a \in D_{\theta}$ iff $(a, 1) \in \theta$ is a deductive system of $A$. Moreover the natural maps associated with the above are mutually inverse and establish an isomorphism between the lattices $Ds(A)$ and $Con(A)$.

For $a \in A$, let $a/D$ be the equivalence class of $a$ modulo $\theta_D$. If we denote by $A/D$ the quotient set $A/\theta_D$, then $A/D$ becomes a residuated lattice with the natural operations induced from those of $A$. Clearly, in $A/D$, $0 = 0/D$ and $1 = 1/D$.

**Proposition 3.10.** Let $D \in Ds(A)$, and $a, b \in A$, then
(i) $a/D = 1/D$ iff $a \in D$, hence $a/D \neq 1$ iff $a \notin D$;
(ii) $a/D = 0/D$ iff $a^* \in D$;
(iii) If $D$ is proper and $a/D = 0/D$, then $a \notin D$;
(iv) $a/D \leq b/D$ iff $a \to b \in D$.

**Remark 3.9.** Let $A, B$ two residuated lattices. We define on $A \times B$, the operations $\land_{x, y}, \lor_{x, y}, \circ_{x, y}, \rightarrow_{x, y}$ for every $(a, b), (a', b') \in A \times B$ by $(a, b) \land_{x, y} (a', b') = (a \land a', b \land b')$, $(a, b) \lor_{x, y} (a', b') = (a \lor a', b \lor b')$, $(a, b) \circ_{x, y} (a', b') = (a \circ a', b \circ b')$, $(a, b) \rightarrow_{x, y} (a', b') = (a \rightarrow a', b \rightarrow b')$. Clearly, $A \times B, \land_{x, y}, \lor_{x, y}, \circ_{x, y}, \rightarrow_{x, y}, (0, 0), (1, 1))$ is a residuated lattice.

**Theorem 3.6.** Let $X$ and $Y$ be nonempty subsets of residuated lattices $A$ and $B$, respectively. Then:

(i): $X^* \times Y^* = (X \times Y)^*$

(ii): $A/X^* \times B/Y^* \cong (A \times B)/(X \times Y)^*$.

**Proof.** (i). We have that $(X \times Y)^* = \{(a, b) \in A \times B : (a, b) \to (x, y) = (x, y)$, for all $(x, y) \in X \times Y\} = \{(a, b) \in A \times B : (a \to x, b \to y) = (x, y)$, for all $(x, y) \in X \times Y\}$. Clearly, $A : a \to x, x \in X \forall x$. Consider the surjective morphisms $p_{X^*} : A \to A/X^*, p_{X^*}(a) = a/X^*$ for every $a \in A$ and $p_{Y^*} : B \to B/Y^*, p_{Y^*}(b) = b/Y^*$ for every $b \in B$. We define $f : (A \times B) \to A/X^* \times B/Y^*$ by $f(a, b) = (a/X^*, b/Y^*)$, for every $(a, b) \in A \times B$. Then $f$ is a surjective morphisms. We denote the filter kernel by $Ker(f) = f^{-1}((1/X^*, 1/Y^*))$ and using Proposition 3.10, $Ker(f) = \{(a, b) \in A \times B : f(a, b) = (1/X^*, 1/Y^*)\} = \{(a, b) \in A \times B : (a/X^*, b/Y^*) = (1/X^*, 1/Y^*)\} = \{(a, b) \in A \times B : a/X^* = 1/X^*, b/Y^* = 1/Y^*\} = \{(a, b) \in A \times B : a \in X^*, b \in Y^*\} = X^* \times Y^*$.

By the first isomorphism theorem and (i), we deduce that $(A \times B)/(X \times Y)^* \cong A/X^* \times B/Y^*$.\[\square\]

Analogously we obtain:
Theorem 3.7. Let $A$ and $B$ two residuated lattices and $D_1 \in Ds(A)$, $D_2 \in Ds(B)$.

Then:
(i): $D_1^\circ \times D_2^\circ = (D_1 \times D_2)^\circ$
(ii): $A/D_1^\circ \times B/D_2^\circ \cong (A \times B)/(D_1 \times D_2)^\circ$.

4. The lattice $Ds^\circ(A)$

We denote by $Ds^\circ(A) = \{ [a]^\circ : a \in A \}$.

Proposition 4.1. If $a, b \in A$, then
(i): $a \leq b \Rightarrow [a]^\circ \subseteq [b]^\circ$
(ii): $[a]^\circ \cap [b]^\circ = [a \land b]^\circ$
(iii): $[a \rightarrow b]^\circ \subseteq [a]^\circ \rightarrow [b]^\circ$
(iv): $[a \lor a]^\circ = [a]^\circ \lor [a]^\circ$

Proof. (i). If $x \in [a]^\circ$ then $x \lor a = 1$, but $a \leq b$, hence $1 = x \lor a \leq x \lor b$, so $x \lor b = 1$. We deduce $x \in [b]^\circ$.

(ii). We have $a \odot b \leq a \land b \leq a, b$, then using (i) we deduce that $[a \odot b]^\circ \subseteq [a \land b]^\circ \subseteq [a]^\circ \land [b]^\circ$, that is, $[a \odot b]^\circ \subseteq [a \land b]^\circ \subseteq [a]^\circ \cap [b]^\circ$.

Let now $x \in [a]^\circ \cap [b]^\circ$, that is, $a \lor x = b \lor x = 1$.

By $c_{10}$, $x \lor (a \odot b) \geq (x \lor a) \cap (x \lor b) = 1$, hence $x \lor (a \odot b) = 1$, that is, $x \in [a \odot b]^\circ$

It follows that $[a]^\circ \cap [b]^\circ \subseteq [a \odot b]^\circ$, hence $[a]^\circ \cap [b]^\circ = [a \odot b]^\circ = [a \land b]^\circ$.

(iii). Let $x \in [a \rightarrow b]^\circ \Rightarrow x \lor (a \rightarrow b) = 1$. We have that $x \in [a]^\circ \rightarrow [b]^\circ \Rightarrow x \lor y \in [b]^\circ$, for any $y \in [a]^\circ$. Let so $y \in [a]^\circ \Rightarrow a \lor y = 1$. We prove that $b \lor (x \lor y) = 1$.

By $c_{12}$ we deduce $1 = x \lor (a \rightarrow b) \leq (x \lor a) \rightarrow (x \lor b) \Rightarrow 1 = (x \lor a) \rightarrow (x \lor b) \Rightarrow x \lor a \leq x \lor b$. Then $x \lor y \lor a \leq x \lor y \lor b \Rightarrow x \lor 1 \leq x \lor y \lor b \Rightarrow x \lor y \lor b = 1 \Rightarrow x \in [a]^\circ \rightarrow [b]^\circ$.

(iv). Since $a, a^* \leq a \lor a^*$ we deduce by (i), that $[a]^\circ, [a]^\circ \subseteq [a \lor a]^\circ \Rightarrow [a]^\circ \lor [a]^\circ \subseteq [a \lor a]^\circ$.

Conversely, let $x \in [a \lor a]^\circ$. We have $x \lor (a \lor a^*) = 1 \Rightarrow (x \lor a) \lor a^* = 1$ and $(x \lor a^*) \lor a = 1 \Rightarrow x \lor a \in [a]^\circ$ and $x \lor a^* \in [a]^\circ$.

By $c_{10}$, $x = x \lor (a \odot a^*) \geq (x \lor a) \odot (x \lor a^*)$. Since $x \lor a \in [a]^\circ$ and $x \lor a^* \in [a]^\circ$ we deduce that $x \in [a]^\circ \lor [a]^\circ$, so $[a \lor a]^\circ \subseteq [a]^\circ \lor [a]^\circ$.

Finally, $[a \lor a]^\circ = [a]^\circ \lor [a]^\circ$.

Remark 4.1. $[a]^\circ \rightarrow [b]^\circ \subseteq [a \rightarrow b]^\circ$. Indeed, if we consider the residuated lattice $A$ from Example 2.2, then $[0]^\circ = [a]^\circ = [b]^\circ = [c]^\circ = [1]$, $[1]^\circ = A$ and $[a]^\circ \rightarrow [b]^\circ = (x \in A : x \lor 1 = 1) = A$ but $[a \rightarrow b]^\circ = [b]^\circ = [1]$.

Proposition 4.2. If $e \in B(A)$ and $[e]^\circ = \{1\}$ then $e = 0$.

Proof. Since by Propositions 3.3 and 3.9, $[e]^\circ = [e]^* = \{x \in A : x \geq e^*\} = \{1\}$ and $e^* \in [e]^*$ we deduce that $e^* = 1$ so $e = 0$.

Remark 4.2. Since for every $a \in A$, $[a]^\circ$ is the pseudocomplement of $[a]$ in the lattice $Ds(A)$, then:
(i): $[a]^\circ = A \Leftrightarrow a = 1$ and $[0]^\circ = \{1\}$
(ii): $[a] \cap [a]^\circ = \{1\}$
(iii): $[a]^\circ \cap [a]^\circ = \{1\}$
(iv): $[a]^\circ = [a]^\circ ^\circ ^\circ ^\circ$.

Definition 4.1. An element $a$ in a residuated lattice $A$ is called nilpotent iff there exists a natural number $n$ such that $a^n = 0$. The minimum $n$ such that $a^n = 0$ is
called nilpotence order of $a$ and will be denoted by $\text{ord}(a)$; if there is no such $n$, then $\text{ord}(a) = \infty$. A residuated lattice $A$ is called locally finite if every $a \in A, a \neq 1$, has finite order.

**Proposition 4.3.** Let $a \in A$ and a natural number $n$. Then $[a]^n \equiv [a^n]$. 

**Proof.** By Proposition 4.1, (i), since $a^n \leq a$ we obtain $[a^n] \subseteq [a]$. Conversely, let $x \in [a]$. Then $a \lor x = 1$. By $c_{11}, 1 = a^n \lor x^n \leq a^n \lor x$. We deduce that $a^n \lor x = 1$, so $x \in [a^n]$ and $[a] \subseteq [a^n]^n$. Finally, $[a]^n = [a^n]$. ■

**Proposition 4.4.** Let $a \in A, a \neq 1$, such that $a$ has a finite order $n$. Then $[a]^n = \{1\}$.

**Proof.** Since $n$ is the finite order of $a$ we have $a^n = 0$. By Proposition 4.3, $[a]^n = [a^n] = [0] = \{1\}$.

**Proof.** By definition, $[a] = \{x \in A : a \lor x = 1\}$. Let $x \in [a]$. Since $1 = a \lor x \leq (a \rightarrow x) \land (x \rightarrow a) \rightarrow a$ we deduce that $(a \rightarrow x) \rightarrow x = (x \rightarrow a) \rightarrow a = 1$, hence $a \rightarrow x = x$ and $x \rightarrow a = a$. Now $x = a \rightarrow x = a \rightarrow (a \rightarrow x) = a^2 \rightarrow x = \ldots = a^n \rightarrow x = 0 \rightarrow x = 1$. We deduce that $[a]^n = \{1\}$. ■

For $a, b \in A$ we denote $\{a\} \lor \{b\} = [a] \rightarrow [b]$. 

**Proposition 4.5.** Let $a, b \in A$. Then $[a] \lor [b] = [a \lor b]^\circ$.

**Proof.** By Lemma 3.1, $[a] \lor [b] = [a] \rightarrow [b] = \{x \in A : x \lor y \in [b]\}$, for every $y \in [a]$ and $\{x \in A : x \lor y \in [x] \lor [y]\} = \{x \in A : x \lor y = 1\}$, for every $y \in [a]$ and $\{x \in A : x \lor y = 1\}$, for every $y \in [a]$. Clearly, $a \in [a]$, so for any $x \in [a] \lor [b]$ we obtain $x \lor a \lor b = 1$. This implies $x \in [a \lor b]^\circ$, hence $[a] \lor [b] \subseteq [a \lor b]^\circ$.

Now, we prove that $[a \lor b] \subseteq [a] \lor [b]$. Let $x \in [a \lor b]$, that is, $x \lor a \lor b = 1$. Let $y \in [a]$. We deduce $y \lor z = 1$, for $z \in A$ such that $z \lor a = 1$. If we denote $t = x \lor b$ we will prove that $(t \lor a) = t \lor y = 1$, for every $y \in [a]$. This implies $x \lor a \lor b = 1$. It is an immediate consequence of Remark 4.2, (iv). ■

**Corollary 4.1.** For $a, b \in A$, $[a] \lor [b] = [a \lor b]^\circ \in \text{Ds}^\circ(A)$.

**Remark 4.3.** If $a, b \in A$, then $[a] \lor [b] \subseteq [a] \lor [b]^\circ$ so, $[a] \lor [b] \subseteq [a] \lor [b]^\circ = [a \lor b]^\circ$.

**Remark 4.4.** By Proposition 4.1, $[a] \lor [a]^\circ = [a] \lor [a]^\circ = [a \lor a]^\circ$.

**Proposition 4.6.** $a \in B(A) \iff [a] \lor [a]^\circ = A$.

**Proof.** By Proposition 4.5, if $a \in B(A)$ then $[a] \lor [a]^\circ = [a \lor a]^\circ = 1 = A$. Conversely, $[a] \lor [a]^\circ = [a \lor a]^\circ = A$ implies $0 \lor (a \lor a) = 1 \Rightarrow a \lor a = 1$. By Proposition 2.2 we deduce that $a \in B(A)$.

**Theorem 4.1.** $(\text{Ds}^\circ(A), \cap, \lor, [1], 1 = [1] = [1])$ is a bounded distributive lattice and $[a] \lor [a]^\circ = [a] \lor [a]^\circ = [a \lor a]^\circ = [a \lor b]^\circ$. for $a, b \in A$.

**Proof.** We shall prove that $\lor$ is the supremum in this lattice.

It is obvious that, by Proposition 4.1, $a, b \leq a \lor b$ implies $[a] \lor [b] \subseteq [a \lor b]$, $a, b \in A$. For $c \subseteq A$ such that $[a] \lor [b] \subseteq [c] \lor [c]$ we will prove that $[a \lor b] \subseteq [c]$. If $c \subseteq [a \lor b]$, then $c \lor a \lor b = 1$, so $c \lor a \lor b = 1$ implies $c \lor 1 = 1$ implies $c \lor 1 = 1$ implies $c \lor 0 = 1$. Thus, $[a \lor b] = [a \lor b]$. ■
Since using Proposition 4.1, \([a]^\circ \cap (b)^\circ \cup [c]^\circ) = [a]^\circ \cap [b \lor c]^\circ \supseteq ([a \cap b) \lor (a \cap c))^\circ = ([a] \cap [b] \lor [a] \cap [c]^\circ)\), for every \(a, b, c \in A\) and \([1] = [0]^\circ, A = [1]^\circ\), we deduce that the lattice \((Ds_p^\circ (A), \cap, \lor, \{1\}, A)\) is distributive and bounded.

Applying Remark 4.2 we get that \([a]^\circ = [a]^{\circ\circ}, [a]^\circ \cap [a]^\circ = \{1\}\).

The equality \([a]^\circ \cap [b]^\circ = ([a] \lor [b])^\circ\), for \(a, b \in A\) is equivalent with \([a]^\circ \cap [b]^\circ = [a \lor b]^\circ\), for \(a, b \in A\).

Let \(x \in [a]^\circ \cap [b]^\circ\). We deduce that \(x \lor y = 1\), for every \(y \in [a]^\circ\), and \(x \lor z = 1\), for every \(z \in [b]^\circ\). Let \(t \in [a \lor b]^\circ\). We obtain \(t \lor a \lor b = 1 \Rightarrow t \lor a \in [b]^\circ \Rightarrow t \lor a \lor a = 1 \Rightarrow x \lor t \in [a]^\circ \Rightarrow x \lor (x \lor t) = 1 \Rightarrow x \lor t = 1\). Thus, \([a]^\circ \cap [b]^\circ \subseteq [a \lor b]^\circ\).

Conversely, let \(x \in [a \lor b]^\circ\). Then \(x \lor z = 1\), for every \(z \in A\) such that \(z \lor a \lor b = 1\). Let \(y_1 \in [a]^\circ\). Then \(y_1 \lor a = 1 \Rightarrow y_1 \lor a \lor b = 1 \Rightarrow x \lor y_1 = 1 \Rightarrow x \in [a]^\circ\). Let \(y_2 \in [b]^\circ\). Then \(y_2 \lor b = 1 \Rightarrow y_2 \lor a \lor b = 1 \Rightarrow x \lor y_2 = 1 \Rightarrow x \in [b]^\circ\). Thus, \(x \in [a]^\circ \cap [b]^\circ\).

Finally, \([a \lor b]^\circ \subseteq [a]^\circ \cap [b]^\circ\), so \([a \lor b]^\circ = [a]^\circ \cap [b]^\circ\).

Remark 4.5. If \(A\) is a chain then \(Ds_p^\circ (A)\) is isomorphic with \(L_2\). the two-elements Boolean algebra. Indeed, for \(a \in A, a \neq 1, [a]^\circ = \{1\}\) and \([1]^\circ = A\).

Remark 4.6. If \(A\) is a locally finite residuated lattice, then every element of \(A\) has a finite order and by Proposition 4.4 we deduce that \(Ds_p^\circ (A)\) is a Boolean algebra isomorphic with \(L_2\).

Remark 4.7. We recall that a residuated lattice is subdirectly irreducible iff it is nontrivial and for any subdirect representation \(f : A \rightarrow \prod_{i \in J} A_i\), there exists a \(j\) such that \(f_j\) is an isomorphism of \(A\) onto \(A_j\). In [12] it is proved that in any subdirectly irreducible residuated lattice, if \(x \lor y = 1\), then \(x = 1\) or \(y = 1\). Obviously, if \(A\) is a subdirectly irreducible residuated lattice, then \(Ds_p^\circ (A)\) is a Boolean algebra isomorphic with \(L_2\).

Remark 4.8. If \(e, f \in B(A)\), then \([e]^\circ \lor [f]^\circ = [e \lor f]^\circ\) and \([e \lor f]^\circ = [e^\circ \lor f^\circ]^\circ = [e^\circ \lor f]^\circ = [e^\circ \lor f^\circ]^\circ = [e^\circ \lor f^\circ]^\circ = [e^\circ \lor f^\circ]^\circ = [e \lor f]^\circ = [1]^\circ = A\).

Remark 4.9. If \(e \in B(A)\), then \([e]^\circ \in B(Ds_p^\circ (A))\), so \(Ds_p^\circ (B(A))\) is a Boolean subalgebra of \(B(Ds_p^\circ (A))\).

In [5] we introduce and characterize the hyperarchimedean residuated lattice.

Definition 4.2. [5] Let \(A\) be a residuated lattice. An element \(a \in A\) is called archimedean if it satisfy the condition : there is \(n \geq 1\) such that \(a^n \in B(A)\), equivalent with \(a \lor (a^\ast)^\circ = 1\). A residuated lattice \(A\) is called hyperarchimedean if all its elements are archimedean.

Proposition 4.7. If \(A\) is a hyperarchimedean residuated lattice then \(Ds_p^\circ (A)\) is a Boolean subalgebra of \(Ds(A)\).

Proof. Since \(A\) is a hyperarchimedean residuated lattice then for every \(a \in A\) there is a natural number \(n \geq 1\) such that \(a^n = e_a \in B(A)\). By Proposition 4.3, \([a]^\circ = [a^n]^\circ = [e_a]^\circ\). We deduce that \(\lor = \lor\) and \(Ds_p^\circ (A)\) is a Boolean algebra.

Theorem 4.2. If \(A\) is a residuated lattice, then the map \(f : (A, \land, \lor, 0, 1) \rightarrow (Ds_p^\circ (A), \cap, \lor, \{1\}, A)\),

\[ f(a) = [a]^\circ, \]

for every \(a \in A\) is an omomorphism of distributive and bounded lattices.
Proof. Let \( a, b \in A \). Applying Proposition 4.1 and Corollary 4.1 we obtain that
\[
(\forall a \in A \& b) = [a \land b]^\circ = [a] \land [b] = \{ f(a) \cap f(b) \mid f(a) \cap f(b) = [a \land b]^\circ = [a] \land [b] \} = \{ f(0) \mid f(0) = \{ 0 \} = \{ 1 \} \} \text{ and } f(1) = [1] = A.
\]

In [1], if \( f : L_1 \rightarrow L_2 \) is a morphism of bounded lattices, then we denote the ideal kernel by \( \text{Ker}(f) = f^{-1}([0]) = \{ x \in L_1 : f(x) = 0 \} \).

Remark 4.10. Using this notation, by Proposition 4.4, if we denote by \( \text{Ord}_{\text{finite}} = \{ x \in A : x \text{ has a finite order} \} \), then \( \text{Ord}_{\text{finite}} \subseteq \text{Ker}(f) \), where \( f : A \rightarrow D_{\text{finite}}^\circ(A) \) is the ontomorphism from Theorem 4.2.

Proposition 4.8. If \( A \) is a hyperarchimedean residuated lattice then \( \text{Ker}(f) = \text{Ord}_{\text{finite}} \) is a proper ideal of \( L(A) \) and \( A/\text{Ker}(f) \approx D_{\text{finite}}^\circ(A) \) as Boolean algebras.

Proof. Let \( a \in \text{Ker}(f) \). Then \( f(a) = \{ 1 \} \Leftrightarrow [a] = \{ 1 \} \). Since \( A \) is a hyperarchimedean residuated lattice then for \( a \in A \) there is a natural number \( n \geq 1 \) such that \( a^n = e_a \in B(A) \). By Proposition 4.3, we deduce that \( [a]^\circ = [a^n]^\circ = [e_a]^\circ \). But Propositions 3.9 and 4.2, \( \{ 1 \} = [e_a]^\circ = [e_a]^\circ \), so \( e_a = a^n = 0 \) and \( a \) has a finite order. We deduce that \( \text{Ker}(f) \subseteq \text{Ord}_{\text{finite}} \). Using Remark 4.10 we deduce that \( \text{Ker}(f) = \text{Ord}_{\text{finite}} \).

By Proposition 4.7, \( A/\text{Ker}(f) \approx D_{\text{finite}}^\circ(A) \) as Boolean algebras.

Corollary 4.2. For every residuated lattice \( A \), \( f_{B(A)} \) is an injective morphism, so \( (B(A), \land, \lor, 0, 1) \) is an isomorphic with a sublattice of \( (D_{\text{finite}}^\circ(A), \land, \lor, \{ 1 \}, A) \).

Proof. To prove the injectivity of \( f \), let \( e, g \in B(A) \) such that \( f(e) = f(g) \). Then \( [e]^\circ = [g]^\circ \). Using Proposition 3.9 we deduce that \( [e]^\circ = [g]^\circ \), so \( e^\circ = g^\circ \). Thus, \( e = g \).

Proposition 4.9. If \( e, f \in B(A) \), then \( [e]^\circ \leadsto [f]^\circ = [e^\circ \lor f]^\circ \in D_{\text{finite}}^\circ(A) \).

Proof. By Proposition 4.5, \( [a]^\circ \leadsto [b]^\circ = [a]^\circ \lor [b]^\circ = [a \lor b]^\circ \), for every \( a, b \in A \).

Applying Propositions 3.9 and Remark 4.2 we have that \( [e]^\circ \leadsto [f]^\circ = [e]^\circ \lor [f]^\circ \in D_{\text{finite}}^\circ(A) \).

Corollary 4.3. \( (D_{\text{finite}}^\circ(B(A)), \land, \lor, \{ 1 \}, A) \) is a Boolean algebra and \( f_{B(A)} : (B(A), \land, \lor, 0, 1) \rightarrow (D_{\text{finite}}^\circ(B(A)), \land, \lor, \{ 1 \}, A) \) defined by \( f_{B(A)}(e) = [e]^\circ = [e]^\circ \), for every \( e \in B(A) \) is an isomorphism of Boolean algebras.

Proof. Apply Theorems 4.1, 4.2, Corollary 4.2 and Proposition 4.9.

Theorem 4.3. Let \( a, b, c \in A \). Then \( [a]^\circ \subseteq [b]^\circ \Leftrightarrow [a] \land [c]^\circ \subseteq [b] \).

Proof. From Lemma 3.1, \( [a]^\circ \leadsto [b]^\circ = \{ x \in A : x \lor y \in [b] \} \), for all \( y \in [a] \).

Suppose that \( [a]^\circ \land [c]^\circ \subseteq [b]^\circ \). Let \( x \in [a] \). We have that \( x \lor c = 1. \) Let \( y \in [a] \), so \( y \lor a = 1 \). By \( e_1, (x \lor y) \lor (a \lor c) \geq (x \lor y \lor a) \lor (x \lor y \lor c) = (x \lor 1) \lor (y \lor 1) = 1 \Rightarrow (x \lor y) \lor (a \lor c) = 1 \Rightarrow x \lor y \in [a \lor c]^\circ \). But \( [a \lor c]^\circ = [a] \land [c]^\circ \subseteq [b]^\circ \), so \( x \lor y \in [b]^\circ \), for any \( y \in [a] \). By definition we deduce that \( x \in [a]^\circ \rightarrow [b]^\circ \) so, \( [c]^\circ \subseteq [a]^\circ \rightarrow [b]^\circ \).

Conversely we suppose that \( [c]^\circ \subseteq [a]^\circ \rightarrow [b]^\circ \), let \( x \in [a] \cap [c]^\circ = [a \lor c]^\circ = [a \land c]^\circ \). So, \( x \lor (a \land c) = 1 \).

We have \( 1 \Rightarrow x \lor (a \land c) \leq (x \lor a) \land (x \lor c) \Rightarrow (x \lor a) \land (x \lor c) = 1 \Rightarrow x \lor a = x \lor c = 1 \Rightarrow x \in [a]^\circ \) and \( x \in [c]^\circ \). But \( [c]^\circ \subseteq [b]^\circ \) so \( x \in [a]^\circ \rightarrow [b]^\circ \). Since \( x \in [a] \) it is easy to show applying Remark 3.2 that \( x \in [b]^\circ \). Obviously, \( [a]^\circ \cap [c]^\circ \subseteq [b]^\circ \).

Remark 4.11. Since \( (D_{\text{finite}}^\circ(A), \land, \lor, \{ 1 \}, A) \) is a commutative monoid using Theorems 4.1 and 4.3 we deduce that \( (D_{\text{finite}}^\circ(B(A)), \land, \lor, \{ 1 \}, A) \) is a residuated lattice.
References