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On the lattice of deductive systems of a residuated lattice

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Abstract. In any residuated lattice A the set $Ds(A)$ of all deductive systems of A forms a pseudo-complemented distributive lattice and we denote by $D^{\diamond}$ the pseudocomplement of D in this lattice (it is proved that $D^{\diamond} = \{a \in A : a \vee x = 1$, for every $x \in D\}$). In this paper we give a characterization for regular deductive systems and we study the lattice $\mathrm{Ds}_p^{\diamond}(A)$ of deductive systems of the form $[a)^{\diamond}$. If A is a hyperarchymedean residuated lattice, then $Ds_p^{\diamond}(A)$ is a Boolean algebra. Also, for $X \subseteq A$ we denote by $X^{*} = \{a \in A : a \to x = x$, for any $x \in X\}$ which is a deductive system and we show that the set $R_{*}(Ds(A)) = \{D \in Ds(A) : D = D^{**}\}$ does a Boolean algebra.

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1. Introduction

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([13]), Dilworth ([7]), Ward and Dilworth ([20]), Ward ([19]), Balbes and Dwinger ([1]) and Pavelka ([16]).

In [10], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: *BCK- latices* in [9], *full BCK- algebras* in [13], *FL_{ew}- algebras* in [14], and *integral, residuated, commutative l-monoids* in [3].

Residuated lattices have been studied extensively and include important classes of algebras such as BL-algebras, introduced by Hájek as the algebraic counterpart of his Basic Logic, and MV-algebras, the algebraic setting for Łukasiewicz propositional logic.

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [2], [4], [7], [12], [15], [19], [20]).

In order to simplify the notation a residuated lattice $(A, \wedge, \vee, \odot, \to, 0, 1)$ will be referred by its support set A.

By $B(A)$ we denote the Boolean algebra of all complemented elements in the lattice $L(A) = (A, \wedge, \vee, 0, 1)$.

In any residuated lattice A the set $Ds(A)$ of all deductive systems of A forms a pseudo-complemented distributive lattice and we denote by $D^{\diamond}$ the pseudocomplement of D in this lattice (it is proved that $D^{\diamond} = \{a \in A : a \vee x = 1$, for every $x \in D\}$). In this paper we give a characterization for regular deductive systems denoted by $R_{\diamond}(Ds(A)) = \{D \in Ds(A) : D = D^{\diamond\diamond}\}$. Also, for $X \subseteq A$ we denote by $X^{*} = \{a \in A : a \to x = x$, for any $x \in X\}$ which is a deductive system and we show that the set $R_{*}(Ds(A)) = \{D \in Ds(A) : D = D^{**}\}$ does a Boolean algebra. We prove that $R_{\diamond}(Ds(A)) \subseteq R_{*}(Ds(A))$ and $D \in R_{\diamond}(Ds(A))$ iff $D = [e)$, with $e \in B(A)$.

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Finally, we study the lattice $Ds_p^{\diamond}(A)$ of deductive systems of the form $[a)^{\diamond}$ with $a \in A$.

If A is a hyperarchymedean residuated lattice, then $Ds_p^{\diamond}(A)$ is a Boolean algebra.

2. Preliminaries

Definition 2.1. *A* residuated lattice *([2], [18]) is an algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$of type (2,2,2,2,0,0) equipped with an order $\leq$ satisfying the following:*

(LR_1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice;*

(LR_2) $(A, \odot, 1)$ *is a commutative ordered monoid;*

(LR_3) $\odot$ *and* $\to$ *form an adjoint pair, i.e.* $c \leq a \to b$ *iff* $a \odot c \leq b$ *for all* $a, b, c \in A$.

The relations between the pair of operations $\odot$ and $\to$ expressed by (LR_3), is a particular case of the *law of residuation* ([2]). Łukasiewicz structure, Gődel structure, Products structure are residuated lattices (see [18]).

Example 2.1. *If* $(A, \vee, \wedge, ', 0, 1)$ *is a Boolean algebra and we define for every* $x, y \in A, x \odot y = x \wedge y$, $x \to y = x' \vee y$, *then* $(A, \vee, \wedge, \odot, \to, 0, 1)$ *become a residuated lattice.*

Remark 2.1. [18]*A residuated lattice* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *is an MV-algebra iff it satisfies the additional condition:* $(x \to y) \to y = (y \to x) \to x$, *for any* $x, y \in A$.

We give an example of finite residuated lattice:

Example 2.2. *([11]) Let* $A = \{0, a, b, c, 1\}$ *with* $0 < a, b < c < 1$, *but* a, b *are incomparable. A become a residuated lattice relative to the following operations:*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

We refer the reader to [4], [12], [18] for basic results in the theory of residuated lattices. In the following, we only present the material needed in the remainder of the paper.

In what follows by A we denote a residuated lattice; for $x \in A$ and a natural number n, we define $x^* = x \to 0, (x^*)^* = x^{**}$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

Theorem 2.1. *([4], [12], [18]) Let* $x, x_1, x_2, y, y_1, y_2, z \in A$. *Then we have the following rules of calculus:*

(c_1) $1 \to x = x, x \to x = 1, y \leq x \to y, x \to 1 = 1, 0 \to x = 1$;

(c_2) $x \odot 0 = 0, x \odot y \leq x, y$, *hence* $x \odot y \leq x \wedge y$ *and* ($x \vee y = 1$ *implies* $x \odot y = x \wedge y$);

(c_3) ($x \leq y$ *iff* $x \to y = 1$) *and* ($x \to y = y \to x = 1$ *iff* $x = y$);

(c_4) $x \to y \leq (z \to x) \to (z \to y)$ *and* $x \to y \leq (y \to z) \to (x \to z)$;

(c_5) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(c_6) $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \leq y^*$;

(c_7) $x \leq x^{**}, x^{**} \leq x^* \to x, 1^* = 0$, $0^* = 1$;

(c_8) $x \to y \leq y^* \to x^*, x^{***} = x^*, (x \odot y)^* = x \to y^* = y \to x^* = x^{**} \to y^*$;

(c_9) $x \odot (y_1 \vee y_2) = (x \odot y_1) \vee (x \odot y_2)$, $(y_1 \vee y_2) \to x = (y_1 \to x) \wedge (y_2 \to x)$ *and* $x \to (y_1 \vee y_2) \geq (x \to y_1) \vee (x \to y_2)$;

(c_{10}) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

Corollary 2.1. *([12]) Let* $a_1, ..., a_n \in A$.

(c_{11}) *If* $a_1 \vee ... \vee a_n = 1$, *then* $a_1^k \vee ... \vee a_n^k = 1$, *for every natural number* k.

Proposition 2.1. *If* A *is a residuated lattice and* $a, b, x \in A$, *then*
(c_{12})**:** $x \vee (a \to b) \leq (x \vee a) \to (x \vee b)$.

Proof. We have $(x \vee a) \to (x \vee b) \overset{c_9}{=} (x \to (x \vee b)) \wedge (a \to (x \vee b)) = 1 \wedge (a \to (x \vee b)) = a \to (x \vee b) \overset{c_9}{\geq} (a \to x) \vee (a \to b) \geq x \vee (a \to b)$.■

Proposition 2.2. ([4]) *For* $e \in A$ *the following are equivalent:*
(i) $e \in B(A)$;
(ii) $e \vee e^* = 1$.

Lemma 2.1. ([4], [12]) *If* $e \in B(A)$, *then*
(c_{13}) $e \odot x = e \wedge x$, *for every* $x \in A$;
(c_{14}) $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$, *for every* $x, y \in A$.

3. The regular deductive systems of a residuated lattice

Definition 3.1. ([12], [18]) *A nonempty subset* $D \subseteq A$ *is called a* deductive system *of* A *if the following conditions are satisfied*:
(Ds_1) $1 \in D$;
(Ds_2) *If* $x, x \to y \in D$, *then* $y \in D$.

Remark 3.1. ([12], [18]) *A nonempty subset* $D \subseteq A$ *is a deductive system of* A *if for all* $x, y \in A$:
(Ds'_1) *If* x, $y \in D$, *then* $x \odot y \in D$;
(Ds'_2) *If* $x \in D, y \in A, x \leq y$, *then* $y \in D$.

Every deductive system of A is a filter for $L(A)$, but a filter of $L(A)$ is not, in general, deductive system of A (see [18]).

We denote by $Ds(A)$ the set of all deductive systems of A.

For a nonempty subset $S \subseteq A$, the smallest deductive system of A which contains S, i.e. $\cap\{D \in Ds(A) : S \subseteq D\}$, is said to be *the deductive system of* A *generated by* S and will be denoted by $[S)$.

If $S = \{a\}$, with $a \in A$, we denote by $[a)$ the deductive system *generated by* $\{a\}$ ($[a)$ is called *principal*).

For $D \in Ds(A)$ and $a \in A$, we denote by $D(a) = [D \cup \{a\})$ (clearly, if $a \in D$, then $D(a) = D$).

Proposition 3.1. ([12], [18]) *Let* $S \subseteq A$ *a nonempty subset of* A, $a \in A$, $D, D_1, D_2 \in Ds(A)$. *Then*
(i) *If* S *is a deductive system, then* $[S) = S$;
(ii) $[S) = \{x \in A : s_1 \odot ... \odot s_n \leq x$, *for some* $n \geq 1$ *and* $s_1, ..., s_n \in S\}$. *In particular,* $[a) = \{x \in A : x \geq a^n$, *for some* $n \geq 1\}$;
(iii) $D(a) = \{x \in A : x \geq d \odot a^n$, *with* $d \in D$ *and* $n \geq 1\}$;
(iv) $[D_1 \cup D_2) = \{x \in A : x \geq d_1 \odot d_2$ *for some* $d_1 \in D_1$ *and* $d_2 \in D_2\}$.

Proposition 3.2. *Let* $D \in Ds(A)$ *and* $a, b \in A$. *Then* $D(a) \cap D(b) = D(a \vee b)$.

Proof. Let $x \in D(a) \cap D(b)$. Then there are $d_1, d_2 \in D$ and $m, n \geq 1$ such that $x \geq d_1 \odot a^m$ and $x \geq d_2 \odot b^n$. Then $x \geq (d_1 \odot a^m) \vee (d_2 \odot b^n) \overset{c_{12}}{\geq} (d_1 \vee d_2) \odot (d_1 \vee b^n) \odot (d_2 \vee a^m) \odot (a \vee b)^{mn}$, hence by Proposition 3.1, $x \in D(a \vee b)$, since $d_1 \vee d_2, d_1 \vee b^n, d_2 \vee a^m \in D$. We deduce that $D(a) \cap D(b) \subseteq D(a \vee b)$.

Conversely, let $x \in D(a \vee b)$, there is $d \in D$ and $m \geq 1$ such that $x \geq d \odot (a \vee b)^m \geq d \odot a^m, d \odot b^m$, that is, $D(a \vee b) \subseteq D(a) \cap D(b)$, so we obtain the desired equality.■

Corollary 3.1. *Let $D \in Ds(A)$ and $a_1, ..., a_n \in A$. Then $D(a_1) \cap ... \cap D(a_n) = D(a_1 \vee ... \vee a_n)$.*

Corollary 3.2. *Let $D \in Ds(A)$ and $a_1, ..., a_n \in A$ such that $a_1 \vee ... \vee a_n \in D$. Then $D(a_1) \cap ... \cap D(a_n) = D$.*

The lattice $(Ds(A), \subseteq)$ is a complete Brouwerian lattice (hence distributive), where for a family $\mathcal{F} = (D_i)_{i \in I}$ of deductive systems, $\inf(\mathcal{F}) = \bigcap_{i \in I} D_i$ and $\sup(\mathcal{F}) = [\bigcup_{i \in I} D_i)$. Clearly, in this lattice $\mathbf{0} = \{1\}$ and $\mathbf{1} = A$.

Proposition 3.3. *([17]) If $a, b \in A$, then*
(i) $[a) = \{x \in A : a \leq x\}$ *iff* $a \odot a = a$;
(ii) $a \leq b$ *implies* $[b) \subseteq [a)$;
(iii) $[a) \cap [b) = [a \vee b)$;
(iv) $[a) \vee [b) = [a \wedge b) = [a \odot b)$;
(v) $[a) = 1$ *iff* $a = 1$.

For D_1, $D_2, D \in Ds(A)$ we denote

$$D_1 \rightsquigarrow D_2 = \{a \in A : D_1 \cap [a) \subseteq D_2\} \text{ and } D^{\diamond} = D \rightsquigarrow \mathbf{0} = D \rightsquigarrow \{1\}.$$

Lemma 3.1. *([6]) If D_1, $D_2 \in Ds(A)$ then*
(i) $D_1 \rightsquigarrow D_2 \in Ds(A)$;
(ii) If $D \in Ds(A)$, then $D_1 \cap D \subseteq D_2$ iff $D \subseteq D_1 \rightsquigarrow D_2$, that is,

$$D_1 \rightsquigarrow D_2 = \sup\{D \in Ds(A) : D_1 \cap D \subseteq D_2\};$$

(iii) $D_1 \rightsquigarrow D_2 = \{x \in A : x \vee y \in D_2,$ *for all* $y \in D_1\}$.

Corollary 3.3. *$(Ds(A), \vee, \cap, \rightsquigarrow, \{1\}, A)$ is a Heyting algebra, where for $D \in Ds(A)$,*

$$D^{\diamond} = \{x \in A : x \vee y = 1, \text{ for every } y \in D\},$$

hence for every $x \in D$ and $y \in D^{\diamond}, x \vee y = 1$. In particular, for every $a \in A$,

$$[a)^{\diamond} = \{x \in A : x \vee a = 1\}.$$

Clearly, $D^{\diamond}$ is the pseudocomplement of D in the lattice $Ds(A)$.

Remark 3.2. *From Lemma 3.1, (ii), we deduce that if D_1, $D_2 \in Ds(A)$ and $x \in A$ such that $x \in D_1$ and $x \in D_1 \rightsquigarrow D_2$, then $x \in D_2$. Also, if $D \in Ds(A)$ then $D \rightsquigarrow D = A$ and $D \subseteq D^{\diamond\diamond}$.*

Proposition 3.4. *$D^{\diamond} = \{a \in A : a \rightarrow x = x$ and $x \rightarrow a = a$, for every $x \in D\}$.*

Proof. Let $a \in D^{\diamond}$. Since $1 = a \vee x \leq [(a \rightarrow x) \rightarrow x] \wedge [(x \rightarrow a) \rightarrow a]$ for every $x \in D$ we deduce that $(a \rightarrow x) \rightarrow x = (x \rightarrow a) \rightarrow a = 1$, hence $a \rightarrow x = x$ and $x \rightarrow a = a$, for every $x \in D$. ■

For $X \subseteq A$ we denote by $X^* = \{a \in A : a \rightarrow x = x$, for any $x \in X\}$.

Proposition 3.5. *$X^* \in Ds(A)$, for every set $X \subseteq A$.*

Proof. Obvious $1 \in X^*$ since by c_1, $1 \rightarrow x = x$, for any $x \in X$. Let $a, b \in X^*$. Then $a \rightarrow x = x$ and $b \rightarrow x = x$, for any $x \in X$. By c_5, we have $(a \odot b) \rightarrow x = a \rightarrow (b \rightarrow x) = a \rightarrow x = x$, hence $a \odot b \in X^*$. If $a \leq b$ and $a \in X^*$ then $a \rightarrow x = x$, for any $x \in X$. By c_4, $1 = a \rightarrow b \leq (b \rightarrow x) \rightarrow (a \rightarrow x)$, so $(b \rightarrow x) \rightarrow (a \rightarrow x) = 1$. Using c_1, $x \leq b \rightarrow x \leq a \rightarrow x = x$, for every $x \in X$, so $b \rightarrow x = x$. We deduce $b \in X^*$.■

Proposition 3.6. *If $D \in Ds(A)$, then $D^{\diamond} \subseteq D^*$.*

Proof. Let $a \in D^{\diamond}$ and $x \in D$. Then $a \vee x = 1 \Rightarrow (a \vee x) \to x = 1 \to x = x \overset{c_9}{\Rightarrow} (a \to x) \wedge (x \to x) = x \Rightarrow (a \to x) \wedge 1 = x \Rightarrow a \to x = x \Rightarrow a \in D^* \Rightarrow D^{\diamond} \subseteq D^*$.■

Remark 3.3. *By Remark 2.1, if the residuated lattice A is a $MV-$algebra then $D^{\diamond} = D^*$.*

Proposition 3.7. *For every subset $X \subseteq A$, we have $X \cap X^* = \emptyset$ or $X \cap X^* = \{1\}$.*

Proof. If $1 \in X$, since $X^* \in Ds(A)$ we deduce that $1 \in X \cap X^*$. Let $x \in X \cap X^*$. Then $x \to x = x$, so $x = 1$ and $X \cap X^* = \{1\}$.

If $1 \notin X$ we prove that $X \cap X^* = \emptyset$. Suppose that exists $x \in X \cap X^*$, obvious, $x \neq 1$. Then $x \to x = x$, so $x = 1$, a contradiction.■

Corollary 3.4. *If $D \in Ds(A)$, then $D \cap D^* = \{1\}$.*

Lemma 3.2. *Let X, Y two subsets of A. If $X \subseteq Y$ then $Y^* \subseteq X^*$.*

Proof. Let $y \in Y^*$. Then $y \to z = z$, for every $z \in Y$. Since $X \subseteq Y$ we deduce that $y \to z = z$, for every $z \in X$, so $y \in X^*$, that is, $Y^* \subseteq X^*$.■

Proposition 3.8. *Let $D_1, D_2 \in Ds(A)$. Then $D_1 \cap D_2 = \{1\}$ iff $D_1 \subseteq D_2^*$.*

Proof. Suppose that $D_1 \cap D_2 = \{1\}$. Let $d_1 \in D_1$. For any $d_2 \in D_2$, $d_2, d_1 \leq (d_1 \to d_2) \to d_2$ so $(d_1 \to d_2) \to d_2 \in D_1 \cap D_2 = \{1\}$. We obtain $d_1 \to d_2 = d_2$, hence $d_1 \in D_2^*$.

Conversely, we assume that $D_1 \subseteq D_2^*$. Since $D_1, D_2 \in Ds(A)$, $1 \in D_1 \cap D_2 \subseteq D_2^* \cap D_2 = \{1\}$, by Remark 3.4, that is, $D_1 \cap D_2 = \{1\}$. ■

Lemma 3.3. *If $D \in Ds(A)$ then $D \subseteq D^{**}$.*

Proof. Let $d \in D$. For any $x \in D^*$, since D, D^* are deductive systems and $x, d \leq (d \to x) \to x$, we deduce that $(d \to x) \to x \in D \cap D^* = \{1\}$, so, $d \to x = x$, hence $D \subseteq D^{**}$. ■

Remark 3.4. *The set of deductive systems $Ds(A)$ forms two pseudocomplemented lattices (with $*$ and with $\diamond$). By Remark 3.3, if the residuated lattice A is a MV-algebra, then the two pseudocomplemented lattices coincide.*

Remark 3.5. *It follows from Glivenko's theorem that the sets $R_*(Ds(A)) = \{D \in Ds(A) : D = D^{**}\}$ and $R_{\diamond}(Ds(A)) = \{D \in Ds(A) : D = D^{\diamond\diamond}\}$ are Boolean algebras. For $D_1, D_2 \in Ds(A)$, $(D_1^* \cap D_2^*)^*$ (respectively, $(D_1^{\diamond} \cap D_2^{\diamond})^{\diamond}$) is the least deductive system including D_1, D_2. Hence for $D_1, D_2 \in Ds(A)$, we have $sup\{D_1, D_2\}$ in $R_*(Ds(A))$ (respectively, $R_{\diamond}(Ds(A))$) is $(D_1^* \cap D_2^*)^*$ (respectively, $(D_1^{\diamond} \cap D_2^{\diamond})^{\diamond}$).*

Remark 3.6. *If $D \in Ds(A)$ then ($D = D^{**}$ iff $D \vee D^* = A$) and ($D = D^{\diamond\diamond}$ iff $D \vee D^{\diamond} = A$).*

Theorem 3.1. $R_{\diamond}(Ds(A)) \subseteq R_*(Ds(A))$.

Proof. By Proposition 3.6, we have $D^{\diamond} \subseteq D^*$. Let $D \in R_{\diamond}(Ds(A))$. Then $D \vee D^{\diamond} = A$. But $A = D \vee D^{\diamond} \subseteq D \vee D^*$, so $D \vee D^* = A$, hence $D \in R_*(Ds(A))$. ■

Proposition 3.9. *The following assertions are equivalent:*

(*i*) $e \in B(A)$;
(*ii*) $[e)^{\diamond} = [e^*)$;
(*iii*) $[e)^{\diamond\diamond} = [e)$.

Proof. $(i) \Rightarrow (ii)$. Let $e \in B(A)$. Since $e \vee e^* = 1$ and $[e)^\diamond = \{x \in A : e \vee x = 1\}$ we deduce that $e^* \in [e)^\diamond$, so $[e^*) \subseteq [e)^\diamond$. If $x \in [e)^\diamond$, since $e \vee x = 1$, we have $e^* = e^* \wedge 1 = e^* \wedge (e \vee x) \overset{c_{13}}{=} e^* \odot (e \vee x) \overset{c_9}{=} (e^* \odot e) \vee (e^* \odot x) \overset{c_{13}}{=} 0 \vee (e^* \wedge x) = e^* \wedge x$, so $e^* \leq x$. It follow that $x \in [e^*)$ and we deduce $[e)^\diamond = [e^*)$.

$(ii) \Rightarrow (i)$. Using Proposition 2.2, $[e)^\diamond = [e^*) \Rightarrow e^* \in [e)^\diamond \Rightarrow e \vee e^* = 1 \Rightarrow e \in B(A)$.

$(i) \Rightarrow (iii)$. $e \in B(A) \Rightarrow [e)^{\diamond\diamond} = [e^*)^\diamond \overset{e^* \in B(A)}{=} [e^{**}) = [e)$.

$(iii) \Rightarrow (i)$. Since $[e)^{\diamond\diamond} = \{x \in A : x \vee y = 1, \text{for every } y \in [e)^\diamond\} = \{x \in A : x \vee y = 1,$ for every $y \in [e^*)\} = \{x \in A : x \vee y = 1,$ for every $y \geq e^*\}$ and $e \in [e) = [e)^{\diamond\diamond}$ we deduce that $e \vee e^* = 1$, so $e \in B(A)$.■

Remark 3.7. *If* $e \in B(A)$, *then* $[e) \in R_\diamond(Ds(A))$.

Theorem 3.2. *Let* $D \in Ds(A)$. *The following assertions are equivalent:*

(i)**:** $D \in R_\diamond(Ds(A))$;

(ii)**:** *there is* $e \in B(A)$ *such that* $D = [e)$.

Proof. $(i) \Rightarrow (ii)$. Let $D \in R_\diamond(Ds(A))$; since $D \vee D^\diamond = A$, there exist $e \in D$, $a \in D^\diamond$ such that $e \odot a = 0$.

Since $a \in D^\diamond$, we have $a \vee e = 1$. Using c_2 we deduce that $a \wedge e = a \odot e = 0$, that is, $e \in B(A)$.

For every $x \in D$, $a \vee x = 1$. We have $e \wedge x = 0 \vee (e \wedge x) = (e \wedge a) \vee (e \wedge x) \overset{c_{14}}{=} e \wedge (a \vee x) = e \wedge 1 = e$, so $e \leq x$, that is, $D = [e)$.

$(ii) \Rightarrow (i)$. By Proposition 3.9, (iii). ■

We say that the inverse image of an deductive system under a morphism of residuated lattices is also a deductive system. Hence we have the following results:

Theorem 3.3. *Let* A, B *two residuated lattices and* $f : A \to B$ *a morphism of residuated lattice. If* Y *is a nonempty subset of* B, *then* $f^{-1}(Y^*)$ *is a deductive system of* A *containing* $[f^{-1}(Y)]^*$. *Moreover, if* D *is deductive system of* B, *then* $f^{-1}(D^\diamond)$ *is a deductive system of* A *containing* $[f^{-1}(D)]^\diamond$.

Theorem 3.4. *Let* A, B *two residuated lattices,* $f : A \to B$ *a morphism of residuated lattice and* $X \subseteq A$ *a nonempty subset of* A. *Then* $f(X^*) \subseteq [f(X)]^*$.

Proof. Let $b \in f(X^*)$ and $y \in f(X)$. Then there exist $a \in X^*$ and $x \in X$ such that $f(a) = b$ and $f(x) = y$. Since $a \in X^*$ and $x \in X$ we deduce that $a \to x = x$. It follows that $b \to y = f(a) \to f(x) = f(a \to x) = f(x) = y$, so, $b \in [f(X)]^*$. We deduce that $f(X^*) \subseteq [f(X)]^*$.■

Theorem 3.5. *Let* A, B *two residuated lattices,* $f : A \to B$ *a surjective morphism of residuated lattice and* $D \in Ds(A)$. *Then*

(i)**:** $f(D^\diamond), f(D^*) \in Ds(B)$;

(ii)**:** $f(D^\diamond) \subseteq [f(D)]^\diamond$ *and* $f(D^*) \subseteq [f(D)]^*$;

(iii)**:** *If* D^* (*respectively* $D^\diamond$) *is a maximal deductive system of* A *such that* $f(D^*)$ (*respectively* $f(D^\diamond)$) *is a proper, then* $f(D^*)$ (*respectively* $f(D^\diamond)$) *is a maximal deductive system of* B.

Proof. (i). Obviously, $1 = f(1) \in f(D^\diamond)$. Let $x, y \in f(D^\diamond)$, that is there are $a, b \in D^\diamond$ such that $f(a) = x$ and $f(b) = y$. Since $D^\diamond \in Ds(A)$, we deduce that $a \odot b \in D^\diamond$ and $x \odot y = f(a) \odot f(b) = f(a \odot b) \in f(D^\diamond)$. Let $x, y \in B$ such that $x \leq y$ and $x \in f(D^\diamond)$. Hence, there is $a \in D^\diamond$ such that $f(a) = x$ and since f is surjective, there exists $b \in A$ such that $f(b) = y$. Then $y = x \vee y = f(a) \vee f(b) = f(a \vee b)$ and $a \vee b \geq a \in D^\diamond$, so $a \vee b \in D^\diamond$ and $y \in f(D^\diamond)$. We obtain that $f(D^\diamond) \in Ds(B)$. Similarly for $f(D^*) \in Ds(B)$.

(ii). Following from Theorem 3.4.

(iii). Let D' be a proper deductive system of B such that $f(D^*) \subseteq D'$. We have that $D^* \subseteq f^{-1}(f(D^*)) \subseteq f^{-1}(D^{'})$ and since $f^{-1}(D^{'})$ is a proper deductive system of A, we must have $D^* = f^{-1}(D^{'})$. We deduce that $f(D^*) = f(f^{-1}(D^{'})) = D'$, since f is a surjective morphism. Similarly for $f(D^\diamond)$.■

Remark 3.8. *For $D \in Ds(A)$, if $D^\diamond$ is a maximal deductive system of A, by Remark 3.6 we deduce that $D^\diamond = D^*$, and by Theorem 3.5 if $f : A \to B$ is a surjective morphism of residuated lattice, then $f(D^*) = f(D^\diamond)$ is a maximal deductive system of B.*

With any deductive system D of A we can (see [12], [18]) associate a congruence θ_D on A by defining : $(a, b) \in \theta_D$ iff $a \to b, b \to a \in D$ iff $(a \to b) \odot (b \to a) \in D$. Conversely, for $\theta \in Con(A)$, the subset D_θ of A defined by $a \in D_\theta$ iff $(a, 1) \in \theta$ is a deductive system of A. Moreover the natural maps associated with the above are mutually inverse and establish an isomorphism between the lattices $Ds(A)$ and $Con(A)$.

For $a \in A$, let a/D be the equivalence class of a modulo θ_D. If we denote by A/D the quotient set A/θ_D, then A/D becomes a residuated lattice with the natural operations induced from those of A. Clearly, in A/D, $\mathbf{0} = 0/D$ and $\mathbf{1} = 1/D$.

Proposition 3.10. *Let $D \in Ds(A)$, and $a, b \in A$, then*
- (i) *$a/D = 1/D$ iff $a \in D$, hence $a/D \neq \mathbf{1}$ iff $a \notin D$;*
- (ii) *$a/D = 0/D$ iff $a^* \in D$;*
- (iii) *If D is proper and $a/D = 0/D$, then $a \notin D$;*
- (iv) *$a/D \leq b/D$ iff $a \to b \in D$.*

Remark 3.9. *Let A, B two residuated lattices. We define on $A \times B$, the operations $\wedge_\times, \vee_\times, \odot_\times, \to_\times$ for every $(a, b), (a', b') \in A \times B$ by $(a, b) \wedge_\times (a', b') = (a \wedge a', b \wedge b')$, $(a, b) \vee_\times (a', b') = (a \vee a', b \vee b')$, $(a, b) \odot_\times (a', b') = (a \odot a', b \odot b')$, $(a, b) \to_\times (a', b') = (a \to a', b \to b')$. Clearly, $(A \times B, \wedge_\times, \vee_\times, \odot_\times, \to_\times, (0, 0), (1, 1))$ is a residuated lattice.*

Theorem 3.6. *Let X and Y be nonempty subsets of residuated lattices A and B, respectively. Then:*

(i)**:** $X^* \times Y^* = (X \times Y)^*$

(ii)**:** $A/X^* \times B/Y^* \approx (A \times B)/(X \times Y)^*$.

Proof. (i). We have that $(X \times Y)^* = \{(a, b) \in A \times B : (a, b) \to (x, y) = (x, y)$, for all $(x, y) \in X \times Y\} = \{(a, b) \in A \times B : (a \to x, b \to y) = (x, y)$, for all $(x, y) \in X \times Y\} = \{(a, b) \in A \times B : a \to x = x$ and $b \to y = y$, for all $(x, y) \in X \times Y\} = \{a \in A : a \to x = x$, for all $x \in X\} \times \{b \in B : b \to y = y$, for all $y \in Y\} = X^* \times Y^*$.

(ii). Note that $X^* \times Y^* \in Ds(A \times B)$. Consider the surjective morphisms $p_{X^*} : A \to A/X^*, p_{X^*}(a) = a/X^*$ for every $a \in A$ and $p_{Y^*} : B \to B/Y^*, p_{Y^*}(b) = b/Y^*$ for every $b \in B$. We define $f : (A \times B) \to A/X^* \times B/Y^*$ by $f(a, b) = (a/X^*, b/Y^*)$, for every $(a, b) \in A \times B$. Then f is a surjective morphisms. We denote the filter kernel by $Ker(f) = f^{-1}((1/X^*, 1/Y^*))$ and using Proposition 3.10, $Ker(f) = \{(a, b) \in A \times B : f(a, b) = (1/X^*, 1/Y^*)\} = \{(a, b) \in A \times B : (a/X^*, b/Y^*) = (1/X^*, 1/Y^*)\} = \{(a, b) \in A \times B : a/X^* = 1/X^*, b/Y^* = 1/Y^*\} = \{(a, b) \in A \times B : a \in X^*, b \in Y^*\} = X^* \times Y^*$.

By the first isomorphism theorem and (i), we deduce that $(A \times B)/(X \times Y)^* \approx A/X^* \times B/Y^*$.■

Analogously we obtain:

Theorem 3.7. *Let A and B two residuated lattices and $D_1 \in Ds(A), D_2 \in Ds(B)$. Then:*
(i)**:** $D_1^{\diamond} \times D_2{}^{\diamond} = (D_1 \times D_2)^{\diamond}$
(ii)**:** $A/D_1^{\diamond} \times B/D_2^{\diamond} \approx (A \times B)/(D_1 \times D_2)^{\diamond}$.

4. The lattice $\mathbf{Ds}_p^{\diamond}(A)$

We denote by $Ds_p^{\diamond}(A) = \{[a)^{\diamond} : a \in A\}$.

Proposition 4.1. *If $a, b \in A$, then*
(i)**:** $a \leq b \Rightarrow [a)^{\diamond} \subseteq [b)^{\diamond}$;
(ii)**:** $[a)^{\diamond} \cap [b)^{\diamond} = [a \odot b)^{\diamond} = [a \wedge b)^{\diamond}$;
(iii)**:** $[a \to b)^{\diamond} \subseteq [a)^{\diamond} \rightsquigarrow [b)^{\diamond}$;
(iv)**:** $[a \vee a^*)^{\diamond} = [a)^{\diamond} \vee [a^*)^{\diamond}$.

Proof. (i). If $x \in [a)^{\diamond}$ then $x \vee a = 1$, but $a \leq b$, hence $1 = x \vee a \leq x \vee b$, so $x \vee b = 1$. We deduce $x \in [b)^{\diamond}$.

(ii). We have $a \odot b \leq a \wedge b \leq a, b$, then using (i) we deduce that $[a \odot b)^{\diamond} \subseteq [a \wedge b)^{\diamond} \subseteq [a)^{\diamond}, [b)^{\diamond}$, that is, $[a \odot b)^{\diamond} \subseteq [a \wedge b)^{\diamond} \subseteq [a)^{\diamond} \cap [b)^{\diamond}$.

Let now $x \in [a)^{\diamond} \cap [b)^{\diamond}$, that is, $a \vee x = b \vee x = 1$.

By c_{10}, $x \vee (a \odot b) \geq (x \vee a) \odot (x \vee b) = 1$, hence $x \vee (a \odot b) = 1$, that is, $x \in [a \odot b)^{\diamond}$.

It follows that $[a)^{\diamond} \cap [b)^{\diamond} \subseteq [a \odot b)^{\diamond}$, hence $[a)^{\diamond} \cap [b)^{\diamond} = [a \odot b)^{\diamond} = [a \wedge b)^{\diamond}$.

(iii). Let $x \in [a \to b)^{\diamond} \Leftrightarrow x \vee (a \to b) = 1$. We have that $x \in [a)^{\diamond} \rightsquigarrow [b)^{\diamond} \Leftrightarrow x \vee y \in [b)^{\diamond}$, for any $y \in [a)^{\diamond}$. Let so $y \in [a)^{\diamond} \Leftrightarrow a \vee y = 1$. We prove that $b \vee (x \vee y) = 1$.

By c_{12} we deduce $1 = x \vee (a \to b) \leq (x \vee a) \to (x \vee b) \Rightarrow 1 = (x \vee a) \to (x \vee b) \Rightarrow x \vee a \leq x \vee b$. Then $x \vee y \vee a \leq x \vee y \vee b \Rightarrow x \vee 1 \leq x \vee y \vee b \Rightarrow x \vee y \vee b = 1 \Rightarrow x \in [a)^{\diamond} \rightsquigarrow [b)^{\diamond}$.

(iv). Since $a, a^* \leq a \vee a^*$ we deduce by (i), that $[a)^{\diamond}, [a^*)^{\diamond} \subseteq [a \vee a^*)^{\diamond} \Rightarrow [a)^{\diamond} \vee [a^*)^{\diamond} \subseteq [a \vee a^*)^{\diamond}$.

Conversely, let $x \in [a \vee a^*)^{\diamond}$. We have $x \vee (a \vee a^*) = 1 \Rightarrow (x \vee a) \vee a^* = 1$ and $(x \vee a^*) \vee a = 1 \Rightarrow x \vee a \in [a^*)^{\diamond}$ and $x \vee a^* \in [a)^{\diamond}$.

By c_{10}, $x = x \vee (a \odot a^*) \geq (x \vee a) \odot (x \vee a^*)$. Since $x \vee a \in [a^*)^{\diamond}$ and $x \vee a^* \in [a)^{\diamond}$ we deduce that $x \in [a)^{\diamond} \vee [a^*)^{\diamond}$, so $[a \vee a^*)^{\diamond} \subseteq [a)^{\diamond} \vee [a^*)^{\diamond}$.

Finally, $[a \vee a^*)^{\diamond} = [a)^{\diamond} \vee [a^*)^{\diamond}$. ■

Remark 4.1. $[a)^{\diamond} \rightsquigarrow [b)^{\diamond} \subsetneq [a \to b)^{\diamond}$. *Indeed, if we consider the residuated lattice A from Example 2.2, then $[0)^{\diamond} = [a)^{\diamond} = [b)^{\diamond} = [c)^{\diamond} = \{1\}, [1)^{\diamond} = A$ and $[a)^{\diamond} \rightsquigarrow [b)^{\diamond} = \{x \in A : x \vee 1 = 1\} = A$ but $[a \to b)^{\diamond} = [b)^{\diamond} = \{1\}$.*

Proposition 4.2. *If $e \in B(A)$ and $[e)^{\diamond} = \{1\}$ then $e = 0$.*

Proof. Since by Propositions 3.3 and 3.9, $[e)^{\diamond} = [e^*) = \{x \in A : x \geq e^*\} = \{1\}$ and $e^* \in [e^*)$ we deduce that $e^* = 1$ so $e = 0$. ■

Remark 4.2. *Since for every $a \in A$, $[a)^{\diamond}$ is the pseudocomplement of $[a)$ in the lattice $Ds(A)$, then:*
(i)**:** $[a)^{\diamond} = A \Leftrightarrow a = 1$ *and* $[0)^{\diamond} = \{1\}$;
(ii)**:** $[a) \cap [a)^{\diamond} = \{1\}$;
(iii)**:** $[a)^{\diamond} \cap [a)^{\diamond\diamond} = \{1\}$;
(iv)**:** $[a)^{\diamond} = [a)^{\diamond\diamond\diamond}$.

Definition 4.1. *An element a in a residuated lattice A is called* nilpotent *iff there exists a natural number n such that $a^n = 0$. The minimum n such that $a^n = 0$ is* *called* nilpotence order *of* a *and will be denoted by* $ord(a)$; *if there is no such* n, *then* $ord(a) = \infty$. *A residuated lattice* A *is called* locally finite *if every* $a \in A, a \neq 1$, *has finite order.*

Proposition 4.3. *Let* $a \in A$ *and a natural number* n. *Then* $[a)^{\diamond} = [a^n)^{\diamond}$.

Proof . By Proposition 4.1, (i), since $a^n \leq a$ we obtain $[a^n)^{\diamond} \subseteq [a)^{\diamond}$. Conversely, let $x \in [a)^{\diamond}$. Then $a \vee x = 1$. By c_{11}, $1 = a^n \vee x^n \leq a^n \vee x$. We deduce that $a^n \vee x = 1$, so $x \in [a^n)^{\diamond}$ and $[a)^{\diamond} \subseteq [a^n)^{\diamond}$. Finally, $[a)^{\diamond} = [a^n)^{\diamond}$. ■

Proposition 4.4. *Let* $a \in A, a \neq 1$, *such that* a *has a finite order* n. *Then* $[a)^{\diamond} = \{1\}$.

Proof 1. Since n is the finite order of a we have $a^n = 0$. By Proposition 4.3, $[a)^{\diamond} = [a^n)^{\diamond} = [0)^{\diamond} = \{1\}$.

Proof 2. By definition, $[a)^{\diamond} = \{x \in A : a \vee x = 1\}$. Let $x \in [a)^{\diamond}$. Since $1 = a \vee x \leq [(a \to x) \to x] \wedge [(x \to a) \to a]$ we deduce that $(a \to x) \to x = (x \to a) \to a = 1$, hence $a \to x = x$ and $x \to a = a$. Now $x = a \to x = a \to (a \to x) = a^2 \to x = ... = a^n \to x = 0 \to x = 1$. We deduce that $[a)^{\diamond} = \{1\}$.■

For $a, b \in A$ we denote

$$[a)^{\diamond} \veebar [b)^{\diamond} = [a)^{\diamond\diamond} \rightsquigarrow [b)^{\diamond}.$$

Proposition 4.5. *Let* $a, b \in A$. *Then* $[a)^{\diamond} \veebar [b)^{\diamond} = [a \vee b)^{\diamond}$.

Proof. By Lemma 3.1, $[a)^{\diamond} \veebar [b)^{\diamond} = [a)^{\diamond\diamond} \rightsquigarrow [b)^{\diamond} = \{x \in A : x \vee y \in [b)^{\diamond}$, for every $y \in [a)^{\diamond\diamond}\} = \{x \in A : x \vee y \vee b = 1$, for every $y \in [a)^{\diamond\diamond}\}$ and $[a)^{\diamond\diamond} = \{x \in A : x \vee y = 1$, for every $y \in [a)^{\diamond}\} = \{x \in A : x \vee y = 1$, for every $y \in A$ such that $y \vee a = 1\}$. Clearly, $a \in [a)^{\diamond\diamond}$, so for any $x \in [a)^{\diamond} \veebar [b)^{\diamond}$ we obtain $x \vee a \vee b = 1$. This implies $x \in [a \vee b)^{\diamond}$, hence $[a)^{\diamond} \veebar [b)^{\diamond} \subseteq [a \vee b)^{\diamond}$.

Now, we prove that $[a \vee b)^{\diamond} \subseteq [a)^{\diamond} \veebar [b)^{\diamond}$. Let $x \in [a \vee b)^{\diamond}$, that is, $x \vee a \vee b = 1$. Let $y \in [a)^{\diamond\diamond}$. We deduce $y \vee z = 1$, for $z \in A$ such that $z \vee a = 1$. If we denote $t = x \vee b$ we will prove that $(t \vee a = 1 \Rightarrow t \vee y = 1$, for every $y \in [a)^{\diamond\diamond})$ equivalent with $(t \in [a)^{\diamond} \Rightarrow t \in [a)^{\diamond\diamond\diamond})$ equivalent with $[a)^{\diamond} \subseteq [a)^{\diamond\diamond\diamond}$. It is an immediate consequence of Remark 4.2, (iv). ■

Corollary 4.1. *For* $a, b \in A$, $[a)^{\diamond} \veebar [b)^{\diamond} = [a \vee b)^{\diamond} \in Ds_p^{\diamond}(A)$.

Remark 4.3. *If* $a, b \in A$, *then* $[a)^{\diamond}, [b)^{\diamond} \subseteq [a)^{\diamond} \veebar [b)^{\diamond}$ *so,* $[a)^{\diamond} \vee [b)^{\diamond} \subseteq [a)^{\diamond} \veebar [b)^{\diamond} = [a \vee b)^{\diamond}$.

Remark 4.4. *By Proposition 4.1,* $[a)^{\diamond} \vee [a^*)^{\diamond} = [a)^{\diamond} \veebar [a^*)^{\diamond} = [a \vee a^*)^{\diamond}$.

Proposition 4.6. $a \in B(A) \Leftrightarrow [a)^{\diamond} \veebar [a^*)^{\diamond} = A$.

Proof. By Proposition 4.5, if $a \in B(A)$ then $[a)^{\diamond} \veebar [a^*)^{\diamond} = [a \vee a^*)^{\diamond} = [1)^{\diamond} = A$. Conversely, $[a)^{\diamond} \veebar [a^*)^{\diamond} = [a \vee a^*)^{\diamond} = A$ implies $0 \vee (a \vee a^*) = 1 \Rightarrow a \vee a^* = 1$. By Proposition 2.2 we deduce that $a \in B(A)$.■

Theorem 4.1. $(Ds_p^{\diamond}(A), \cap, \veebar, \{1\}, A = [1)^{\diamond})$ *is a bounded distributive lattice and* $[a)^{\diamond} = [a)^{\diamond\diamond\diamond}, [a)^{\diamond} \cap [a)^{\diamond\diamond} = \{1\}, [a)^{\diamond\diamond} \cap [b)^{\diamond\diamond} = ([a)^{\diamond} \veebar [b)^{\diamond})^{\diamond} = [a \vee b)^{\diamond\diamond}$, *for* $a, b \in A$.

Proof. We shall prove that $\veebar$ is the supremum in this lattice.

It is obvious that, by Proposition 4.1, $a, b \leq a \vee b$ implies $[a)^{\diamond}$, $[b)^{\diamond} \subseteq [a \vee b)^{\diamond}$, $a, b \in A$. For $c \in A$ such that $[a)^{\diamond}$, $[b)^{\diamond} \subseteq [c)^{\diamond}$ we will prove that $[a \vee b)^{\diamond} \subseteq [c)^{\diamond}$. If $t \in [a \vee b)^{\diamond}$, then $t \vee a \vee b = 1$, so $t \vee a \in [b)^{\diamond} \subseteq [c)^{\diamond}$. We deduce that $(t \vee c) \vee a = 1$, so $t \vee c \in [a)^{\diamond}$. But $[a)^{\diamond} \subseteq [c)^{\diamond}$, implies $t \vee c \in [c)^{\diamond}$ implies $t \vee c = 1$ implies $t \in [c)^{\diamond}$.

Thus, $[a \vee b)^{\diamond} \subseteq [c)^{\diamond}$.

Since using Proposition 4.1, $[a)^{\diamond} \cap ([b)^{\diamond} \veebar [c)^{\diamond}) = [a)^{\diamond} \cap [b \vee c)^{\diamond} = [a \odot (b \vee c))^{\diamond} \overset{c_9}{=} [(a \odot b) \vee (a \odot c))^{\diamond} = [a \odot b)^{\diamond} \veebar [a \odot c)^{\diamond} = ([a)^{\diamond} \cap [b)^{\diamond}) \veebar ([a)^{\diamond} \odot [c)^{\diamond})$, for every $a, b, c \in A$ and $\{1\} = [0)^{\diamond}, A = [1)^{\diamond}$, we deduce that the lattice $(Ds_p^{\diamond}(A), \cap, \veebar, \{1\}, A)$ is distributive and bounded.

Applying Remark 4.2 we get that $[a)^{\diamond} = [a)^{\diamond\diamond\diamond}, [a)^{\diamond} \cap [a)^{\diamond\diamond} = \{1\}$.

The equality $[a)^{\diamond\diamond} \cap [b)^{\diamond\diamond} = ([a)^{\diamond} \veebar [b)^{\diamond})^{\diamond}$, for $a, b \in A$ is equivalent with $[a)^{\diamond\diamond} \cap [b)^{\diamond\diamond} = [a \vee b)^{\diamond\diamond}$, for $a, b \in A$.

Let $x \in [a)^{\diamond\diamond} \cap [b)^{\diamond\diamond}$. We deduce that $x \vee y = 1$, for every $y \in [a)^{\diamond}$ and $x \vee z = 1$, for every $z \in [b)^{\diamond}$. Let $t \in [a \vee b)^{\diamond}$. We obtain $t \vee a \vee b = 1 \Rightarrow t \vee a \in [b)^{\diamond} \Rightarrow x \vee t \vee a = 1 \Rightarrow x \vee t \in [a)^{\diamond} \Rightarrow x \vee (x \vee t) = 1 \Rightarrow x \vee t = 1$. Thus, $[a)^{\diamond\diamond} \cap [b)^{\diamond\diamond} \subseteq [a \vee b)^{\diamond\diamond}$.

Conversely, let $x \in [a \vee b)^{\diamond\diamond}$. Then $x \vee z = 1$, for every $z \in A$ such that $z \vee a \vee b = 1$. Let $y_1 \in [a)^{\diamond}$. Then $y_1 \vee a = 1 \Rightarrow y_1 \vee a \vee b = 1 \Rightarrow x \vee y_1 = 1 \Rightarrow x \in [a)^{\diamond\diamond}$. Let $y_2 \in [b)^{\diamond}$. Then $y_2 \vee b = 1 \Rightarrow y_2 \vee a \vee b = 1 \Rightarrow x \vee y_2 = 1 \Rightarrow x \in [b)^{\diamond\diamond}$. Thus, $x \in [a)^{\diamond\diamond} \cap [b)^{\diamond\diamond}$.

Finally, $[a \vee b)^{\diamond\diamond} \subseteq [a)^{\diamond\diamond} \cap [b)^{\diamond\diamond}$, so $[a \vee b)^{\diamond\diamond} = [a)^{\diamond\diamond} \cap [b)^{\diamond\diamond}$. ■

Remark 4.5. *If A is a chain then $Ds_p^{\diamond}(A)$ is isomorphic with L_2, the two-elements Boolean algebra. Indeed, for $a \in A, a \neq 1$, $[a)^{\diamond} = \{1\}$ and $[1)^{\diamond} = A$.*

Remark 4.6. *If A is a locally finite residuated lattice, then every element of A has a finite order and by Proposition 4.4 we deduce that $Ds_p^{\diamond}(A)$ is a Boolean algebra isomorphic with L_2.*

Remark 4.7. *We recall that a residuated lattice is subdirectly irreducible iff it is nontrivial and for any subdirect representation $f : A \to \prod_{i \in I} A_i$, there exists a j such that f_j is an isomorphism of A onto A_j. In* [12] *it is proved that in any subdirectly irreducible residuated lattice, if $x \vee y = 1$, then $x = 1$ or $y = 1$. Obviously, if A is a subdirectly irreducible residuated lattice, then $Ds_p^{\diamond}(A)$ is a Boolean algebra isomorphic with L_2.*

Remark 4.8. *If $e, f \in B(A)$, then $[e)^{\diamond} \veebar [f)^{\diamond} = [e \vee f)^{\diamond} \overset{e \vee f \in B(A)}{=} [(e \vee f)^*) = [e^* \wedge f^*) = [e^* \odot f^*) = [e^*) \vee [f^*) = [e)^{\diamond} \vee [f)^{\diamond}$, and $[e)^{\diamond} \veebar [e)^{\diamond\diamond} = [e)^{\diamond} \veebar [e^*)^{\diamond} = [e \vee e^*)^{\diamond} = [1)^{\diamond} = A$.*

Remark 4.9. *If $e \in B(A)$, then $[e)^{\diamond} \in B(Ds_p^{\diamond}(A))$, so $Ds_p^{\diamond}(B(A))$ is a Boolean subalgebra of $B(Ds_p^{\diamond}(A)$.*

In [5] we introduce and characterize the hyperarchimedean residuated lattice.

Definition 4.2. [5] *Let A be a residuated lattice. An element $a \in A$ is called* archimedean *if it satisfy the condition : there is $n \geq 1$ such that $a^n \in B(A)$,(equivalent with $a \vee (a^n)^* = 1$). A residuated lattice A is called* hyperarchimedean *if all its elements are archimedean.*

Proposition 4.7. *If A is a hyperarchimedean residuated lattice then $Ds_p^{\diamond}(A)$ is a Boolean subalgebra of $Ds(A)$.*

Proof. *Since A is a hyperarchimedean residuated lattice then for every $a \in A$ there is a natural number $n \geq 1$ such that $a^n = e_a \in B(A)$. By Proposition 4.3, $[a)^{\diamond} = [a^n)^{\diamond} = [e_a)^{\diamond}$. We deduce that $\vee = \veebar$ and $Ds_p^{\diamond}(A)$ is a Boolean algebra.*

Theorem 4.2. *If A is a residuated lattice, then the map*

$$f : (A, \wedge, \vee, 0, 1) \to (Ds_p^{\diamond}(A), \cap, \veebar, \{1\}, A),$$

defined by $f(a) = [a)^{\diamond}$, for every $a \in A$ is an ontomorphism of distributive and bounded lattices.

Proof. Let $a, b \in A$. Applying Proposition 4.1 and Corollary 4.1 we obtain that $f(a \wedge b) = [a \wedge b)^{\diamond} = [a)^{\diamond} \cap [b)^{\diamond} = f(a) \cap f(b)$, $f(a \vee b) = [a \vee b)^{\diamond} = [a)^{\diamond} \veebar [b)^{\diamond} = f(a) \veebar f(b)$, $f(0) = [0)^{\diamond} = \{1\}$ and $f(1) = [1)^{\diamond} = A$.■

In [1], if $f : L_1 \to L_2$ is a morphism of bounded lattices, then we denote the ideal kernel by $Ker(f) = f^{-1}(\{0\}) = \{x \in L_1 : f(x) = 0\}$.

Remark 4.10. *Using this notation, by Proposition 4.4, if we denote by* $Ord_{finite} = \{x \in A : x \text{ has a finite order}\}$*, then* $Ord_{finite} \subseteq Ker(f)$*, where* $f : A \to Ds_p^{\diamond}(A)$ *is the ontomorphism from Theorem 4.2.*

Proposition 4.8. *If A is a hyperarchimedean residuated lattice then* $Ker(f) = Ord_{finite}$ *is a proper ideal of* $L(A)$ *and* $A/Ker(f) \approx Ds_p^{\diamond}(A)$ *as Boolean algebras.*

Proof. Let $a \in Ker(f)$. Then $f(a) = \{1\} \Leftrightarrow [a)^{\diamond} = \{1\}$. Since A is a hyperarchimedean residuated lattice then for $a \in A$ there is a natural number $n \geq 1$ such that $a^n = e_a \in B(A)$. By Proposition 4.3, we deduce that $[a)^{\diamond} = [a^n)^{\diamond} = [e_a)^{\diamond}$. But Propositions 3.9 and 4.2, $\{1\} = [e_a)^{\diamond} = [e_a^*)$, so $e_a = a^n = 0$ and a has a finite order. We deduce that $Ker(f) \subseteq Ord_{finite}$. Using Remark 4.10 we deduce that $Ker(f) = Ord_{finite}$.

By Proposition 4.7, $A/Ker(f) \approx Ds_p^{\diamond}(A)$ as Boolean algebras.■

Corollary 4.2. *For every residuated lattice A,* $f_{|B(A)}$ *is an injective morphism, so* $(B(A), \wedge, \vee, 0, 1)$ *is a isomorphic with a sublattice of* $(Ds_p^{\diamond}(A), \cap, \veebar, \{1\}, A)$.

Proof. To prove the injectivity of f, let $e, g \in B(A)$ such that $f(e) = f(g)$. Then $[e)^{\diamond} = [g)^{\diamond}$. Using Proposition 3.9 we deduce that $[e^*) = [g^*)$, so $e^* = g^*$. Thus, $e = g$. ■

Proposition 4.9. *If* $e, f \in B(A)$*, then* $[e)^{\diamond} \rightsquigarrow [f)^{\diamond} = [e^* \vee f)^{\diamond} \in Ds_p^{\diamond}(A)$.

Proof. By Proposition 4.5, $[a)^{\diamond\diamond} \rightsquigarrow [b)^{\diamond} = [a)^{\diamond} \veebar [b)^{\diamond} = [a \vee b)^{\diamond}$, for every $a, b \in A$.

Applying Propositions 3.9 and Remark 4.2 we have that $[e)^{\diamond} \rightsquigarrow [f)^{\diamond} = [e)^{\diamond\diamond\diamond} \rightsquigarrow [f)^{\diamond} = [e^*)^{\diamond\diamond} \rightsquigarrow [f)^{\diamond} = [e^* \vee f)^{\diamond} \in Ds_p^{\diamond}(A)$. ■

Corollary 4.3. $(Ds_p^{\diamond}(B(A)), \cap, \veebar, ^{\diamond}, \{1\}, A)$ *is a Boolean algebra and*

$$f_{|B(A)} : (B(A), \wedge, \vee, ^*, 0, 1) \to (Ds_p^{\diamond}(B(A)), \cap, \veebar, ^{\diamond}, \{1\}, A)$$

defined by $f_{|B(A)}(e) = [e)^{\diamond} = [e^*)$*, for every* $e \in B(A)$ *is an isomorphism of Boolean algebras.*

Proof. Apply Theorems 4.1, 4.2, Corollary 4.2 and Proposition 4.9 .■

Theorem 4.3. *Let* $a, b, c \in A$*. Then* $[c)^{\diamond} \subseteq [a)^{\diamond} \rightsquigarrow [b)^{\diamond} \Leftrightarrow [a)^{\diamond} \cap [c)^{\diamond} \subseteq [b)^{\diamond}$.

Proof. From Lemma 3.1, $[a)^{\diamond} \rightsquigarrow [b)^{\diamond} = \{x \in A : x \vee y \in [b)^{\diamond}$, for all $y \in [a)^{\diamond}\}$.

Suppose that $[a)^{\diamond} \cap [c)^{\diamond} \subseteq [b)^{\diamond}$ and let $x \in [c)^{\diamond}$. We have that $x \vee c = 1$. Let $y \in [a)^{\diamond}$, so $y \vee a = 1$. By c_{10}, $(x \vee y) \vee (a \odot c) \geq (x \vee y \vee a) \odot (x \vee y \vee c) = (x \vee 1) \odot (y \vee 1) = 1 \Rightarrow (x \vee y) \vee (a \odot c) = 1 \Rightarrow x \vee y \in [a \odot c)^{\diamond}$. But $[a \odot c)^{\diamond} = [a)^{\diamond} \cap [c)^{\diamond} \subseteq [b)^{\diamond}$, so $x \vee y \in [b)^{\diamond}$, for any $y \in [a)^{\diamond}$. By definition we deduce that $x \in [a)^{\diamond} \rightsquigarrow [b)^{\diamond}$ so, $[c)^{\diamond} \subseteq [a)^{\diamond} \rightsquigarrow [b)^{\diamond}$.

Conversely if we suppose that $[c)^{\diamond} \subseteq [a)^{\diamond} \rightsquigarrow [b)^{\diamond}$, let $x \in [a)^{\diamond} \cap [c)^{\diamond} = [a \odot c)^{\diamond} = [a \wedge c)^{\diamond}$. So, $x \vee (a \wedge c) = 1$.

We have $1 = x \vee (a \wedge c) \leq (x \vee a) \wedge (x \vee c) \Rightarrow (x \vee a) \wedge (x \vee c) = 1 \Rightarrow x \vee a = x \vee c = 1 \Rightarrow x \in [a)^{\diamond}$ and $x \in [c)^{\diamond}$. But $[c)^{\diamond} \subseteq [a)^{\diamond} \rightsquigarrow [b)^{\diamond}$ so $x \in [a)^{\diamond} \rightsquigarrow [b)^{\diamond}$. Since $x \in [a)^{\diamond}$ it is easy to show applying Remark 3.2 that $x \in [b)^{\diamond}$. Obviously, $[a)^{\diamond} \cap [c)^{\diamond} \subseteq [b)^{\diamond}$.■

Remark 4.11. *Since* $(Ds_p^{\diamond}(A), \cap, [1)^{\diamond} = A)$ *is a commutative monoid using Theorems 4.1 and 4.3 we deduce that* $(Ds_p^{\diamond}(B(A)), \cap, \veebar, \rightsquigarrow, \{1\}, A)$ *is a residuated lattice.*

References

[1] R. Balbes, Ph. Dwinger: *Distributive Lattices*, University of Missouri Press, 1974.
[2] T. S. Blyth, M. F. Janovitz: *Residuation Theory*, Pergamon Press, 1972.
[3] W. J. Blok, D. Pigozzi: *Algebraizable Logics*, Memoirs of the American Mathematical Society, No. 396, Amer. Math. Soc, Providence, 1989.
[4] D. Buşneag, D. Piciu: *Residuated lattice of fractions relative to an* $\wedge-$*closed system*, Bulletin mathematique de la Société de Science Mathematiques de Roumanie, tome 49(97), no.1 (2006), 13-24.
[5] D. Buşneag, D. Piciu: *Archimedean residuated lattice,* submitted.
[6] R. Cretan, A. Jeflea: *On the lattice of congruence filters of a residuated lattice*, Annals of University of Craiova, Math. Comp. Sci. Ser., Vol.33 (2006), 174-188.
[7] R. P. Dilworth: *Non-commutative residuated lattices*, Transactions of the American Mathematical Society 46 (1939), 426-444.
[8] P. Hájek: *Metamathematics of Fuzzy Logic*, Kluwer Academic Publ., Dordrecht, 1998.
[9] U. Hőhle: *Commutative residuated monoids*, in: U. Hőhle, P. Klement (eds), Non-classical Logics and Their Aplications to Fuzzy Subsets, Kluwer Academic Publishers, 1995.
[10] P. M. Idziak: *Lattice operations in BCK-algebras*, Mathematica Japonica, 29(1984), 839-846.
[11] A. Iorgulescu: *Algebras of logic as BCK algebras,* Ed. ASE, Bucharest, 2008.
[12] T. Kowalski, H. Ono: *Residuated lattices: an algebraic glimpse at logic without contraction*, 2001.
[13] W. Krull: *Axiomatische Begründung der allgemeinen Ideal theorie*, Sitzungsberichte der physikalisch medizinischen Societäd der Erlangen 56 (1924), 47-63.
[14] M. Okada, K. Terui: *The finite model property for various fragments of intuitionistic linear logic*, Journal of Symbolic Logic, 64 (1999), 790-802.
[15] H. Ono, Y. Komori: *Logics without the contraction rule*, Journal of Symbolic Logic, 50 (1985), 169-201.
[16] J. Pavelka: *On fuzzy logic II. Enriched residuated lattices and semantics of propositional calculi*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik 25 (1979), 119-134.
[17] D. Piciu: *Algebras of fuzzy logic*, Ed. Universitaria, Craiova, 2007.
[18] E. Turunen: *Mathematics Behind Fuzzy Logic*, Physica-Verlag, 1999.
[19] M. Ward: *Residuated distributive lattices*, Duke Mathematical Journal 6 (1940), 641-651.
[20] M. Ward, R. P. Dilworth: *Residuated lattices*, Transactions of the American Mathematical Society 45 (1939), 335-354.

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