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A BCC-algebra as a subclass of K-algebras

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Abstract. Our main purpose in this note is to prove the class of BCC-algebras as a subclass of K-algebras. Further it is shown that:(i) the class of B-algebras is equivalent to the class of $BCCI$-algebras, (ii) a $BCCI$-algebra is a member of the subclass of K-algebras in which B-algebras lies, if the group G is non-abelian.

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1. Introduction

The notion of a K-algebra $(G, \cdot, \odot, e)$ was first introduced by Dar and Akram [4] in 2003 and published in 2005. A K-algebra is an algebra built on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G which is attached to an abstract K-algebra $(G, \cdot, \odot, e)$. Obviously, this system is non-commutative and non-associative with a right identity e, if $(G, \cdot, e)$ is non-commutative. It was proved in [1, 4] that a K-algebra on an abelian group is equivalent to a p-semisimple BCI-algebra. For a given group G, the K-algebra is proper if G is not an elementary abelian 2-group. Thus, a K-algebra is abelian and non-abelian purely depends on the base group G. Dar and Akram further renamed *a K-algebra on a group G* as a $K(G)$-algebra [5] due to its structural basis G. The $K(G)$-algebras have also been characterized by their left and right mappings in [2, 5]. Recently, Dar and Akram have proved in [7] that the class of $K(G)$-algebras is a generalized class of B-algebras [16] when G is a non-abelian group, and they also have proved that the K-algebra is a generalized class of the class of $BCH/BCI/BCK$-algebras [11, 12, 13] when G is an abelian group. In this paper we prove the class of BCC-algebras as a subclass of K-algebras. Further more it is shown that:(i) the class of B-algebras is equivalent to the class of $BCCI$-algebras, (ii) a $BCCI$-algebra is a member of the subclass of K-algebras in which B-algebras lies, if the group G is non-abelian.

2. K-algebras

In this section we give review of K-algebras.

Definition 2.1. [4] *Let $(G, \cdot, e)$ be a group in which each non-identity element is not of order 2. Then a K- algebra is a structure $\mathcal{K} = (G, \cdot, \odot, e)$ on a group G in which induced binary operation $\odot : G \times G \to G$ is defined by $\odot(x, y) = x \odot y = x.y^{-1}$ and satisfies the following axioms:*

(K1) $(x \odot y) \odot (x \odot z) = (x \odot ((e \odot z) \odot (e \odot y))) \odot x$,

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(K2) $x \odot (x \odot y) = (x \odot (e \odot y)) \odot x$,
(K3) $(x \odot x) = e$,
(K4) $(x \odot e) = x$,
(K5) $(e \odot x) = x^{-1}$
for all $x, y, z \in G$.

If the group $(G, \cdot, e)$ is abelian , then the above axioms $(K1)$ and $(K2)$ can be replaced by:
$(\overline{K1})$ $(x \odot y) \odot (x \odot z) = z \odot y$.
$(\overline{K2})$ $x \odot (x \odot y) = y$.
In what follows, we denote a K-algebra by $\mathcal{K}$ unless otherwise specified.

Proposition 2.1. [4, 5] *Let $\mathcal{K}$ be a K- algebra on abelian group G which is not elementary abelian 2-group, then the following results hold within K-algebra $\mathcal{K}$ for all $x, y, z \in G$:*
(K6) $(x \odot y) \odot z = (x \odot z) \odot y$.
(K7) $(e \odot x) \odot (e \odot y) = y \odot x = e \odot (x \odot y)$.
(K8) $(x \odot z) \odot (y \odot z) = x \odot y$.
(K9) $e \odot (e \odot x) = x$.
(K10) $x \odot (e \odot x) = x^2$.
(K11) $x \odot (e \odot y) = y \odot (e \odot x)$.
(K12) $x \odot y = e \Longleftrightarrow x = y$.
(K13) $e \odot x = e \Longleftrightarrow x = e$.
(K14) $e \odot x = x \Longleftrightarrow x$ *is of order 2 in* G.

Theorem 2.1. [2] *Let $\mathcal{K}$ be a K-algebra on non-abelian group G. Then the following identities hold in $\mathcal{K}$ for all $x, y, z \in G$:*
(K15) $(x \odot y) \odot z = x \odot (z \odot (e \odot y))$.
(K16) $x \odot (y \odot z) = (x \odot y) \odot (e \odot z)$.
(K17) $e \odot (x \odot y) = y \odot x$.
(K18) $x \odot y = e = y \odot x \Longrightarrow x = y$.
(K19) $(x \odot y) \odot (z \odot y) = x \odot z$.
(K20) $(x \odot y) \odot (e \odot y) = x$.

Theorem 2.2. [4] *Let $\mathcal{K}$ be a K-algebra on abelian group G. Then the following properties of interaction of $\odot$ on $\cdot$ hold within K-algebra $\mathcal{K}$ for all $g, h, k, x \in G$:*
(P1) $(x \odot g) \cdot g = x = (x \cdot g)g^{-1} = (x \cdot g) \odot g$, *i.e.,* $\odot$ *and* $\cdot$ *are right inversion of each other.*
(P2) $g \odot (h \cdot k) = (g \odot h) \odot k = (g \odot k) \odot h$.
(P3) $(h \cdot k) \odot g = h \cdot (k \odot g) = k \cdot (h \odot g)$.

Definition 2.2. [4, 5] *Let $\mathcal{K}$ be a K-algebra. For a fixed element $x \in \mathcal{K}$, the mapping $L_x : \mathcal{K} \to \mathcal{K}$ defined by $L_x(y) = x \odot y$ for all $y \in \mathcal{K}$, is called left map on $\mathcal{K}$.*

Definition 2.3. [4, 5] *Let $\mathcal{K}$ be a K-algebra. For a fixed element $x \in \mathcal{K}$, the mapping $R_x : \mathcal{K} \to \mathcal{K}$ defined by $R_x(y) = y \odot x$ for all $y \in \mathcal{K}$, is called right map on $\mathcal{K}$.*

Theorem 2.3. [4] *Aut $(G) \cong$ Aut $(\mathcal{K})$.*

Theorem 2.4. [5] *The left map L_e is an automorphism of a K-algebra $\mathcal{K}$.*

Lemma 2.1. [5] *Let $\mathcal{K}$ be a K-algebra on abelian group G. Let $R = \{R_x : x \in G\}$ be the set of all right mappings of K-algebra with the operation of composition $(\circ)$ of the right mappings defined by*

$$R_x \circ R_y = R_{x \odot (e \odot y)}.$$

Then system $(R, \circ)$ *forms an abelian group which is isomorphic to the group* G.

Theorem 2.5. [5] *Let* $\mathcal{K}$ *be a* K*-algebra on abelian group* G. *Let* $R = \{R_x : x \in G\}$ *be a set of all right mappings of* K*-algebra* $\mathcal{K}$ *defined by* $(g)R_x = g \odot x$ *for all* g, $x \in \mathcal{K}$. *Then the algebra is a* K*-algebra* $\mathcal{K}$ *on* G *if and only if the system* $(R, \circ)$ *on* $\mathcal{K}$ *is isomorphic to the group* G.

K-algebras have been extensively studied by authors since 2004 (see [1-8]).

3. Basic Definitions

J. Neggers and S. H. Kim introduced the notion of B-algebras in 2002.

Definition 3.1. [16] *An algebra* $(X, *, \circ)$ *of type* $(2, 0)$ *is called* B*-algebra if it satisfies the following axioms:*
(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B3) $(x * y) * z = x * (z * (0 * y))$.
for all $x, y, z \in X$.

Komori [14] introduced a notion of BCC-algebras, and Dudek [9] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori.

Definition 3.2. *An algebra* $(X, *, 0)$ *of type (2, 0) with the special element* 0 *is called a BCC-algebra if it satisfies the following axioms:*
(BCC1) $((x * y) * (z * y)) * (x * z) = 0$,
(BCC2) $x * x = 0$,
(BCC3) $0 * x = 0$,
(BCC4) $x * 0 = x$,
(BCC5) $x * y = 0, y * x = 0$ *implies that* $x = y$
for all $x, y, z \in X$.

In a BCC-algebra, the following holds

$$(x * y) * x = 0.$$

4. A BCC-algebra as a subclass of K-algebras

We realize a generalization class of the class of BCC-algebras $(X, *, 0)$ except the axiom $BCC3$, i.e., $(0 * x = 0)$ and call it a $BCCI$-algebra. Thus we define it as follow:

Definition 4.1. *An algebra* $(X, *, 0)$ *is called a* $BCCI$*-algebra if it satisfies the following axioms:*
(BCCI1) $x * x = 0$,
(BCCI2) $x * 0 = x$,
(BCCI3) $x * y = 0 = y * x \Rightarrow x = y$,
(BCCI4) $((x * y) * (z * y)) * (x * z) = 0$
for all $x, y, z \in X$.

Theorem 4.1. *The class of* B*-algebras is equivalent to the class of* $BCCI$*-algebras.*

Proof. Suppose that $(X,*,0)$ is a B-algebra generating with axioms $B1$, $B2$ and $B3$, if $x,y,z\in X$. Then

$$\begin{aligned}((x*y)*(z*y))*(x*z) &= (x*((z*y)*(0*y)))*(x*z),\ [by\ B3]\\ &= (x*(z*((0*y)*(0*y))))*(x*z),\ [by\ B3]\\ &= (x*(z*0)*(x*z),\ [by\ B1]\\ &= (x*z)*(x*z),\ [by\ B2]\\ &= 0,\ [by\ B1]\end{aligned}$$

which proves that axiom $(BCCI4)$ hold in B-algebra. The axioms $(BCCI1-BCCI3)$ are common to axioms $B1$ and $B2$ of B-algebra. Thus $BCCI$-algebra is a subalgebra of B-algebra, i.e.,

$$BCCI-algebra \ \subseteq \ B-algebra. \qquad (a)$$

Conversely, suppose that $(X,*,0)$ is a $BCCI$-algebra.If $x,\ y\ ,z,\ u\in X$ such that $z=u*y$, then $u=z*(0*y)$ by (a).
B3 $\Longrightarrow (x*y)*z=x*(z*(0*y))$
$\Longrightarrow ((x*y)*(u*y)=x*u$
$\Longrightarrow BCCI4:((x*y)*(u*y))*(x*u)=0.$ Thus

$$B-algebra \ \subseteq \ BCCI-algebra \qquad (b)$$

By combining inequalities (a) and (b) together the proof of the theorem is complete. □

Example 4.1. *Consider K-algebra $\mathcal{K}=(G,\cdot,\odot,e)$ on the non-abelian group $G=S_3=\{<a,y>: a^3=e=y^2=(ay)^2; xy=z\}$where, $a^2=b$, $ay=x$, $a^2y=z=by$ and $\odot$ is given by the following Cayley's table:*

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

(1) *The K-algebra on S_3 is non-abelian since*
$$x\odot(e\odot a)=y\neq z=a\odot(e\odot x).$$
(2) *It is easy to verify that the K-algebra on S_3 is a B-algebra.*

(3) *The K-algebra on S_3 is a $BCCI$-algebra since*
$$((x*a)*(y*a))*(x*y)=0$$
is verified easily for all x,a,y in S_3.

Since B-algebra is K-algebra built on a non-abelin group as proved in [7], it consequently elaborate that:

Theorem 4.2. *Let $(X,*,0)$ be a $BCCI$-algebra. Then $BCCI$-algebra is a member of the subclass of a K-algebra in which B-algebra lies.*

Since BCC-algebra is a subclass of the class of a $BCCI$-algebra which is further equivalent to the class of B-algebras, and the class of B-algebras is a subclass of the K-algebras built on non-abelian group, we conclude that BCC-algebra is a subclasses of the class of K-algebras on non-abelian groups. Thus we elaborate that:

Theorem 4.3. *Let $(X, *, 0)$ be a $BCCI$-algebra. Then the class of $BCCI$-algebra $(X, *, 0)$ is a subclass of non-abelian K-algebras.*

Corollary 4.1. [7] *A BCK-algebra is a member of abelian K-algebras.*

Corollary 4.2. [10, 14] *A BCC-algebra is a generalization of BCK-algebras.*

Thus we restrict the idea to a subclass of $BCCI$-algebra together with additional axiom $BCC3$ and assimilate that:

Theorem 4.4. *Let $(X, *, 0)$ be a BCC-algebra. The class of BCC-algebras $(X, *, 0)$ is a subclass of non-abelian K-algebras $\mathcal{K}$.*

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