Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 36(1), 2009, Pages 17–23
ISSN: 1223-6934

Existence of nontrivial solutions for a class of elliptic systems

Lotfi LASSOUED and Ali MAALAOUI

Abstract. We establish several existence results for a class of nonlinear elliptic systems of Schrödinger type. The proofs are mainly based on topological degree arguments.

2000 Mathematics Subject Classification. Primary 35J55; Secondary 47H11.
Key words and phrases. nonlinear elliptic system, topological degree.

1. Preliminaries

Let E a Banach space. Let $P \subset E$ be a cone, that is, if $x, y \in P$, $\lambda \in \mathbb{R}_+$ then $\lambda x \in P$ and $x + y \in P$, with $P \cap (-P) = \{0\}$. We know that E become an ordered Banach space by the order induced by P, i.e., $x \geq y \Longleftrightarrow x - y \in P$.

A cone is said to be solid if $\overset{\circ}{P} \neq 0$, and total if $\overline{P - P} = E$. The cone P is called normal, if there exist $C > 0$ such that if $x \leq y$ then $\|x\| \leq C \|y\|$.

E is said to have the lattice structure if for $x, y \in P$, $\inf \{x, y\}$ and $\sup \{x, y\}$ exist and belong to E. We denote then $x^+ = \sup \{x, 0\}$ and $x^- = \sup \{-x, 0\}$ and we define $|x| = x^+ + x^-$.

In all that will follow, we will call a set $\mathcal{M}$ or an operator T is admissible if the topological degree $d(I-T, \mathcal{M}, 0)$ is well defined *i.e.* $0 \in (I - T)(\partial \mathcal{M})$. And we remind the reader that a completely continuous operator in a Banach space is a continuous operator that transforms bounded sets on relatively compact sets. Also if T is a smooth completely continuous operator then the differential of the operator on any point where it is smooth, is a compact linear operator.

Lemma 1.1. *Let $\mathcal{M}$ be a bounded open set which contain 0, of a Banach spaces and let $T : \overline{\mathcal{M}} \longmapsto E$ be a compact operator, If*

$$Tu \neq \lambda u, \ \forall u \in \partial \mathcal{M}, \ \forall \lambda \geq 1$$

Then $d(I - T, \mathcal{M}, 0) = 1$.

Proof. Let H be the homotopy defined by $H(t, u) = u - tTu$; by the assumption imposed on $\partial \mathcal{M}$ we have the compatibility hypothesis of H so we have

$$d(H(1, .), \mathcal{M}, 0) = d(H(0, .), \mathcal{M}, 0) = d(id, \mathcal{M}, 0) = 1.$$

This completes our proof. □

Lemma 1.2. *Let T be a linear compact operator in E such that $1 \in \sigma(T)$, then*

$$d(I - T, \mathcal{M}, 0) = (-1)^{\alpha},$$

where α the sum of multiplicity of the eigenvalues of $DT(x_0)$ in $]1, +\infty[$.

Received: 11 June 2009.

Proof. Let $\lambda_1, ..., \lambda_k$ be the eigenvalues of T greater than one and $E_1, ..., E_k$ the corresponding eigenspaces. Let $P_1, ..., P_k$ denote the projectors on the previous subspaces. We consider $P = \sum_{i=1}^{k} P_i$, then P commutes with T and is a projector on E_0 the direct sum of the E_i. Remark that dim $E_0 = \alpha$. Now we take the following homotopy:

$$H(t,x) = x - [2tPx + (1-t)TPx] - (1-t)T(I-P)x$$

it is easy to see that this homotopy is admissible on $\overline{B(0,1)}$. Thus, we have

$$d(I-T, \overline{B(0,1)}, 0) = d(I-2P, \overline{B(0,1)}, 0),$$

and since P is a finite rank operator, then we have

$$d(I-T, \overline{B(0,1)}, 0) = d(I-2P, \overline{B(0,1)}, 0) = (-1)^{\alpha}.$$

This completes the proof. ☐

Lemma 1.3 (Homotopy criteria)**.** *Let $\mathcal{M}$ be a bounded set, let T_1 and T_2 be two completely continuous operator, without fixed point on Ω. If*

$$\|T_1x - T_2x\| \leq \|x - T_2x\|, \ \forall x \in \mathcal{M},$$

Then $I - T_1$ and $I - T_2$ are homotopic in $\mathcal{M}$.

Proof. Consider the linear homotopy

$$H(t,x) = x - (tT_1x + (1-t)T_2x), \forall x \in \mathcal{M}, \ \forall t \in [0,1]$$

and notice that

$$\|H(t,x)\| = \|x - T_2x + t(T_2x - T_1x)\| \geq (1-t)\|x - T_2x\|, \forall x \in \mathcal{M}, \ \forall t \in [0,1],$$

thus H is compatible, which proves the result. ☐

Now, we focus on the research of a non trivial solution for the problem

$$\begin{array}{c} -\Delta u = f(u) \text{ on } \Omega \\ u = 0 \text{ on } \partial\Omega \end{array} \tag{1}$$

where f is in $C^1(\mathbb{R}^n, \mathbb{R}^n)$, and Ω is a bounded domain of $\mathbb{R}^d$. And we let E denote the space $C_0^1(\Omega, \mathbb{R}^n)$.

here we suppose that f satisfies the following assumptions :

H1) $\limsup_{s_i \longrightarrow +\infty} \frac{f_i(s)}{s_j} = a_{i,j}$, $\liminf_{s_i \longrightarrow -\infty} \frac{f_i(s)}{s_j} = b_{i,j}$.

H2) $\frac{\partial f_i}{\partial x_j}(0) = a^0_{i,j} \geq 0$.

Let A denote the matrix $\left(a^0_{i,j}\right)_{1 \leq i,j \leq n}$. And $\beta_1 > \beta_2 \geq ... \geq \beta_2$ be the eigenvalues of A.

We add now the remaining assumptions :

H3)$\beta_1 > \lambda_1 > \beta_2$.

H4)$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \beta_2 & \beta_1 \end{vmatrix} < 0$

Theorem 1.1. *Suppose that $r(|a_{i,j}|) < \lambda_1$ and $r(|b_{i,j}|) < \lambda_1$, then, under the assumptions H1-H4 the problem* (1) *has at least one positive solution.*

2. Proof of theorem 4

We split the proof into several steps. STEP 1.

Let $B = (\max(|a_{i,j}|, |b_{i,j}|))_{1\leq i,j\leq n}$, $T = (-\Delta)^{-1} f : C\left(\overline{\Omega}, \mathbb{R}^n\right) \longrightarrow C\left(\overline{\Omega}, \mathbb{R}^n\right)$, and $H = (-\Delta)^{-1} B$, then we have,

$$|Tx| \leq H|x| + x_0, \forall x \in C\left(\overline{\Omega}, \mathbb{R}^n\right),$$

Since $r(H) < 1$, the operator $I - H$ is a positive invertible operator, let $K = \left\{x \in C\left(\overline{\Omega}, \mathbb{R}^n\right); Tx = \mu x, \mu \geq 1\right\}$ then K is bounded, in fact, consider $x \in K$, then there exist $\mu \geq 1$ such that

$$Tx = \mu x$$

so we have

$$|x| \leq \mu |x| = |Tx| \leq H|x| + x_0$$

so $(I - H)|x| \leq x_0$ and it follows that $|x| \leq (I - H)^{-1} x_0$ which implies the boundedness of K, and if we take $R_0 > 0$ such that $K \subset B(0, R_0)$ we have

$$Tx \neq \mu x, \forall x \in \partial B(0, R_0), \forall \mu \geq 1,$$

finally using lemma (1) we have the existence of $R_0 > 0$ such that

$$d(I - T, B(0, R), 0) = 1, \forall R \geq R_0.$$

STEP 2. Let us consider now the operator L defined by

$$Lx = DT(0)x = (-\Delta)^{-1} Df(0)x, \forall x \in C\left(\overline{\Omega}, \mathbb{R}^n\right).$$

Lemma 2.1. *$I - L$ is homotopic to T in a small neighborhood of* 0.

Proof. : Remark first that the assumptions imply that 1 is not an eigenvalue of L. Thus there exist $\delta > 0$ such that

$$\|x - Lx\| \geq \delta \|x\|, \ \forall x \in E,$$

because of the Fredholm alternative applied to the compact operator L. In the other hand we have

$$Tx = Lx + o(\|x\|),$$

therefore there exist $r_0 > 0$ such that for every $0 < r < r_0$,

$$\|x - Tx\| \geq \|x - Lx\| - \|o(\|x\|)\| \geq \frac{\delta}{2} \|x\|, \ \forall x \in B(0, r),$$

And that proves that the domain $B(0, r)$ for r sufficiently small is admissible. Now we have

$$\|Tx - Lx\| = \|o(\|x\|)\| \leq \frac{\delta}{2} \|x\| \leq \|x - Lx\|,$$

therefore, using lemma(3) we get the desired result. □

STEP 3. Let us study the spectrum of L.

Here we denote $\Phi_i = \varphi_i \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, where φ_i, $i \geq 1$ are the eigenfunctions of the 1 dimensional Laplace operator. Since they form an orthonormal basis of $L^2(\Omega)$ we can write every $u \in L^2(\Omega, \mathbb{R}^n)$ as

$$u = \sum_{j=1}^{\infty} A_j \Phi_j, \tag{2}$$

where the $A_j \in M_n(\mathbb{R})$ are diagonal.

Note that the decomposition (2) is not unique if we remove the diagonal condition. But we can give it a more settled sens in the following way :

Let $F : M_n(\mathbb{R}) \longmapsto \mathbb{R}^n$, defined for every $M = (m_{i,j}) \in M_n(\mathbb{R})$ by

$$F(M) = \begin{pmatrix} \sum_{j=1}^{n} m_{1,j} \\ \vdots \\ \sum_{j=1}^{n} m_{n,j} \end{pmatrix},$$

and consider the space

$$\widetilde{M_n(\mathbb{R})} = M_n(\mathbb{R}) \diagup \ker F,$$

Then (2) is unique if we take $A_j \in \widetilde{M_n(\mathbb{R})}$. So for $M \in M_n(\mathbb{R})$ we will write $\widetilde{M}$ its class in $\widetilde{M_n(\mathbb{R})}$.

Now let us consider the following eigenvalue problem:

$$\begin{cases} -\Delta u = \mu A u \text{ in } \Omega \\ u = 0 \text{ in } \partial\Omega \end{cases} \tag{3}$$

We know that this problem has an increasing sequence of eigenvalue $(\mu_i)_{i\geq 1}$. Let Ψ_i be an eigenfunction that correspond to the eigenvalue μ_i, then we have $\Psi_i = \sum A_j \Phi_j$ and if we plug it in the equation (3) we get :

$$\sum \lambda_j A_j \Phi_j = \sum \mu_i A A_j \Phi_j \tag{4}$$

, thus

$$\widetilde{AA_j} = \frac{\lambda_j}{\mu_i} \widetilde{A_j}.$$

Now consider the operator operators $\widetilde{\mathcal{L}} : \widetilde{M_n(\mathbb{R})} \longmapsto \widetilde{M_n(\mathbb{R})}$ defined by $\widetilde{\mathcal{L}}\widetilde{M} = \widetilde{AM}$ and $\mathcal{L} : M_n(\mathbb{R}) \longmapsto M_n(\mathbb{R})$ defined by $\mathcal{L}M = AM$. We can easily verify that the operator $\widetilde{\mathcal{L}}$ is well defined and thus $\frac{\lambda_j}{\mu_i}$ are eigenvalues of $\widetilde{\mathcal{L}}$ so let us find its spectrum. Let β be an eigenvalue of $\widetilde{\mathcal{L}}$, then there exist $\widetilde{B} \in \widetilde{M_n(\mathbb{R})} \diagdown \{0\}$ such that

$$\widetilde{AB} = \beta \widetilde{B}.$$

Thus

$$AB = \beta B + C$$

where $C \in \ker F$, therefore

$$(A - \beta I) B = C,$$

so if β is not an eigenvalue of $\mathcal{L}$ then $B = (A - \beta I)^{-1} C$, thus $F(B) = F(C) = 0$ and $\widetilde{B} = 0$ which yields to a contradiction, so β is an eigenvalue of $\mathcal{L}$ but we know that the eigenvalues of $\mathcal{L}$ are the same as A, therefore the spectrum of $\widetilde{\mathcal{L}}$ is included in the spectrum of A and the other inclusion is obvious, thus the eigenvalues μ_i have the form $\dfrac{\lambda_j}{\beta_k}$, where $j \geq 1$ and $1 \leq k \leq n$. Therefore,

$$\mu_1 = \frac{\lambda_1}{\beta 1}.$$

Notice that since $\widetilde{\mathcal{L}}$ acts on a finite dimensional vector space then there exist $j_0 \geq 1$ such that $\widetilde{A_j} = 0$, $\forall j \geq j_0$.

Let us now find the multiplicity of μ_1. Let Ψ be an eigenfunction corresponding to μ_1, using the same process as in (4), we get

$$\widetilde{AA_j} = \frac{\lambda_j}{\mu_1}\widetilde{A_j}$$

therefore, $\widetilde{A_j} = 0$, $\forall j \geq 2$ and thus $\Psi = \widetilde{A_1}\Phi_1$ where $\widetilde{A_1} \in \widetilde{M_n(\mathbb{R})}$ is an eigenvector of $\widetilde{\mathcal{L}}$ associated to $\beta 1$. So μ_1 has the same multiplicity as $\beta 1$ but since A is a positive irreductible matrix the spectral radius $r(A) = \beta 1$ is of multiplicity one, thus the multiplicity of μ_1 is 1.

Now using H3) and H4) we get $\mu_2 = \frac{\lambda_1}{\beta_2} > 1$ thus, the only eigenvalue of L in $]1, +\infty[$ is $\frac{1}{\mu_1}$. Therefore using lemma(2) we get the existence of $\varepsilon > 0$ such that

$$d(I - T, B(0, \varepsilon), 0) = -1.$$

Now using the topological degree excision property we have the existence of a non trivial solution for the problem (1).

Remark 2.1. *We can replace the hypothesis H3) by* $\beta_1 > \lambda_1$ *and* $\max\limits_{j,k} \sum\limits_i |a_{i,j} - a_{ik}| < \lambda_2$.

Now let us assume there exist $A_0 = (a_{i,j})_{1 \leq i,j \leq n}$ such that $\frac{|f(s)-As|}{|s|} \longrightarrow 0$ and $r(A_0) > \lambda_1$.

Theorem 2.1. *Under the previous assumption and the assumptions H2-H4, the problem (1) has a non trivial solution.*

Proof. The proof is almost the same as the previous one. In fact we need only to worry about the degree of the operator T for a large neighborhood of zero. But we have the existence of R sufficiently large such that

$$d(I - T, B(0, R), 0) = d(I - \widetilde{T}, B(0, R), 0)$$

where $\widetilde{T} = (-\Delta)^{-1} \circ A_0$. In fact, we have the existence of $\delta > 0$ such that

$$\left\| x - \widetilde{T}x \right\| \geq \delta, \ \forall x \in E.$$

We also have for R sufficiently large

$$\left\| Tx - \widetilde{T}x \right\| \leq \delta \left\| x \right\| \leq \left\| x - \widetilde{T}x \right\|, \ \forall x; \|x\| > R.$$

Using lemma(3) And since $\sigma(\widetilde{T}) \subset]0,1[$ we have using lemma(2) that $d(I-T, B(0,R), 0) = d(I - \widetilde{T}, B(0, R), 0) = 1$. And we conclude by comparing with the degree near the origin. □

Theorem 2.2. *Suppose that there exist a partition* I, J *of* $\{1, ..., n\}$ *such that* $u_I \longrightarrow f(u_I, v_J)$ *is nondecreasing and* $v_J \longrightarrow f(u_I, v_J)$ *is nonincreasing. We suppose in addition that there exist* $1 > \alpha > 0$ *such that for* $t \in]0,1[$, $f(tu_I, \frac{1}{t}v_J) \geq t^\alpha f(u_I, v_J)$. *then the problem* (1) *have one and only one positive solution.*

3. Proof of theorem 8.

Here E is equipped with the slotwise partial order on $C_0^1(\Omega)$.

Let $T : E \times E \longrightarrow E$ defined by

$$T(u, v) = (-\Delta)^{-1} \circ f(u_I, v_J),$$

then T satisfies

$$T(u,v) \geq T(tu, \frac{1}{t}v) \geq t^{\alpha}T(u,v),\ \forall t \in]0,1[\,,\ \forall (u,v) \in E \times E.$$

we construct now the approximation sequence (u_n, v_n) defined by

$$\begin{cases} u_{n+1} = T(u_n, v_n) \\ v_{n+1} = T(v_n, u_n) \end{cases},$$

with $(u_0, v_0) \in E$

We choose $t_0 > 0$ sufficiently small such that

$$\begin{cases} t_0 x_0 < x_0 < \frac{1}{t_0} x_0 \\ t_0 x_0 < y_0 < \frac{1}{t_0} x_0 \\ (t_0)^{1-\alpha} x_0 < T(x_0, x_0) < \left(\frac{1}{t_0}\right)^{1-\alpha} x_0 \end{cases}$$

let $u_0 = t_0 x_0$ and $v_0 = \frac{1}{t_0} x_0$, it follows that $u_0 \leq T(u_0, v_0)$ and $T(v_0, u_0) \leq v_0$. And finally we have

$$u_0 \leq u_1 \leq ... \leq u_n \leq v_n \leq ... \leq v_1 \leq v_0,\ \forall n \in \mathbb{N},$$

Lets show that the two sequences are Cauchy sequences.

for any $n \geq 0$ there exist $\lambda > 0$ such that $\lambda v_n \leq u_n \leq v_n$, so lets take

$$t_n = \sup \{1 \geq \lambda > 0; \lambda v_n \leq u_n \leq v_n\},$$

we see that $t_n \in]0,1]$, and t_n is increasing, so let

$$t^* = \lim_{n \longrightarrow +\infty} t_n,$$

suppose that $t^* < 1$ then using the fact that $t_n v_n \leq u_n$ and $v_n \leq \frac{1}{t_n} u_n$ we have

$$\begin{aligned} u_{n+1} &= T(u_n, v_n) \geq T(t_n v_n, \frac{1}{t_n} u_n) = T(\frac{t_n}{t^*} t^* v_n, \frac{t^*}{t_n} \frac{1}{t^*} u_n) \\ &\geq \left(\frac{t_n}{t^*}\right)^{\beta} T(t^* v_n, \frac{1}{t^*} u_n) \geq \frac{t_n}{t^*} T(t^* v_n, \frac{1}{t^*} u_n) \\ &\geq \frac{t_n}{t^*} (t^*)^{\beta} T(v_n, u_n) \geq t_n (t^*)^{\beta - 1} v_{n+1} \end{aligned}$$

so we have $t_n (t^*)^{\beta-1} \leq t_{n+1}$, and if $n \longrightarrow +\infty$ we have $(t^*)^{\beta} \leq t^*$ which is impossible, so we have $t^* = 1$.

For any positive integer n and p we have

$$0 \leq u_{n+p} - u_n \leq v_n - u_n \leq v_n - t_n v_n \leq (1 - t_n) v_0 \tag{5}$$

so we have

$$\|u_{n+p} - u_n\|_{\infty} \leq (1 - t_n) \|v_0\|_{\infty}$$

and it follows that (u_n) is a Cauchy sequence, the same for (v_n). And since E is a Banach space we have the convergence of (u_n) to u^* and (v_n) to v^*. and using (5) we have $v^* = u^*$ and by continuity of T we have we obtain the desired fixed point.

Unicity : Let u^* and v^* be two positive fixed points of T, and denote

$$a = \sup \left\{ \lambda > 0; \lambda v^* \leq u^* \leq \frac{1}{\lambda} v^* \right\}$$

suppose that $a < 1$, then we have

$$u^* = T(u^*, u^*) \geq T\left(av^*, \frac{1}{a} v^*\right) \geq a^{\beta} T(v^*, v^*) = a^{\beta} v^*$$

it follows from the definition of a that $a \geq a^{\beta}$ which is impossible so $a = 1$ and we have $v^* = u^*$.

References

[1] A. Abramovich, C. D. Aliprantis, An Invitation to Operator Theory, American Mathematical Society, Providence, 2002.

[2] M.A. Krasnosel'skii, P.P. Zabreiko, Geometric Methods of Nonlinear Analysis, Springer-Verlag, 1984.

[3] Z. Liang, L. Zhang, S. Li, Fixed point theorems for a class of mixed monotone operators, Z. Anal. Anwendungen 22 (2003), 529-542.

[4] J. Sun, X. Liu, Computation of topological degree in ordered Banach spaces with lattice structure and its application to superlinear differential equations, J. Math. Anal. Appl. 348 (2008), 927-937.

[5] J. Sun, X. Liu, Computation for topological degree and its applications, J. Math. Anal. Appl. 202 (1996), 785-796.

(Lotfi LASSOUED) Department of Mathematics, University of El Manar, F.S.T. Tunis, Campus Universitaire 2029, Tunisia
E-mail address: lassoued.lotfi@gmail.com

(Ali MAALAOUI) Department of Mathematics, University of El Manar, F.S.T. Tunis, Campus Universitaire 2029, Tunisia
E-mail address: ali.maalaoui@gmail.com