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Existence of weak solution for nonlinear elliptic system

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Abstract. In this paper, we prove the existence of weak solutions for the following nonlinear elliptic system

$$\begin{aligned}&-\operatorname{div}\mathcal{A}(x,\nabla u) \;=\; -a(x)|u|^{p(x)-2}u \;-\; b(x)|u|^{\alpha(x)}|v|^{\beta(x)}v \;+\; f(x) \quad in \;\; \Omega,\\ &-\operatorname{div}\mathcal{B}(x,\nabla v) \;=\; -c(x)|v|^{q(x)-2}v \;-\; d(x)|v|^{\beta(x)}|u|^{\alpha(x)}u \;+\; g(x) \quad in \;\; \Omega,\\ &u \;=\; v \;=\; 0 \quad on \;\; \partial\Omega,\end{aligned}$$

where Ω is an open bounded domains of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. The existence of weak solutions is proved using the theory of monotone operators.

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1. Introduction

In this study, Ω is a bounded domain in $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$. The purpose of this paper is to study the existence of weak solutions for the following nonlinear elliptic system involving the $p(x)$-Laplacian

$$\begin{cases} -\operatorname{div}\mathcal{A}(x,\nabla u) = a(x)|u|^{p(x)-2}u - b(x)|u|^{\alpha(x)}|v|^{\beta(x)}v + \; f(x) \; in\,\Omega, \\ -\operatorname{div}\mathcal{B}(x,\nabla v) = c(x)|v|^{q(x)-2}v - d(x)|v|^{\beta(x)}|u|^{\alpha(x)}u + g(x) \; in\,\Omega, \\ u \;=\; v \;=\; 0 \quad \text{on} \;\; \partial\Omega. \end{cases} \tag{1.1}$$

The operator $\mathcal{A}$ (resp. $\mathcal{B}$) are with $p(x)$ (resp. $q(x)$)-type nonstandard structural conditions.
Examples of the operator $\mathcal{A}$ considered here arise from variational integrals like

$$\int |\nabla u|^{p(x)} dx, \tag{1.2}$$

the Euler-Lagrange equation of (1.2) is the $p(x)$-Laplacian

$$\operatorname{div}(\,p(x)|\nabla u|^{p(x)-2}\nabla u) = 0, \tag{1.3}$$

where

$$\mathcal{A}(x,\xi) = p(x)|\xi|^{p(x)-2}\xi.$$

The study of various mathematical problems with variable exponent has received considerable attention in recent years. There is an extensive literature on partial differential equations and the calculus of variations with various nonstandard growth conditions, for examples we cite works of X-L Fan, M. Mihailescu and V. Radulescu [14], [22, 23]. The operator $p(x)$-Laplacian turns up in many mathematical settings, e.g., non-Newtonian fluids, reaction-diffusion problems, porous media, astronomy, quasi-conformal mappings. see [3, 4, 9].

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Problems including this operator for bounded domains have been studied in [14, 22] and for unbounded domains in [12, 10]. Many authors have studied semilinear and non linear elliptic systems, as a reference we cite [12, 19, 27].

The generalized formulation for many stationary boundary value problems for partial differential equations leads to operator equation of type

$$L(u) = f$$

on a Banach space. Indeed, the weak formulation consists in looking for an unknown function u from a Banach space H such that an integral identity containing u holds for each test function v from the space H. Since the identity is linear in v, we can take its sides as values of continuous linear functionals at the element $v \in H$. Denoting the terms containing the unknown u as the value of an operator A, we obtain

$$(L(u), v) = (f, v) \quad \forall v \in H,$$

which is equivalent to the equality of functionals on H, i.e. the equality of elements of H' (the dual space of H): $L(u) = f$.
In this paper, we consider nonlinear systems with model L of the form

$$\begin{aligned} L\{u,v\} \quad &= \quad \{-\operatorname{div} \mathcal{A}(x, \nabla u) - a(x)|u|^{p(x)-2}u + b(x)|u|^{\alpha(x)}|v|^{\beta(x)}v \ , \\ &\qquad -\operatorname{div} \mathcal{B}(x, \nabla v) + c(x)|u|^{\alpha(x)}|v|^{\beta(x)}u - d(x)|v|^{q(x)-2}v\}. \end{aligned}$$

When $\operatorname{div} \mathcal{A}(x, \nabla u) = \Delta_p u$ and $\operatorname{div} \mathcal{B}(x, \nabla v) = \Delta_q v$, the existence of solutions for such systems was proved, using the method of sub and super solutions in [6, 7, 13].
In this study, we generalize several cases dealing with existence of solutions for nonlinear elliptic systems. To this end, we introduce the following intermediary problem

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u) = -a(x)|u|^{p(x)-2}u + f(x), \quad x \in \Omega, \\ u = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $p(\cdot) \in C^0(\overline{\Omega})$ such that $\inf_{x \in \Omega} p(x) > 1$ and $a(\cdot)$ is a non negative function satisfying condition (F0).
The following Theorems are our majors results.

Theorem 1.1. *The nonlinear elliptic problem* (1.4) *has a non trivial weak solution.*

Theorem 1.2. *Under assumptions (F0), (F1), (F2) and (F3) below, The nonlinear elliptic system (1.1) have a non trivial weak solution.*

This paper consists of five sections. First, we recall some elementary proprieties of the generalized Lebesgue-Sobolev spaces and introduce the notations needed in this work. Section 3 is devoted to the study of some preliminary results which allows us to prove the existence of weak solutions of our problem. Particulary we give the proof of Theorem 1.1. In the fourth section, we justify the existence of weak solutions in the case of bounded domains. The goal of the last section is the main result, when $\Omega = \mathbb{R}^N$.

2. Generalized Lebesgue-Sobolev Spaces Setting.

In order to discuss problem (1.1), we need some results about the spaces $W^{1,p(x)}(\Omega)$ which we call generalized Lebesgue- Sobolev spaces. Let us shortly recall some basic facts about the setup for generalized Lebesgue- Sobolev spaces, for more details see

for instance [14], [25], [15] and [30].
Let

$$C_+(\overline{\Omega}) = \{h \,/\, h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\}.$$

For $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u \,/\; u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

We define the so-called Luxemburg norm, on this space by the formula

$$|u|_{L^{p(x)}} = \inf\{\alpha > 0,\; \int_\Omega \Big|\frac{u(x)}{\alpha}\Big|^{p(x)} dx \le 1\}.$$

It's well known, that $(L^{p(x)}(\Omega); |.|_{L^{p(x)}})$ is a is a separable, uniformly convex Banach space.
$(L^{p(x)}(\Omega); |.|_{L^{p(x)}})$ is termed a generalized Lebesgue space. Moreover, its conjugate space is $L^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, one has the following inequality

$$\Big|\int_\Omega u(x)v(x)dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{L^{p(x)}}|v|_{L^{p'(x)}} \le 2|u|_{L^{p(x)}}|v|_{L^{p'(x)}}, \tag{2.1}$$

where, $p^- = \min\limits_{\overline{\Omega}} p(x)$ and $p'^- = \min\limits_{\overline{\Omega}} p'(x)$.
Note that $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, for every functions p_1 and p_2 in $C(\overline{\Omega})$ satisfying $p_1(x) \le p_2(x)$, for any $x \in \overline{\Omega}$. In addition this imbedding is continuous.
An important role in manipulating the generalized Lebesgue spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}dx.$$

If $(u_n), u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true.

$$|u|_{L^{p(x)}} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}} \le \rho_{p(x)}(u) \le |u|^{p^+}_{L^{p(x)}}, \tag{2.2}$$

$$|u|_{L^{p(x)}} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}} \le \rho_{p(x)}(u) \le |u|^{p^-}_{L^{p(x)}}, \tag{2.3}$$

$$|u_n - u|_{L^{p(x)}} \to 0 \text{ if and if } \rho_{p(x)}(u_n - u) \to 0. \tag{2.4}$$

Another interesting property of the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is

Proposition 2.1. (*see* [11]) *Let $p(x)$ and $q(x)$ be measurable functions such that $p \in L^\infty(\mathbb{R}^N)$ and $1 \le p(x)q(x) \le \infty$, for a.e. $x \in \mathbb{R}^N$ Let $u \in L^{q(x)}(\mathbb{R}^N)$, $u \ne 0$. Then*

$$|u|_{p(x)q(x)} \le 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \le ||u|^{p(x)}|_{q(x)} \le |u|^{p^-}_{p(x)q(x)},$$

$$|u|_{p(x)q(x)} \ge 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \le ||u|^{p(x)}|_{q(x)} \le |u|^{p^+}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ is constant, then

$$||u|^p|_{q(x)} = |u|^p_{pq}.$$

The generalized Lebesgue-Sobolev space is defined by:

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega)\}.$$

$W^{1,p(x)}(\Omega)$ can be equipped with the norm defined as follow

$$\|u\|_{p(x)} = |u|_{L^{p(x)}} + |\nabla u|_{L^{p(x)}}, \text{ for all } u \in W^{1,p(x)}(\Omega). \tag{2.5}$$

In this paper, we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.
It follows from Fan and Zhao [14] that the generalized Lebesgue- Sobolev spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces. On the other hand if $q \in C_+(\overline{\Omega})$ satisfys $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, the imbedding from $W^{1,p(x)}(\Omega)$ into $L^{q(x)}(\Omega)$ is compact and continuous. Note that Poincaré inequality is also satisfied and we have the existence of a constant $C > 0$ such that

$$|u|_{L^{p(x)}} \leq C|\nabla u|_{L^{p(x)}}, \text{ for all } u \in W_0^{1,p(x)}(\Omega). \tag{2.6}$$

In view of (2.5), it follows that $|\nabla u|_{L^{p(x)}}$ and $\|u\|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We refer to [21] for more properties of Lebesgue and Sobolev spaces with variable exponent. We also refer to the recent papers [22, 23, 24] for the treatment of nonlinear boundary value problems in Lebesgue-Sobolev spaces with variable exponent.

Definition 2.1. $1 < p(x) < N$ *and for* $x \in \mathbb{R}^N$, *let us define*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & p(x) < N, \\ +\infty & p(x) > N, \end{cases}$$

where $p^*(x)$ *is the so-called critical Sobolev exponent of* $p(x)$.

Proposition 2.2. (*see* [11]) *Let* $p(\cdot) \in C_+^{0,1}(\mathbb{R}^N)$, *that is Lipshitz-continuous function defined on* $\mathbb{R}^N$, *then there exists a positive constant c such that*

$$|u|_{p^*(x)} \leq \|u\|_{p(x)},$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

We use the following compactness properties of $W_0^{1,p(x)}(\Omega)$ in our existence proof. The limit function v belongs Mazur's Lemma, the first follows from the reflexivity and the second form the fact that $W_0^{1,p(x)}(\Omega)$ embeds compactly into $L^{p(x)}(\Omega)$ [21].

Through this paper we suppose that the following assumptions are satisfied.

(F0) $\quad a(x), c(x)$ are resp. in $L^\infty(\Omega) \cap L^{p'(x)}(\Omega)$ and $L^\infty(\Omega) \cap L^{q'(x)}(\Omega)$.

(F1) $$s(x) = \frac{p(x)p^*(x)q^*(x)}{p(x)p^*(x)q^*(x) - pq^*(x) - p^*(x)q^*(x)}, \qquad b(x) \in L^{s(x)}(\Omega),$$

(F2) $$r(x) = \frac{q(x)p^*(x)q^*(x)}{q(x)p^*(x)q^*(x) - qq^*(x) - p^*(x)q^*(x)}, \qquad d(x) \in L^{r(x)}(\Omega),$$

$$\widetilde{p}(x) = \frac{p(x)p^*(x)}{p^*(x) - p(x)}; \qquad \widetilde{q}(x) = \frac{q(x)q^*(x)}{q^*(x) - q(x)},$$

(F3) $\quad 1 < p^-, q^- \;\; \alpha^+ < p^- - 1, q^- - 1, \;\; \beta^+ < p^- - 1, q^- - 1, p^+ < p^* > 2.$

Further notation will be introduced as need.

3. Proof of Theorem 1.1

This section is devoted to the study of problems of type: $Lu = f$, where L is an operator from H (Banach space) into it's dual $H^{'}$. The methods used are variational, more precisely, we use the theory of monotone operators.
To this end, we introduce some technical results [5, 7, 29] which allows us to the proof of Theorem1.1. Note that hypothesis F0) and F3) will be used in this section. First, we give conditions needed for the operators $\mathcal{A}$ (resp. $\mathcal{B}$).
In this note the operators $\mathcal{A}(x,\xi);\mathcal{B}(x,\xi):\Omega\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ are such that:
(A1) The mapping $x\mapsto\mathcal{A}(x,\xi)$ (resp.$\mathcal{B}(x,\xi)$) is measurable for all $\xi\in\mathbb{R}^N$.
(A2) The mapping $x\mapsto\mathcal{A}(x,\xi)$ (resp.$\mathcal{B}(x,\xi)$) is is continuous for all $x\in\Omega$.
(A3) For all $x\in\Omega$ and $\xi\in\mathbb{R}^N$; $\mathcal{A}(x,-\xi)=\mathcal{A}(x,\xi)$ and $\mathcal{B}(x,-\xi)=\mathcal{B}(x,\xi)$.
(A4) There exist a positive constant $\gamma_1,\delta_1>0$ such that

$$\mathcal{A}(x,\xi).\xi\geq\gamma_1|\xi|^{p(x)},\ \ ;\ \ \forall\ \ x\in\Omega\ \text{ and }\ \xi\in\mathbb{R}^N,$$

$$\mathcal{B}(x,\xi).\xi\geq\delta_1|\xi|^{q(x)}\ \ ;\ \ \forall\ \ x\in\Omega\ \text{ and }\ \xi\in\mathbb{R}^N.$$

(A5) There exist a positive constant $\gamma_2\geq\gamma_1>0,\delta_2>\delta_1$ such that

$$\mathcal{A}(x,\xi).\xi\leq\gamma_2|\xi|^{p(x)-1},\ \ ;\ \ \forall\ \ x\in\Omega\ \text{ and }\ \xi\in\mathbb{R}^N,$$

$$\mathcal{B}(x,\xi).\xi\leq\delta_2|\xi|^{q(x)-1},\ \ ;\ \ \forall\ \ x\in\Omega\ \text{ and }\ \xi\in\mathbb{R}^N.$$

(A6) For all $x\in\Omega$ and $\xi,\eta\in\mathbb{R}^N$; $\xi\neq\eta$ the following inequalities hold

$$\Big(\mathcal{A}(x,\xi)-\mathcal{A}(x,\eta)\Big)(\xi-\eta)>0,$$

$$\Big(\mathcal{B}(x,\xi)-\mathcal{B}(x,\eta)\Big)(\xi-\eta)>0.$$

These are called the structure conditions of $\mathcal{A}$ (resp. $\mathcal{B}$).

Definition 3.1. *Let $L:H\to H^{'}$ be an operator on a Banach space H. L is:*
- *Monotone, if $\langle L(u_1)-L(u_2),u_1-u_2\rangle\geq 0$ for all u_1,u_2.*
- *Strongly continuous, if $u_n\rightharpoonup u$ implies $L(u_n)\to L(u)$.*
- *Weakly continuous, if $u_n\rightharpoonup u$ implies $L(u_n)\rightharpoonup L(u)$.*
- *Demi-continuous, if $u_n\to u$ implies $L(u_n)\rightharpoonup L(u)$.*
- *Said to satisfy the M_0-condition, if $u_n\rightharpoonup u$, $L(u_n)\rightharpoonup f$ and $\langle L(u_n),u_n\rangle\to\langle f,u\rangle$ imply $L(u)=f$.*

The following Proposition plays an important role in the present paper. Precisely, it gives a sufficient conditions to the existence of weak solutions for the problems $Lu=f$.

Proposition 3.1. *Let H be a reflexive, separable Banach space and $L:H\to H'$ an operator which is: coercive, bounded, demicontinuous, and satisfies the M_0-condition. Then the equation $L(u)=f$ admits a solution for each $f\in H'$.*

Below we write $X=W_0^{1,p(\cdot)}$ and $\|u\|=|\nabla u|_{p(x)}$.
In the sequel, we introduce the operator L defined on $W^{1,p(x)}(\Omega)$ by

$$Lu=-\text{div}\,\mathcal{A}(x,\nabla u)+a(x)|u|^{p(x)-2}u, \tag{3.1}$$

where $a(x)$ is a non negative function satisfying assumption (F0).

Definition 3.2. *Let $u \in X$. u is called a weak solution of problem (1.4) if*

$$\int_\Omega \mathcal{A}(x,\nabla u)\nabla v dx + \int_\Omega a(x)|u|^{p(x)-2}uvdx = \int_\Omega fvdx; \quad \forall\, v \in X.$$

Remark 3.1. *Using equation (3.1), L is the sum of L_1 and L_2, where*

$$(L_1(u),v) = \int_\Omega \mathcal{A}(x,\nabla u)\nabla v dx \ \ and \ \ (L_2(u),v) = \int_\Omega a(x)|u|^{p(x)-2}uvdx.$$

In order to prove the existence of weak solutions of the problem (1.4), we will employ variational methods. Precisely, we justify that the operator L satisfies the hypothesis of Proposition 3.1. To this end, we introduce a series of Lemmas dealing with continuity, boundness, coercivity and monotonicity.

Lemma 3.1. *The operator L is a bounded.*

Proof. Denot by

$$\Omega_1 = \{x \in \Omega;\, |u(x)| \geq 1\} \ \text{ and } \ \Omega_2 = \{x \in \Omega;\, |u(x)| < 1\},$$

then

$$(L_2(u),v) = \int_{\Omega_1} a(x)|u|^{p(x)-2}uvdx + \int_{\Omega_1} a(x)|u|^{p(x)-2}uvdx.$$

In view of the assumption $p^+ - 1 < p^*(x)$, the following embeddings hold true:

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{(p^+-1)p(x)}(\Omega) \ \text{ and } \ W_0^{1,p(x)}(\Omega) \hookrightarrow L^{(p^--1)p(x)}(\Omega).$$

Due to Proposition 2.1, we obtain

$$||u|^{p^+-1}|_{p(x)} = |u|^{p^+-1}_{p(x)} \leq c_1\|u\|^{p^+-1}_{p(x)}. \tag{3.2}$$

Take the function $a(x)$ in $L^{p^*(x)/(p^*(x)-2)}(\Omega)$, $|u|^{p^+-1}, v \in L^{p^*(x)}(\Omega)$, and applying Holder inequality, we get

$$\begin{aligned}\Big|\int_{\Omega_1} a(x)|u|^{p(x)-2}uvdx\Big| &\leq c_1|\,a(x)|_{p^*/(p^*-2)}||u|^{p^+-1}|_{p^*(x)}|v|_{p^*(x)}\\ &\leq c_2|\,a(x)|_{p^*/(p^*-2)}|u|^{p^+-1}_{(p^+-1)p^*(x)}|v|_{p^*(x)}.\\ &\leq c_3|\,a(x)|_{p^*/(p^*-2)}\|u\|^{p^+-1}_{p(x)}\|v\|_{p(x)} < \infty.\end{aligned}$$

Similarly,

$$\Big|\int_{\Omega_2} a(x)|u|^{p(x)-2}uvdx\Big| \leq c_4|\,a(x)|_{p^*/(p^*-2)}\|u\|^{p^--1}_{p(x)}\|v\|_{p(x)} < \infty.$$

It follows that the operator L_2 bounded.
Using the structure of $\mathcal{A}$, Holder inequality, equations (2.2)and (2.3), we infer that

$$\begin{aligned}\Big|(L_1(u),v)\Big| &\leq \gamma_2\int_\Omega |\nabla u|^{p(x)-1}|\nabla v|dx\\ &\leq 2\gamma_2\Big\||\nabla u|^{p(x)-1}\Big\|_{p'(x)}\Big\|\nabla v_{p(x)}\\ &\leq 2\gamma_2\max\Big(\|u\|^{p^+}_{1,p(x)}, \|u\|^{p^-}_{1,p(x)}\Big)\|v\|_{1,p(x)}\end{aligned}$$

Hence the operator L is bounded, as we hope. □

Lemma 3.2. *The operator L is dimicontinuous.*

Proof. The proof of the demicontinuity of L_2 will be deduced from the following assumptions.

First step. For all $u, v \in L^{p(x)}(\Omega)$, $|u-v|_{p(x)} \to 0 \Rightarrow ||u-v|^{\frac{p(x)}{p(x)-1}}|_{p(x)-1} \to 0$.

Let $\varepsilon > 0$, $\eta < \varepsilon$ and $u, v \in L^{p(x)}(\Omega)$ such that $|u-v|_{p(x)} < \eta$, then we have

$$|u-v|_{p(x)} = \inf\{\mu \in]0, \eta[;\ \int_\Omega \frac{|u-v|^{p(x)}}{\mu^{p(x)}}\,dx \le 1\}.$$

On the other hand $0 < \mu < \eta < 1$, it follows

$$\int_\Omega \frac{|u-v|^{p(x)}}{\mu^{p(x)-1}}\,dx \le \int_\Omega \frac{|u-v|^{p(x)}}{\mu^{p(x)}}\,dx$$

and consequently

$$\inf\{\mu \in]0, \eta[;\ \int_\Omega \frac{|u-v|^{p(x)}}{\mu^{p(x)-1}}\,dx \le 1\} \le \inf\{\mu \in]0, \eta[;\ \int_\Omega \frac{|u-v|^{p(x)}}{\mu^{p(x)}}\,dx \le 1\}.$$

Since the last term of this inequality represent $|u-v|_{p(x)} < \eta < \epsilon$. The proof of the first claim will be immediately deduced if we consider the fact

$$||u-v|^{\frac{p(x)}{p(x)-1}}|_{p(x)-1} \le \inf\{\mu \in]0, \eta[;\ \int_\Omega \frac{|u-v|^{p(x)}}{\mu^{p(x)-1}}\,dx \le 1\}.$$

Second step. We claim that the map $u \in L^{p(x)}(\Omega) \longmapsto |u|^{p(x)-2} \in L^{\frac{p(x)}{p(x)-1}}(\Omega)$ is continuous. To this end we will use the convention

$$u^{p(x)} = \begin{cases} u^{p(x)}, & \text{for} \quad u \ge 0; \\ -(-u)^{p(x)} & \text{for} \quad u \le 0. \end{cases}$$

Our intention is to show the following identity:

$$|u-v|_{p(x)} \to 0 \Rightarrow |u^{p(x)-1} - v^{p(x)-1}|_{\frac{p(x)}{p(x)-1}} \to 0.$$

The result is trivial when $p(x) = 2$. We claim to prove the result for $p(x) > 2$.

$$\rho_{\frac{p(x)}{p(x)-1}}(u^{p(x)-1} - v^{p(x)-1}) := \int_\Omega |u^{p(x)-1} - v^{p(x)-1}|^{\frac{p(x)}{p(x)-1}}\,dx,$$

then, for $x \in \Omega$, by Lagrange theorem applied to the function $g(y) = y^{p(x)-1}$, there exists $c(x)$ satisfying

$$\frac{g(u(x)) - g(v(x))}{u(x) - v(x)} = g'(c(x)).$$

Due to the fact that $(u-v) \in L^{p(x)}(\Omega)$, we have $|u-v|^{\frac{p(x)}{p(x)-1}} \in L^{p(x)-1}(\Omega) = (L^{\frac{p(x)-1}{p(x)-2}}(\Omega))^*$ and $|u|, |v| \in L^{p(x)}(\Omega)$ imply $|u|^{\frac{p(x)(p(x)-2)}{p(x)-1}}$, $|v|^{\frac{p(x)(p(x)-2)}{p(x)-1}} \in L^{\frac{p(x)-1}{p(x)-2}}(\Omega)$. Hence

$$\rho_{\frac{p(x)}{p(x)-1}}(u^{p(x)-1} - v^{p(x)-1}) \le p^{+\frac{p^+}{p^- -1}} \int_\Omega |u-v|^{\frac{p(x)}{p(x)-1}} \sup(|u|, |v|)^{\frac{p(x)(p(x)-2)}{p(x)-1}}\,dx.$$

Thus the proof of the continuity by using (2.1), (2.4) and the second claim.

Next we are going to prove the demicontinuity of the operator L_1. For this purpose let $(u_n) \subset W_0^{1,p(x)}(\Omega)$ such that $u_n \longrightarrow u$ in $W_0^{1,p(x)}(\Omega)$. Passing to a subsequence we may assume that $u_n \longrightarrow u$ and $\nabla u_n \longrightarrow \nabla u$ pointwise almost everywhere. In view of the continuity of the map $\xi \longrightarrow \mathcal{A}(x, \xi)$, it follows that $\mathcal{A}(x, \nabla u_n) \longrightarrow \mathcal{A}(x, \nabla u)$ almost everywhere. Using condition **(A5)**, we obtain

$$\int_\Omega \Big|\mathcal{A}(x, \nabla u_n)\Big|^{\frac{p(x)}{p(x)-1}} dx \le \gamma_2 \int_\Omega \Big|\nabla u_n\Big|^{p(x)} dx < \infty$$

By the convergence of the sequence $(u_n), (\mathcal{A}(x, \nabla u_n))$ is bounded in $L^{p'(x)}(\Omega)$. Thus we may pass again to a further subsequence and assume that $\mathcal{A}(x, \nabla u_n) \longrightarrow \mathcal{A}(x, \nabla u)$ weakly in $L^{p'(x)}(\Omega)$.

Let us argue by contradiction that the whole sequence converges weakly. Suppose that we can find a neighbourhood U of $\mathcal{A}(x, \nabla u)$ and a subsequence such that $\mathcal{A}(x, \nabla u_{n_k}) \subset L^{p'(x)}(\Omega) \backslash U$. We may assume pointwise convergence by passing to a further subsequence and this sub-subsequence converges weakly in $L^{p'(x)}(\Omega)$ to $\mathcal{A}(x, \nabla u)$ by the earlier argument, which is a contradiction. Consequently,

$$(L_1 u_n, v) = \int_\Omega \mathcal{A}(x, \nabla u_n) \nabla v dx \longrightarrow \int_\Omega \mathcal{A}(x, \nabla u) \nabla v dx = (L_1 u, v).$$

□

Lemma 3.3. *The operator L is strictly monotone.*

Proof. The strict monotonicity of L will be immediately deduced for the monotonicity condition of $\mathcal{A}$, precisely assumption **(A6)** and the following elementary identity [20] and [28].

$$2^{2-p}|a-b|^p \leq \Big(a|a|^{p-2} - b|b|^{p-2}\Big).(a-b), \text{ if } p(x) \geq 2, \tag{3.3}$$

$$(p-1)|a-b|^2\Big(|a|+|b|\Big)^{p-2} \leq \Big(a|a|^{p-2} - b|b|^{p-2}\Big).(a-b), \text{ if } 1 < p(x) < 2. \tag{3.4}$$

for all $a, b \in \mathbb{R}^n$, where . denotes the standard inner product in $\mathbb{R}^n$. □

The last Lemma in this section deal with coercivity, in particular we have

Lemma 3.4. *The operator L is coercive.*

Proof. Using the positivity of the function $a(\cdot)$, the definition of L, we get

$$(Lu, u) \geq \int_\Omega \mathcal{A}(x, \nabla u) \nabla u dx.$$

In view of assumption **(A4)**, we can write

$$(Lu, u)\gamma_1 \geq \gamma_1 \int_\Omega |\nabla u|^{p(x)} dx := \rho_{p(x)}(|\nabla u|). \tag{3.5}$$

Combining equations (2.2), (2.3) and (3.5), one has

$$(Lu, u) \geq \gamma_1(|\nabla u|_{p(x)}^{p^-}, |\nabla u|_{p(x)}^{p^+}) = \gamma_1 \inf(\|u\|^{p^-}, \|u\|^{p^+}).$$

Using the fact that $p^- > 1$, one writes

$$(Lu, u)/\|u\| \geq \gamma_1 \inf(\|u\|_{p(x)}^{p^- -1}, \|u\|_{p(x)}^{p^+ -1}) \to \infty \quad \text{as} \quad \|u\| \to \infty.$$

Hence, the operator L is coercive as required. □

Remark. Using previous Lemmas, all conditions of Proposition 3.1 are fulfilled. hence, the proof of Theorem 1.1 is completed.

4. Nonlinear systems on bounded domains

The goal of this section is to prove existence of weak solutions for the system (1.1). The operators $\mathcal{A}$ and $\mathcal{B}$ are such that conditions **(A1)-(A6)** fulfulled. We suppose that $p(x)$ and $q(x)$ are Lipshitz-continuous functions defined on $\mathbb{R}^N$. In addition, we take $p(x), q(x) \in C^{0,1}(\Omega)$. We denote by $p'(x), q'(x)$ the conjugate exponents of $p(x), q(x)$ respectively. i.e.

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = \frac{1}{q(x)} + \frac{1}{q'(x)} = 1.$$

a, b, c, d are non negative functions satisfying condition (F0), (F1) and (F2). Finally, α and β are regular nonnegative functions such that the assumption F3) will be satisfied.
In the following discussions, we will use the product space

$$W_{p(x),q(x)} := W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega), \tag{4.1}$$

which is equipped with the norm

$$\|(u,v)\|_{p(x),q(x)} := \max\{\|u\|_{p(x)}; \|v\|_{q(x)}\};\ \forall\,(u,v) \in W_{p(x),q(x)}, \tag{4.2}$$

where $\|u\|_{p(x)}$ (resp., $\|u\|_{q(x)}$) is the norm of $W_0^{1,p(x)}(\Omega)$ (resp., $W_0^{1,q(x)}(\Omega)$).
The space $W^*_{p(x),q(x)}$ denotes the dual space of $W_{p(x),q(x)}$ and equipped with the norm

$$\|.\|_{*,p(x),q(x)} := \|.\|_{*p(x)} + \|.\|_{*,q(x)},$$

where $\|.\|_{*p(x)}, \|.\|_{*,q(x)}$ are respectively the norm of $W_0^{-1,p'(x)}(\Omega)$ and $W_0^{-1,q'(x)}(\Omega)$, dual resp. of $W_0^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$.
First, we recall the following definition.

Definition 4.1. *The pair $(u,v) \in W_{p(x),q(x)}$ is called a weak solution of the system (1.1), if*

$$\int_\Omega (\mathcal{A}(x,\nabla u)\nabla\Phi_1 + \mathcal{B}(x,\nabla v)\nabla\Phi_2)dx = \int_\Omega (F_1(x,u,v)\Phi_1 + F_2(x,u,v)\Phi_2)dx,$$

for all $(\Phi_1,\Phi_2) \in W_{p(x),q(x)}$, where F and G are defined by

$$F_1(x,u,v) = -a(x)|u|^{p(x)-2}u - b(x)|u|^{\alpha(x)}|v|^{\beta(x)}v + f(x),$$

$$F_2(x,u,v) = -c(x)|v|^{q(x)-2}u - d(x)|u|^{\alpha(x)}|v|^{\beta(x)}u + g(x).$$

The weak formulation of the system (1.1) is reduced to the operator form identity

$$L_1(u,v) + L_2(u,v) + B(u,v) = F, \tag{4.3}$$

where L_1, L_2, B and F are defined on $W_{p(x),q(x)}$ as follow:

$$\begin{aligned}
(L_1(u,v),(\Phi_1,\Phi_2)) : &= \int_\Omega \mathcal{A}(x,\nabla u)\nabla\Phi_1 dx + \int_\Omega \mathcal{B}(x,\nabla v)\nabla\Phi_2 dx,\\
(L_2(u,v),(\Phi_1,\Phi_2)) : &= \int_\Omega a(x)|u|^{p(x)-2}u\Phi_1 dx + \int_\Omega c(x)|v|^{q(x)-2}v\Phi_2 dx,\\
(B(u,v),(\Phi_1,\Phi_2)) : &= \int_\Omega b(x)|u|^{\alpha(x)}|v|^{\beta(x)}v\Phi_1 dx + \int_\Omega d(x)|v|^{\beta(x)}|u|^{\alpha(x)}u\Phi_2 dx,\\
(F,\Phi) : &= ((f,g),(\Phi_1,\Phi_2)) = \int_\Omega f\Phi_1\, dx + \int_\Omega g\Phi_2\, dx.
\end{aligned}$$

Proof. of Theorem 1.2. To prove the existence of weak solutions of the system $(\mathcal{S})$, we are going to study properties of the operators L_1, L_2, B and F.
1. In view of the previous section, in particular Lemmas 3.1, 3.2, 3.3 we have similar properties to the operators L_1 and L_2, i.e. L_1 and L_2 are demi-continuous, bounded and strictly monotone, hence their sum.
2. The second remark consist in the proof of coercivity of the operator $\widetilde{L}$ defined on the space $W_{p(x),q(x)}$ by: $(\widetilde{L}(u,v),(\Phi_1,\Phi_2)) = ((L_1 - L_2 + B)(u,v),(\Phi_1,\Phi_2))$, for all $(\Phi_1,\Phi_2) \in W_{p(x),q(x)}$. Let $(u,v) \in W_{p(x),q(x)}$, then

$$\begin{aligned}(\widetilde{L}(u,v),(u,v)) \quad &\geq \quad \gamma_1 \int_\Omega |\nabla u|^{p(x)} + \int_\Omega a(x)|u|^{p(x)} + \delta_1 \int_\Omega |\nabla v|^{q(x)} + \int_\Omega c(x)|v|^{q(x)} \\ &\quad + \int_\Omega b(x)|u|^{\alpha(x)}|v|^{\beta(x)+1} + \int_\Omega d(x)|u|^{\alpha(x)+1}|v|^{\beta(x)}.\end{aligned}$$

Since, the functionals $a(x), b(x), c(x)$ and $d(x)$ are positive on Ω, we have

$$(\widetilde{L}(u,v),(u,v)) \geq \int_\Omega |\nabla u|^{p(x)} + \int_\Omega |\nabla v|^{q(x)}.$$

Using inequalities (2.2) and (2.3), we obtain

$$(\widetilde{L}(u,v),(u,v)) \geq \min(|\nabla u|^{p^+}_{p(x)}; |\nabla u|^{p^-}_{p(x)}) + \min(|\nabla v|^{q^+}_{q(x)}; |\nabla v|^{q^-}_{q(x)}).$$

Since $\|u\|_{p(x)} = |\nabla u|_{p(x)}$, $\|v\|_{q(x)} = |\nabla v|_{q(x)}$ and $p^-, q^- > 1$, therefore

$$\frac{(\widetilde{L}(u,v),(u,v))}{\|(u,v)\|_{p(x),q(x)}} \to \infty \quad \text{as} \quad \|(u,v)\|_{p(x),q(x)} \to \infty.$$

The proof of the coercivity of the operator $\widetilde{L}$ is verified.
3. The operator $B(u;v)$ is well defined; indeed, if

$$\Omega_1 = \{x \in \Omega;\ |u(x)| \geq 1,\ |v(x)| \geq 1\}, \qquad \Omega_2 = \{x \in \Omega;\ |u(x)| < 1,\ |v(x)| < 1\},$$
$$\Omega_3 = \{x \in \Omega;\ |u(x)| \geq 1,\ |v(x)| \leq 1\} \text{ and } \Omega_4 = \{x \in \Omega;\ |u(x)| < 1,\ |v(x)| \geq 1\},$$

we have

$$\int_\Omega b(x)|u|^{\alpha(x)}|v|^{\beta(x)} v\phi_1 dx = \sum_{i=1}^{4} \Big(\int_{\Omega_i} b(x)|u|^{\alpha(x)}|v|^{\beta(x)} v dx \phi_1 \Big).$$

Furthermore,

$$\Big| \int_{\Omega_1} b(x)|u|^{\alpha(x)}|v|^{\beta(x)} v\phi_1 dx \Big| \leq \int_{\Omega_1} b(x)|u|^{\alpha^+}|v|^{\beta^+ +1}|\phi_1| dx.$$

Since $\alpha^+ + 1 < p^*(x)$, $\beta^+ + 1 < q^*(x)$, then the following embeddings hold true

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha^+ p(x)}(\Omega) \text{ and } W_0^{1,q(x)}(\Omega) \hookrightarrow L^{(\beta^+ +1)q(x)}(\Omega).$$

Then, we obtain

$$||u|^{\alpha^+}|_{\alpha^+ p(x)} \leq c_1 |u|_{p(x)} \leq c_2 ||u|^{\alpha^+}|_{p^*(x)}, \text{ and } ||v|^{\beta^+ +1}|_{q(x)} \leq c_3 ||v|^{\beta^+ +1}|_{q^*(x)}.$$

If we apply (2.2), (2.3) and Proposition 2.1 and take the functionals $b \in L^{s(x)}(\Omega)$; $d \in L^{r(x)}(\Omega)$, then we have

$$\Big| \int_{\Omega_1} b(x)|u|^{\alpha(x)}|v|^{\beta(x)} v\phi_1 dx \Big| \leq |b(x)|_{s(x)} |u^{\alpha^+}|_{p^*(x)} ||v|^{\beta^+ +1}|_{q^*(x)} |\phi_1|_{\widetilde{p}(x)} < \infty,$$

$$\Big| \int_{\Omega_1} d(x)|v|^{\beta(x)}|u|^{\alpha(x)} u\phi_2 dx \Big| \leq |d(x)|_{r(x)} ||u|^{\alpha^+ +1}|_{p^*(x)} ||v|^{\beta^+}|_{q^*(x)} |\phi_2|_{\widetilde{q}(x)} < \infty,$$

$$\Big|\int_{\Omega_2} b(x)|u|^{\alpha(x)}|v|^{\beta(x)}v\phi_1 dx\Big| \le |b(x)|_{s(x)}|u^{\alpha^-}|_{p^*(x)}||v|^{\beta^-+1}|_{q^*(x)}|\phi_1|_{\widetilde{p}(x)} < \infty,$$

$$\Big|\int_{\Omega_2} d(x)|v|^{\beta(x)}|u|^{\alpha(x)}u\phi_2 dx\Big| \le |d(x)|_{r(x)}||u|^{\alpha^-+1}|_{p^*(x)}||v|^{\beta^-}|_{q^*(x)}|\phi_2|_{\widetilde{q}(x)} < \infty.$$

Repeating the same arguments we deduce

$$\Big|\int_{\Omega_i} b(x)|u|^{\alpha(x)}|v|^{\beta(x)}v\phi_1 dx\Big| < \infty,\ \Big|\int_{\Omega_i} d(x)|v|^{\beta(x)}|u|^{\alpha(x)}u\phi_2 dx\Big| < \infty, \text{ for } i = 3, 4.$$

Hence, $|(B(u;v),(\Phi_1,\Phi_2))| < \infty$. The operator $B(u;v)$ is well defined on $W_{p(x),q(x)}$.
4. It remainds to prove the continuity of the operator B. To this end we will show the compactness of B.
Let $\{(u_n,v_n)\} \subset W_{p(x),q(x)}$ be a sequence such that $(u_n,v_n) \rightharpoonup (u,v)$ weakly in $W_{p(x),q(x)}$. We claim that $B(u_n,v_n) \to B(u,v)$ strongly in $W_{p(x),q(x)}$, i.e. for all $(\Phi_1,\Phi_2) \in W_{p(x),q(x)}$ we have

$$\Big|(B(u_n,v_n) - B(u,v);\ (\Phi_1,\Phi_2))\Big| = \circ(1) \text{ as } n \to \infty.$$

Clearly

$$B(u_n,v_n) - B(u,v) = (B_u(u_n,v_n) - B_u(u,v)) + (B_v(u_n,v_n) - B_v(u,v)),$$

where

$$(B_u(u_n,v_n) - B_u(u,v);\ (\Phi_1,\Phi_2)) = \int_\Omega b(x)(|u_n|^{\alpha(x)}|v_n|^{\beta(x)}v_n - |u|^{\alpha(x)}|v|^{\beta(x)}v)\Phi_1 dx,$$

and

$$(B_v(u_n,v_n) - B_v(u,v);\ (\Phi_1,\Phi_2)) = \int_\Omega d(x)(|v_n|^{\beta(x)}|u_n|^{\alpha(x)}u_n - |v|^{\beta(x)}|u|^{\alpha(x)}u)\Phi_2 dx.$$

Then it's sufficient to prove the compactness of $B_u(u,v)$ and $B_v(u,v)$.

$$\begin{aligned}(B_u(u_n,v_n) - B_u(u,v);\ (\Phi_1,\Phi_2)) &= \int_\Omega b(x)|v_n|^{\beta(x)+1}(|u_n|^{\alpha(x)} - |u|^{\alpha(x)})\Phi_1\, dx \\ &\quad + \int_\Omega b(x)|u|^{\alpha(x)}(|v_n|^{\beta(x)+1} - |v|^{\beta(x)}v)\Phi_1\, dx.\end{aligned}$$

In view of item **3.** one writes

$$\begin{aligned}\Big|(B_u(u_n,v_n) - B_u(u,v);\ (\Phi_1,\Phi_2))\Big| &\le c_1|b(x)|_{s(x)}\Big(|v_n|^{\beta(x)+1}_{q^*}\Big||u_n|^{\alpha(x)} - |u|^{\alpha}\Big|_{p^*} \\ &\quad ||u|^{\alpha(x)}|_{p^*}||v_n|^{\beta(x)+1} - |v|^{\beta(x)}v|_{q^*}\Big)|\Phi_1|_{\widetilde{p}}.\end{aligned}$$

A similar calculation gives us the following inequality

$$\begin{aligned}\Big|(B_v(u_n,v_n) - B_v(u,v);\ (\Phi_1,\Phi_2))\Big| &\le c_2|d(x)|_{r(x)}\Big(|u_n|^{\alpha+1}_{p^*(x)}\Big||v_n|^{\beta(x)} - |v|^{\beta(x)}\Big|_{q^*} \\ &\quad ||v|^{\beta}|_{p^*}||u_n|^{\alpha(x)+1} - |u|^{\alpha(x)}u|_{p^*}\Big)|\Phi_2|_{\widetilde{q}}.\end{aligned}$$

Due to the continuity of Nemytskii operators $u \to |u|^{\alpha(x)}$ (resp. $v \to |v|^{\beta(x)}v$) from $L^{p(x)}(\Omega)$ into $L^{p^*(x)}(\Omega)$ (resp. from $L^{q(x)}(\Omega)$ into $L^{q^*(x)}(\Omega)$), there exists $n_0 \ge 0$ such that for all $n \ge n_0$ we have

$$\Big||u_n|^{\alpha(x)} - |u|^{\alpha(x)}\Big|_{p^*(x)} = \circ(1), \tag{4.4}$$

$$\Big||v_n|^{\beta(x)+1} - |v|^{\beta(x)}v\Big|_{q^*(x)} = \circ(1). \tag{4.5}$$

Finally from equations (4.4) and (4.5), we have the claim and the operator B will be compact and completely continuous. Hence, B satisfies the M_0-condition and the system $(\mathcal{S})$ possess a weak solution $(u, v) \in W_{p(x),q(x)}$, for all (f, g) in the dual of $W_{p(x),q(x)}$. The proof of the main result on bounded domains is completed.

$\square$

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