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The relation between the weight factor and the number of steps in a projection algorithm

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Abstract. This paper gives an implementation of a projection algorithm for solving the convex feasibility problem. We analyze the influence of the weight factor from the projection algorithm on the total number of steps needed to obtain a solution of a convex feasibility problem. We solve a linear system of inequations using a projection algorithm and we determine how the weight factor influence the total number of iterations calculated until the system solution is obtain.

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1. Introduction

Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$ and let $T : C \to C$ be a nonlinear mapping with nonempty fixed point set $F(T)$ in C. Let $\{x_k\}_{k\geq 0}$ be a sequence defined by the following *Mann-type* iterative process

$$x_{k+1} = (1 - t_k)x_k + t_k T(x_k),\ x_0 \in C, \tag{1}$$

where $t_k \in \mathcal{R}^+,\ k = 0, 1, ...$ If $0 < t_k < 1$, then x_{k+1} is a convex combination between x_k and $T(x_k)$. This restriction concerning $\{t_k\}_{k\geq 0}$ is not always satisfied; the typical case is, for example, the projection algorithm for convex feasibility problem, algorithm which have just the form (1) and $0 < t_k < 2$.

A straightforward application of the Mann-type iteration is the projection algorithms for solving the convex feasibility problem, particularly because such algorithm have the form (1) and the projection mapping has the properties required in the Mann-type iteration. The geometric idea of the projection method is to projects the current iteration onto certain set form the intersecting family and to take the next iteration on the straight line connecting the current iteration and this projection. A weight factor gives the exact position of the next iteration. Different strategies concerning the selection of the set onto which the current iteration will be projected, will give particular projection types algorithms.

The projection algorithm was used in [1], [16] for solving a system of linear inequalities (the authors referred their method as *relaxation algorithm*). Generalizations for convex sets in real n-dimensional spaces were given in [10], [12]. Bergman [5] considered the classical projection method for the case of m intersecting closed convex sets M_i in a real Hilbert space. He showed that, given an arbitrary starting point x_0, the sequence generated by the projection algorithm converges weakly to a point

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in $M = \cap_{i=1}^{m} M_i$. In [11] certain *regularity conditions* of the family of sets were described that guarantee the strong convergence of the iterations. In its essence, this regularity conditions means that $int \cap_{i=1}^{m} M_i \neq \emptyset$; more exactly, the main condition considered in [11] is: $M_{\overline{\alpha}} \cap (Int \cap_{\alpha \in A} M_\alpha) \neq \emptyset$, where A is a set of indexes and $M_{\overline{\alpha}}$ is a certain set of the family. Such condition is necessary for the affirmative answer to the following very simple problem: *Suppose that a sequence is getting closer to every set of an intersecting family of sets; does the sequence get also closer to their intersection?*. In [11] an affirmative answer is done, provided that the family satisfies the above regularity condition. Note that the boundedness of the sequence is also required. Bausche and Borwein [2] considered such a property as a definition for the *regular n-tuple of sets*, namely, a family of n sets is regular if

$$\forall_{\epsilon>0}\ \exists_{\delta>0}\ \forall_{\substack{x \in \mathcal{H} \\ max\{d(x,M_i),\ i=1,\dots,n\} \leq \delta}}\ d(x, \cap M_i) \leq \epsilon$$

In some recent papers, other conditions for strong convergence have been given, for example in [2], [3], [4], [8]. A complete and exhaustive study on algorithms for solving convex feasibility problem, including comments about their applications and an excellent bibliography, was given by Bausche and Borwein [2].

2. Definitions and some former and recent results

The convergence properties of (1), both weak and strong, are related with the structural properties of T.

Definition 2.1. *The mapping T is said to be*

(a1) quasi-nonexpansive if

$$\|T(x) - x^*\|^2 \leq \|x - x^*\|^2,\ \forall x \in C,\ x^* \in F(T);$$

(b1) demicontractive if for certain constant $\mathbf{p} \in \mathcal{R}$ the following inequality holds

$$\|T(x) - x^*\|^2 \leq \|x - x^*\|^2 + \mathbf{p}\|x - T(x)\|^2,\ \forall x \in C,\ x^* \in F(T).$$

Definition 2.2. *The mapping T is said to be*

(a2) firmly nonexpansive if

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 - \|x - y - (T(x) - T(y))\|^2,\ \forall x, y \in C;$$

(b2) firmly quasi-nonexpansive if

$$\|T(x) - x^*\|^2 \leq \|x - x^*\|^2 - \|x - T(x)\|^2,\ \forall x \in C,\ x^* \in F(T).$$

Usually, the convergence of (1) requires some additional smoothness properties of the mapping T, like continuity or demiclosedness.

Definition 2.3. *A mapping T is said to be demiclosed, if for any sequence $\{x_k\}_{k\geq 0}$ which converges weakly to y, and if the sequence $\{T(x_k)\}_{k\geq 0}$ converges strongly to z, then $T(y) = z$.*

In what follows, only the particular case of demiclosedness *at zero* will be used, which is the particular case when $z = 0$.

A typical result for real Hilbert spaces states that if $T : C \to C$ is quasi-nonexpansive and $I - T$ is demiclosed then the sequence $\{x_k\}_{k\geq 0}$ defined by (1) with $0 < a \leq t_k \leq b < 1$ converges weakly to a fixed point of T ([9]). For strong convergence some additional conditions must be imposed, for instance, that T be continuous and $lim_{k\to\infty} d(x_k, F(T)) = 0$, where $d(x, M)$ is the distance from x to M ([17]).

In the paper [14] we have considered a class of mappings which satisfies the following condition: There exists a strict positive number λ such that

$$\langle x - T(x), x - x^* \rangle \geq \lambda \|x - T(x)\|^2, \ \forall x \in C, \ x^* \in F(T). \tag{2}$$

For this class of mappings the weak convergence of the sequence $\{x_k\}_{k\geq 0}$ generated by (1) is shown, provided that $I - T$ is demiclosed at zero. For the strong convergence of the same sequence, additional condition on T and on the starting iteration point is needed ([14]). These results were extended to some more general spaces (Banach spaces, uniformly smooth Banach spaces, etc.) in some papers [6], [7], [18]. The almost identic conditions were used in [15] for proving the weak convergence of a Mann and Ishikawa iteration processes with errors to a fixed point of T, processes considered earlier in [13] and [19] for nonlinear strongly accretive operators.

C. Moore [15] observed that the class of maps satisfying (2) coincide with the class of demicontractive mappings. Indeed, it can be seen that (2) is equivalent with the condition (b1), where $\lambda = \frac{1-\mathbf{p}}{2}$.

3. The convex feasibility problem

Let $M_i \subset H, \ i = 1, ..., m$ be a family of closed convex subsets of $\mathcal{H}$ with nonempty intersection, $\bigcap M_i \neq \emptyset$. The convex feasibility problem is:

$$Find \ a \ point \ of \ \bigcap M_i.$$

Let x be a point in $\mathcal{H}$ and let $P(x, i)$ be the projection of x onto M_i (if $x \in M_i$, then $P(x, i) = x$). Let i_x be the least index such that

$$\|x - P(x, i_x)\| = \max_i \|x - P(x, i)\|.$$

Define the mapping $T : \mathcal{H} \to \mathcal{H}$ by $T(x) = P(x, i_x)$. It is clear that $x \in \bigcap M_i$ if and only if $T(x) = x$, hence if and only if x is a fixed point of T, that is $\bigcap M_i = F(T)$. For any $x \in \mathcal{H}$ and $x^* \in F(T)$, the following Kolmogorov condition $\langle x - P(x, i_x), P(x, i_x) - x^* \rangle \geq 0$ is satisfied and it is routine to see that T is firmly quasi-nonexpansive. The Mann iteration in this case converges strongly to an element of $F(T)$ if and only if $\{x_k\}_{k\geq 0}$ is regular with respect to $F(T)$. In its turn this property of $\{x_k\}_{k\geq 0}$ is in connection with the regularity property of the family $\{M_i\}$.

Let us consider a family of n sets: $M_i, \ i = \overline{1, \ldots, n}$. Each set contains the points that verify the inequation $a_i x + b_i y + c_i \geq 0$ where $a_i, b_i, c_i \in \mathcal{R}$. This way we obtain a n inequations system with the solution given by the intersection of those n sets. Thus, the solution of the system represents the solution for the following convex feasibility problem:

$$Find \ a \ point \ of \ M = \bigcap_{i=1}^{n} M_i.$$

4. The application

We have implemented a projection algorithm for solving the following convex feasibility problem:

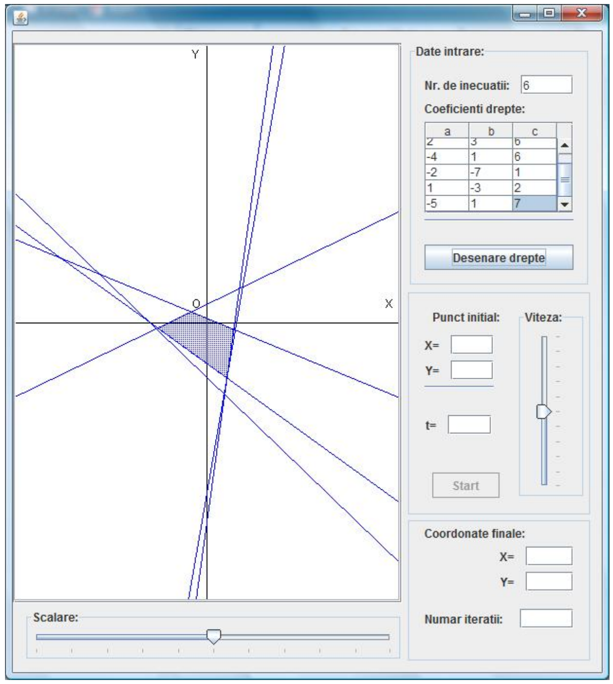

FIGURE 1. The application

$$\begin{cases} M_i = \{(x,y) \in \mathcal{R} \times \mathcal{R}, a_i x + b_i y + c_i \geq 0\}, i = \overline{1,\ldots,n} \\ find(x,y) \in M = \bigcap_{i=1}^{n} M_i \end{cases}$$

The sets M_i are half-spaces determinate by $a_i x + b_i y + c_i = 0$, $a_i, b_i, c_i \in \mathcal{R}$, $i = \overline{1,\ldots,n}$. So the sets are convex subsets of the space $\mathcal{R}^2$.

The application 1 permits to solve a convex feasibility problem with a variable number of sets. The only condition imposed by the application is that for any $i = \overline{1,\ldots,n}$ $c_i \in \mathcal{R}_+$.

From the following examples we can observe that the initial point does not influence the number of iterations calculated in order to obtain a solution of the convex feasibility problem.

The application permits for the same initial data (the same system and the same initial point) to introduce different weight factor in order to analyze its influence on the number of iterations.

In the application the operator used is the classical projection:

$$x_{n+1} = (1-t)x_n + tP_{M_i}x_n, \forall n \geq 0.$$

If $P_{M_i}(x)$ denote the projection of x onto M_i then the classical projection method is

$$x_{k+1} = (1-t_k)x_k + t_k P_{M_{\alpha(k)}}(x_k),$$

where t_k is the weight factor, $0 < t_k < 2$, and the function $\alpha : \mathcal{N} \to \{1, ..., N\}$ defines the strategy. The usual strategy is the cyclic covering the sets of family, that is $\alpha(k) = mod_N(k) + 1$.

TABLE 1. The number of iterations for the initial approximation (1,4)

initial approximation	weight factor	number of iterations
(1,4)	0.1	257
(1,4)	0.2	200
(1,4)	0.3	169
(1,4)	0.4	129
(1,4)	0.5	93
(1,4)	0.6	80
(1,4)	0.7	57
(1,4)	0.8	35
(1,4)	0.9	18
(1,4)	1.0	17
(1,4)	1.1	17
(1,4)	1.2	17
(1,4)	1.3	13
(1,4)	1.4	7
(1,4)	1.5	4
(1,4)	1.6	4
(1,4)	1.7	3
(1,4)	1.8	4
(1,4)	1.9	5

In the current step the sets for the projection are considered cyclic, the sets are analyzed starting from the first set M_1 to the last M_n. This means that for the current iteration x_n, the sets, M_i $i = \overline{1, \ldots, n}$, are checked in order. We start with M_1, and we check if the point x_n is in this set or no. If $x_n \in M_1$, then we go to the next set M_2. Else we apply the projection and we calculate the next iteration x_{n+1}.

```
try {
 do {
  for (int i=0;i<DH1.n;i++) {
   if (D1.testPunct(i,P)==false){
    double x=P.x-D1.getCoord(i,0)*((D1.getCoord(i,0)*P.x+
      +D1.getCoord(i,1)*P.y+D1.getCoord(i,2))/
      (Math.pow(D1.getCoord(i,0),2)+
      +Math.pow(D1.getCoord(i, 1), 2)))*DH1.getT();
    double y=P.y-D1.getCoord(i,1)*((D1.getCoord(i,0)*P.x+
     +D1.getCoord(i,1)*P.y+
     +D1.getCoord(i,2))/(Math.pow(D1.getCoord(i,0),2)+
     +Math.pow(D1.getCoord(i,1),2)))*DH1.getT();
       P=new Punct(x,y);
       DH1.AddFinalPoint(x, y);
       parent.UpdateGrafic();
       itmax++;
       Thread.sleep(sleep);}
     }
      ok=true;
```

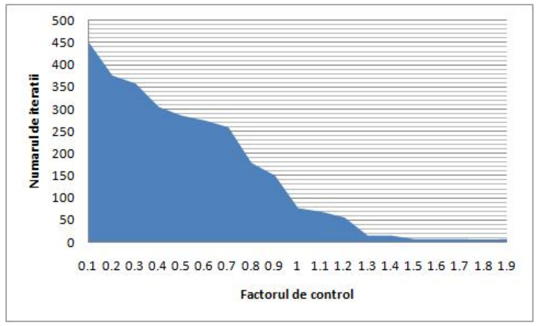

FIGURE 2. The weight factor and the number of iterations

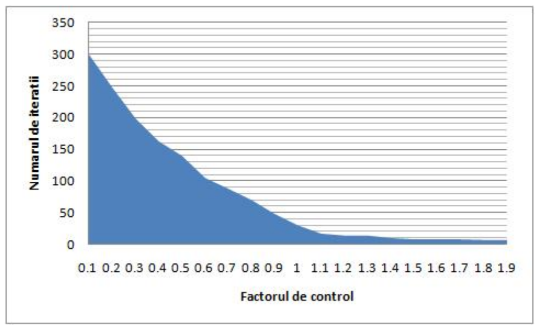

FIGURE 3. The weight factor and the iterations number

```
      for (int j=0;j<DH1.n;j++)
         if (D1.testPunct(j,P)==false)
              ok=false;
     }while (ok==false);
  } catch (InterruptedException ex) {
  Logger.getLogger(Executor.class.getName()).
     log(Level.SEVERE,null,ex);
}
```

From the following examples we can observe that the weight factor influence the total number of iterations calculated until the solution if the system in find.

Let's consider the following example:

$$\begin{cases} x+2y+3\geq 0\\ 2x+3y+6\geq 0\\ -4x+y+6\geq 0\\ -2x-7y+1\geq 0\\ x-3y+2\geq 0\\ -5x+y+7\geq 0 \end{cases}$$

The results for this system, for diferent weight factor $t\in(0,2)$ and for different initial approximations, are given in the next tables 1 and 2.

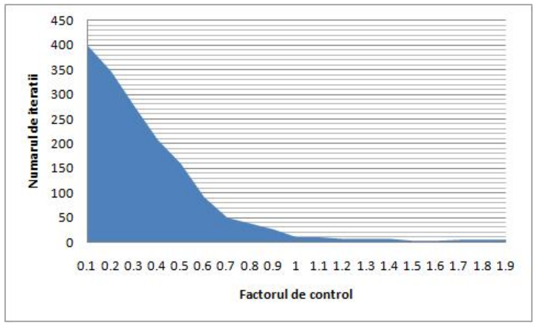

Figure 4. The weight factor versus the iterations number

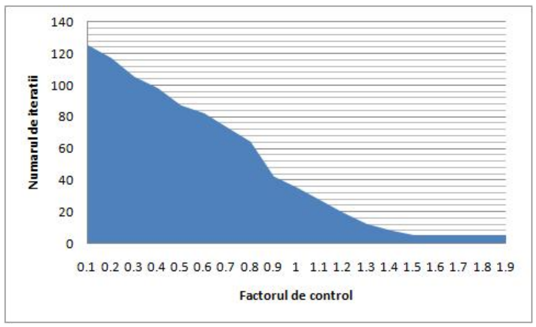

Figure 5. The influence of the weight factor on the iterations number

We can observe that for the weight factor $t = 1.5$ we obtain a minimum number of iterations. In some cases we can observe that the iterations number is constant for the weight factor $t \in [1.5, 2]$, and in other cases for $t \in [1, 1.5]$.

For small values of the weight factor $t \in [0.1, 1)$ the number of calculated iterations is very big and it is getting smaller very fast when the weight factor is closing to $t = 1.5$.

Table 2. The influence of the weight factor on the iterations number

weight factor	number of iterations	number of iterations	number of iterations
0.1	407	138	205
0.2	293	102	183
0.3	247	81	148
0.4	215	72	97
0.5	193	57	69
0.6	179	49	55
0.7	158	36	27
0.8	130	31	17
0.9	117	18	17
1.0	88	15	17
1.1	60	12	15
1.2	5	9	13
1.3	4	5	10
1.4	4	3	6
1.5	4	2	2
1.6	4	2	2
1.7	4	3	4
1.8	4	4	7
1.9	4	5	9

Table 3. The connection between the weight factor and the iterations number

weight factor	number of iterations	number of iterations
0.1	268	192
0.2	221	179
0.3	197	158
0.4	168	131
0.5	139	102
0.6	101	82
0.7	72	82
0.8	49	41
0.9	37	17
1.0	26	13
1.1	15	5
1.2	9	4
1.3	4	2
1.4	2	2
1.5	2	2
1.6	2	2
1.7	4	2
1.8	5	1
1.9	5	1

5. Conclusions

The initial approximation does not influence, at least not significantly, the number of iterations calculated until the solution for the convex feasibility is found. The parameter that influence this number is the weight factor from the projection algorithm.

It can be observed that for the weight factor $t = 1.5$ we obtain a minimum iterations number. The iterations number in this case in very small and it is influence by the system and the initial approximation, but this influence is minor.

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