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Some categorical properties of Hilbert algebras

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Abstract. We show that the category of Hilbert algebras is complete. We also show that it has equalizers, coequalizers and kernel pairs. Coproducts of Hilbert algebras are characterized and finally it is proved that the category of Hilbert algebras is cocomplete.

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1. Introduction

The concept of Hilbert algebras was introduced in the 50-ties by L. Henkin ([7]) and T. Skolem for investigations in intuitionistic and other non-clasical logics, as an algebraic counterpart of Hilbert's positive implicative propositional calculus ([6]). Hilbert algebras were intensively studied by A. Diego ([5]) and this theory was further developed by D. Busneag ([2], [3]).

Among category theoretic constructs, limits and colimits are ones of the fundamental importance. We will prove that the category of Hilbert algebras is both complete and cocomplete. Firstly, we will show that in the category of Hilbert algebras there are equalizers and coequalizers (section 3). The existence of direct products is obvious and in the next section we shall prove the dual: the existence of coproducts. Finally, combining these results with a theorem from [8], we obtain the completeness and cocompleteness of the category of Hilbert algebras.

We will follow standard definitions. Our categorical concepts will be those of standard texts ([1], [3], [8]) and are included in the following section.

2. Preliminaries

In this section, we include some general categorical concepts. For the following notions we refer to [3] and [8].

Let $\mathcal{C}$ be a category and $(M_i)_{i\in I}$ a family of objects from $\mathcal{C}$.

Definition 2.1. *The direct product of the family $(M_i)_{i\in I}$ is a pair $(P,(p_i))_{i\in I}$ where $P \in \mathcal{C}$ and $(p_i)_{i\in I}$ is a family of morphisms in $\mathcal{C}$, $p_i : P \to M_i$ such that for every other pair $(P',(p'_i))_{i\in I}$ composed by an object $P' \in \mathcal{C}$ and a family of morphisms $(p'_i)_{i\in I}$, $p'_i : P' \to M_i$, there is an unique morphism $u : P' \to P$ such that $p_i \circ u = p'_i$, for every $i \in I$.*

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Definition 2.2. *The direct coproduct of the family* $(M_i)_{i\in I}$ *is a pair* $(S,(\alpha_i))_{i\in I}$ *where* $P \in \mathcal{C}$ *and* $(\alpha_i)_{i\in I}$ *is a family of morphisms in* $\mathcal{C}$, $\alpha_i : M_i \to S$ *such that for every other pair* $(S',(\alpha_i'))_{i\in I}$ *composed by an object* $P' \in \mathcal{C}$ *and a family of morphisms* $(\alpha_i')_{i\in I}$, $\alpha_i' : M_i \to S'$, *there is an unique morphism* $u : S \to S'$ *such that* $u \circ \alpha_i = \alpha_i'$, *for every* $i \in I$.

By a couple of morphisms (f, g) in $\mathcal{C}$, we understand two morphisms $f, g : A \to B$, where A, B are objects in $\mathcal{C}$.

Definition 2.3. *A pair* (K, i) *with* $K \in \mathcal{C}$ *and* $i : K \to A$ *a morphism will be called the equalizer of the couple* (f, g) *if* $f \circ i = g \circ i$ *and for every other pair* (K', i') *with* $K' \in \mathcal{C}$ *and* $i' : K' \to A$ *with* $f \circ i' = g \circ i'$, *there exists an unique morphism* $u : K' \to K$ *such that* $i' = i \circ u$. *In this case we note* $(K, i) = Ker(f, g)$.

Definition 2.4. *A pair* (P, p) *with* $P \in \mathcal{C}$ *and* $p : B \to P$ *a morphism will be called the coequalizer of the couple* (f, g) *if* $p \circ f = p \circ g$ *and for every other pair* (P', p') *with* $P' \in \mathcal{C}$ *and* $p' : B \to P'$ *with* $p' \circ f = p' \circ g$, *there exists an unique morphism* $u : P \to P'$ *such that* $p' = u \circ p$. *In this case we note* $(P, p) = Coker(f, g)$.

We will say that the category $\mathcal{C}$ has *products (coproducts)* if there exists the direct product (coproduct) of any family of objects from $\mathcal{C}$. Also we will say that $\mathcal{C}$ has *equalizers (coequalizers)* if there exists the equalizer (coequalizer) for any couple of morphisms in $\mathcal{C}$.

For the notions of monomorphism and epimorphism we refer also to [1] and [3]. Let $f : A \to B$ be a morphism in $\mathcal{C}$. f is said to be a *monomorphism* if for any $C \in \mathcal{C}$ and every two morphisms $\alpha, \beta : C \to A$ such that $f \circ \alpha = f \circ \beta$, then $\alpha = \beta$. Similarly, f is called an *epimorphism* if for any $C \in \mathcal{C}$ and every two morphisms $\alpha, \beta : B \to C$ such that $\alpha \circ f = \beta \circ f$, then $\alpha = \beta$.

Remark 2.1. *It is easy to prove that if* $(K, i) = Ker(f, g)$, *then* i *is a monomorphism and if* $(P, p) = Coker(f, g)$, *then* p *is an epimorphism.*

Let $f : A \to B$ a morphism in $\mathcal{C}$. We refer to f as an *equalizer* if there exists a couple of morphisms (α, β), $\alpha, \beta : B \to C$, such that $(A, f) = Ker(\alpha, \beta)$. f will be called a *coequalizer* if it exists a couple of morphisms (α, β), $\alpha, \beta : C \to A$, such that $(B, f) = Coker(\alpha, \beta)$.

Remark 2.2. *Following Remark 2.1, every equalizer in* $\mathcal{C}$ *is a monomorphism and every coequalizer is an epimorphism.*

Definition 2.5. *(*[5], [6]*) A Hilbert algebra is an algebra* $(A, \to, 1)$ *of type* $(2, 0)$ *such that the following axioms are verified for every* $x, y, z \in A$:

(a_1) $x \to (y \to x) = 1$;

(a_2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;

(a_3) *If* $x \to y = y \to x = 1$, *then* $x = y$.

For examples of Hilbert algebras see [2]-[5]. In this paper, we will note with $\mathcal{H}$ the category of Hilbert algebras. In [5] it is proved that $\mathcal{H}$ has direct products and it is equational. Then every monomorphism in $\mathcal{H}$ will be injective (see [1], p.31), hence every equalizer will be an injective morphism. The existence of equalizers in $\mathcal{H}$ will be proved in section 2. Using some results from [4], we will offer an example of monomorphism which is not an equalizer.

For the case of epimorphisms, by Remark 2.2, every coequalizer is an epimorphism. Although in $\mathcal{H}$ not every epimorphism is surjective (see [4]), we will see that every coequalizer is surjective. Conversely, every surjective morphism will be a coequalizer and $\mathcal{H}$ will have coequalizers.

An important notion that will be used regarding Hilbert algebras is the notion of deductive system ([5]). Let A be a Hilbert algebra. A nonempty subset D of A is called a *deductive system* if $1 \in D$ and for every $x, y \in A$ such that $x, x \to y \in D$, we have $y \in D$. The importance of deductive systems results from the fact that the congruences of Hilbert algebras can be given in terms of deductive systems (see [3], p, 183). If D is a deductive system of the Hilbert algebra A, then the set

$\Phi(D) = \{(x, y) \in A \times A : x \to y, y \to x \in D\}$ is a congruence of A. We note with $A/_D$ the quotient algebra $A/_D = A/_{\Phi(D)} = \{[x]_D : x \in A\}$, where for every $x \in A$, $[x]_{\Phi(D)} = [x]_D = \{y \in A : x \to y, y \to x \in D\}$ is the equivalence class of x relative to $\Phi(D)$.

A particular case of deductive system is the kernel of a morphism: if $f : A \to B$ is a morphism of Hilbert algebras, then the set $Ker(f) = \{x \in A : f(x) = 1\}$ is called the *kernel* of f and it is a deductive system in A.

3. Equalizers and coequalizers in the category $\mathcal{H}$ of Hilbert algebras

We are starting this section with our first result:

Theorem 3.1. *The category $\mathcal{H}$ has equalizers.*

Proof. Let (f, g) a pair of morphism in $\mathcal{H}$, $f, g : A \to B$. Then the nonempty set $K = \{x \in A : f(x) = g(x)\}$ $(1 \in K)$ is a subalgebra of A and if we consider the embedding $i : K \to A$, we have $f \circ i = g \circ i$. We prove that $(K, i) = Ker(f, g)$.

Let K' be other Hilbert algebra and a morphism $i' : K' \to A$ such that $f \circ i' = g \circ i'$. We define $u : K' \to K$, $u(x) = i'(x)$, for all $x \in K'$. u is well defined since from $f \circ i' = g \circ i'$ we have $i'(x) \in K$ for every $x \in K'$. It is clearly that u is a morphism and $i \circ u = i'$.

To prove the uniqueness of u, let $u' : K' \to K$ be other morphism with $i \circ u' = i'$. Then $i \circ u' = i' = i \circ u$ and since i is injective and hence a monomorphism, we obtain $u = u'$. □

Proposition 3.1. *In $\mathcal{H}$, every equalizer is injective (see Remark 2.2).*

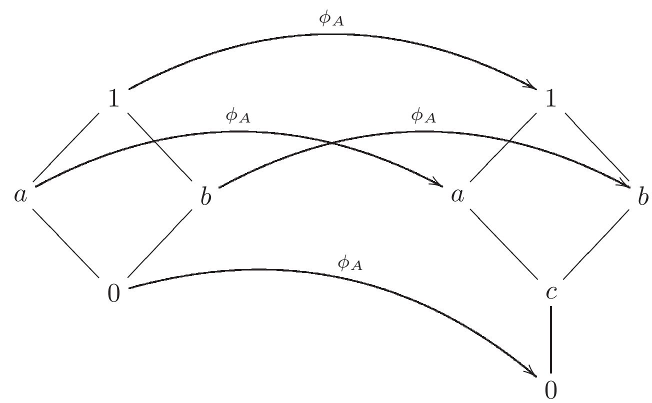
Example 3.1. *In $\mathcal{H}$, monomorphisms and injective morphisms coincide and by the above proposition, every equalizer is injective. The converse of this result is not always true. In* [4] *it is proved that the category $\mathcal{H}$ is a category with proper epic subalgebras (this means that there exists epimorphisms which are not surjective functions). The epimorphisms which are considered in that paper are also monomorphism. We are considering the following morphism of Hilbert algebras which by a theorem from* [4] *will be both an epimorphism and a monomorphism, but will not be surjective. We will prove that such morphism can not be an equalizer for a couple of morphisms.*

We are considering the two Hilbert algebras $A = \{0, a, b, 1\}$ and $H_A = \{0, c, a, b, 1\}$ with the implication from the following tables:

$\to$	*0*	*a*	*b*	*1*
0	*1*	*1*	*1*	*1*
a	*0*	*1*	*b*	*1*
b	*0*	*a*	*1*	*1*
1	*0*	*a*	*b*	*1*

$\to$	*0*	*c*	*a*	*b*	*1*
0	*1*	*1*	*1*	*1*	*1*
c	*0*	*1*	*1*	*1*	*1*
a	*0*	*b*	*1*	*b*	*1*
b	*0*	*a*	*a*	*1*	*1*
1	*0*	*c*	*a*	*b*	*1*

The system (A, ϕ_A, H_A) will be the free Hertz extension of A (see [4]*), where $\phi_A : A \to H_A$ is the following morphism of Hilbert algebras:*



Then ϕ_A is a monomorphism, an epimorphism but not surjective. We prove that ϕ_A can not be an equalizer. Suppose that $(A, \phi_A) = Ker(\alpha, \beta)$, where $\alpha, \beta : H_A \to C$ are two morphisms. We should have $\alpha \circ \phi_A = \beta \circ \phi_A$. Since ϕ_A is an epimorphism, we obtain $\alpha = \beta$. Also, for every other pair (K, i), where $i : K \to H_A$ is a morphism with $\alpha \circ i = \beta \circ i$, we should have an unique morphism $u : K \to A$ such that $\phi_A \circ u = i$.

We consider $K = H_A$ and $i : K \to H_A$ the inclusion. There will be no morphism u such that $\phi_A \circ u = i$, since the set $i(K)$ has 5 elements and the set $(\phi_A \circ u)(K)$ can not have more than 4 elements.

Theorem 3.2. *The category $\mathcal{H}$ has coequalizers.*

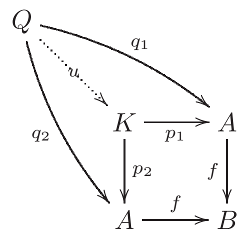
Proof. Let $f, g : A \to B$ a couple of morphism in $\mathcal{H}$. We consider the set $S = \{(f(x), g(x) : x \in A)\} \subseteq B \times B$ and the smallest congruence R on B which contains S, $R = \bigcap_{\theta \in Con(B), S \subseteq \theta} \theta$. Let be $p : B \to B/_R$ the canonical surjection. Then from $(f(x), g(x)) \in R$, we obtain $[f(x)]_R = [g(x)]_R$ for every $x \in A$. This means that $p \circ f = p \circ g$. We prove that $(B/_R, p) = Coker(f, g)$.

Let C be a Hilbert algebra and a morphism $p' : B \to C$ such that $p' \circ f = p' \circ g$. Let $R_1 = \{(y, y') \in B \times B : p'(y) = p'(y')\} = \{(y, y') \in B \times B) : y \to y', y' \to y \in Ker(p')\}$, where $Ker(p') = \{y \in B : p'(y) = 1\}$. Then R_1 is the congurence of B generated by the deductive system $Ker(p')$. Since for every $x \in A$ we have $p'(f(x)) = p'(g(x))$, we obtain $(f(x), g(x)) \in R_1$. This means that $S \subseteq R_1$. Then $R \subseteq R_1$. We can define now $u : B/_R \to C$, with $u([y]_R) = p'(y)$. Then u is well defined because for $[y_1]_R = [y_2]_R$, we have $(y_1, y_2) \in R \subseteq R_1 \Rightarrow p'(y_1) = p'(y_2)$. Clearly u is a morphism and $u \circ p = p'$.

To prove the uniqueness of u, let $u' : B \to C$ be other morphism such that $u' \circ p = p'$. Then $u' \circ p = u \circ p$ and since p is surjective, hence an epimorphism, we obtain $u = u'$. □

To obtain some useful results regarding coequalizers, we will need the following notion.

Definition 3.1. *([8], p.53) Let $\mathcal{C}$ be a category and $f : A \to B$ a morphism in $\mathcal{C}$. A system $(K; p_1, p_2)$ formed by an object K from $\mathcal{C}$ and two morphisms $p_1, p_2 : K \to A$ is said to form a kernel pair for f if $f \circ p_1 = f \circ p_2$ and for any other system $(Q; q_1, q_2)$ with morphisms $q_1, q_2 : Q \to A$ such that $f \circ q_1 = f \circ q_2$, there exists an unique morphism $u : Q \to K$ such that $p_1 \circ u = q_1$ and $p_2 \circ u = q_2$. A category in which every morphism has a kernel pair, is called a category with kenrel pairs.*



In [8](p.50-53), can be found results regarding some special limits and colimits in a category, limits as kernel pairs. The existence of kernel pairs in $\mathcal{H}$ can be deduced from there: *in a category with finite products, the existence of kernels is equivalent with the existence of fibred products* (proposition 8.6), and *a category with fibred products is a category with kernel pairs.* Since in $\mathcal{H}$, we have kernels, we deduce that H has kernel pairs (see the following corollary for how the kernel pair can be constructed).

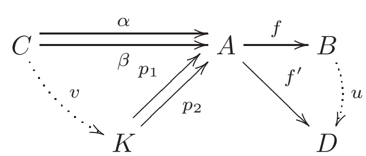
Corollary 3.1. *The category $\mathcal{H}$ has kernel pairs.*

Proof. Since the existence of kernel pairs has already been explained, we will offer just a short sketch of the proof, we show how the kernel pair of a morphism $f : A \to B$ can be constructed.

We consider $K = \{(x_1, x_2) \in A \times A : f(x_1) = f(x_2)\}$ the subalgebra of $A \times A$ and the canonical projections $p_1, p_2 : K \to A$, $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$ for every $(x_1, x_2) \in K$. Clearly $f \circ p_1 = f \circ p_2$. It is easy to prove that $(K; p_1, p_2)$ becomes e kernel pair for f. $\square$

Remark 3.1. *Let $f : A \to B$ be a coequalizer in $\mathcal{H}$. Then f is the coequalizer of its kernel pair.*

Proof. Indeed, let be $\alpha, \beta : C \to A$ such that $f = Coker(\alpha, \beta)$ and $(K; p_1, p_2)$ the kernel pair of f. Since $f \circ p_1 = f \circ p_2$, it is necessary to prove that for any other morphism $f' : A \to D$ such that $f' \circ p_1 = f' \circ p_2$, there exists an unique morphism $u : B \to D$ with $u \circ f = f'$.



From $f \circ \alpha = f \circ \beta$ and the fact that $(K; p_1, p_2)$ is the kernel pair of f, we deduce the existence of an unique morphism $v : C \to K$ such that $\alpha = p_1 \circ v$ and $\beta = p_2 \circ v$. Since $f' \circ p_1 = f' \circ p_2$, we deduce that $f' \circ \alpha = f' \circ \beta$, f being the equalizer of (α, β), we obtain the existence of an unique $u : B \to D$ with $f' = u \circ f$. $\square$

The following two results will help us to show that in $\mathcal{H}$ surjective morphisms and coequalizers coincide.

Proposition 3.2. *Let $f : A \to B$ be a surjective morphism in $\mathcal{H}$. Then f is a coequalizer.*

Proof. We show that f is the coequalizer of its kernel pair $(K; p_1, p_2)$. Let $f' : A \to C$ a morphism such that $f' \circ p_1 = f' \circ p_2$. Since f is surjective, for every $y \in B$, there exists an element $x \in A$ such that $f(x) = y$. We define $u : B \to C$, $u(y) = f'(x)$. u is well defined because from $f(x_1) = f(x_2) = y$, we have $(x_1, x_2) \in K$, then $(f' \circ p_1)(x_1, x_2) = (f' \circ p_2)(x_1, x_2)$, so $u(y) = f'(x_1) = f'(x_2)$. Clearly u is a morphism and $u \circ f = f'$. To prove its uniqueness, let be $u' : B \to C$ with $u' \circ f = f'$. Then we have $u' \circ f = u \circ f$; f being surjective, it becomes an epimorphism, so we obtain $u = u'$. □

Lemma 3.1. *In $\mathcal{H}$, we consider a surjective morphism $f : A \to B$ and morphism $g : A \to C$ such that $Ker(f) \subseteq Ker(g)$. Then there exists an unique morphism $h : B \to C$ sucht that $h \circ f = g$.*

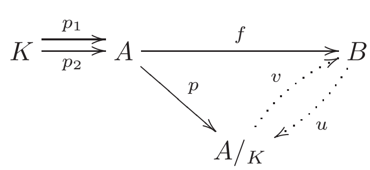
Proof. Let be $(K; p_1, p_2)$ the kernel pair of f. Since f surjective, by Proposition 3.2, we deduce that f is a coequalizer. Then, by Remark 3.1, $f = Coker(p_1, p_2)$.

Let $(x_1, x_2) \in K$. Then $x_1 \to x_2, x_2 \to x_1 \in Ker(f) \subseteq Ker(g)$, so $g(x_1) = g(x_2) \Rightarrow g \circ p_1 = g \circ p_2$. Then there exists an unique morphism $h : B \to C$ such that $h \circ f = g$. □

Proposition 3.3. *Let $f : A \to B$ be a coequalizer in H. Then f is surjective.*

Proof. By Remark 3.1, $f = Coker(p_1, p_2)$, where $(K; p_1, p_2)$ is the kernel pair of f. By Corollary 1, $K = \{(x_1, x_2) \in A \times A : f(x_1) = f(x_2)\} = \{(x_1, x_2) \in A \times A : x_1 \to x_2, x_2 \to x_1 \in Ker(f)\}$. Then K will be the congruence of A generated by the deductive system $Ker(f)$.

Let $p : A \to A/_K$ the canonical surjection. Since for every $(x_1, x_2) \in K$, $(p \circ p_1)(x_1, x_2) = [x_1]_K = [x_2]_K = (p \circ p_2)(x_1, x_2)$, we obtain $p \circ p_1 = p \circ p_2$ and since $f = Coker(p_1, p_2)$, there will exist an unique morphism $u : B \to A/_K$ such that $u \circ f = p$.



We observe that for every $x \in Ker(p) \Rightarrow p(x) = [1]_K$. Then $(x, 1) \in K$, so $x \in Ker(f)$. This means that $Ker(p) \subseteq Ker(f)$. By Lemma 3.1, we will obtain an unique morphism $v : A/_K \to B$ with $v \circ p = f$.

We have the following equalities: $(u \circ v) \circ p = u \circ (v \circ p) = u \circ f = p = 1_{A/_K} \circ p$ and since p is surjective and hence an epimorphism, we obtain $u \circ v = 1_{A/_K}$. Also $(v \circ u) \circ f = v \circ (u \circ f) = v \circ p = f = 1_B \circ f$. Since f is a coequalizer, by Remark 2.2, f is an epimorphism, hence $v \circ u = 1_B$. This way, u, v are isomorphism, one the inverse of the other. f will be surjective since $f = v \circ p$ and both v and p are surjective. □

Corollary 3.2. *In $\mathcal{H}$, surjective morphisms and coequalizers coincide.*

Example 3.2. *The morphism ϕ_A considered in Example 3.1 is an example of an epimorphism (not a surjective one) which is not an equalizer. So in $\mathcal{H}$, not every epimorphism is a coequalizer.*

4. Coproducts of Hilbert algebras

In this section we will prove that the category $\mathcal{H}$ of Hilbert algebras has coproducts. In this scope, we will need the following notions and for more details we refer to [1] and [3].

Let $\mathcal{C}$ be a category of algebras of the same type.

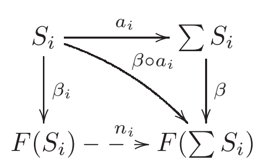
Definition 4.1. *([1], p.7) Let $A \in \mathcal{C}$ and S a subset of A. If there exists a smallest subalgebra of A that contains S, then it is called the subalgebra of A generated by S and is denoted by $[S]$ ($[S]$ exists whenever $S \neq \phi$).*

Lemma 4.1. *([1], p.8) Let $A, B \in \mathcal{C}$, $S \subseteq A$ and two morphisms $f, g : [S] \to B$ such that $f/_S = g/_S$. Then $f = g$.*

Definition 4.2. *([1], p.14) $A \in \mathcal{C}$ is said to be free over a class K of algebras of $\mathcal{C}$ if there exists a subset $X \subseteq A$ such that $[X] = A$ and for any other $B \in K$ and any function $f : X \to B$, there exists a morphism $f' : A \to B$ with $f'|_X = f$. In this case, X is said to be a free generating set for A. By Lemma 4.1, we remark that f' is uniquely determined.*

By a Theorem of Birkhoff (see [1], Th.4, p.19), the existence of free algebras is assured in an equational class. We are starting the proof of existence of the coproducts in the category of Hilbert algebras, by proving the existence of coproducts of free Hilbert algebras. The proof follows the steps from [9], where the existence of coproducts of BCK algebras is proved. For a set X, we will note with $F(X)$ a free Hilbert algebra over X.

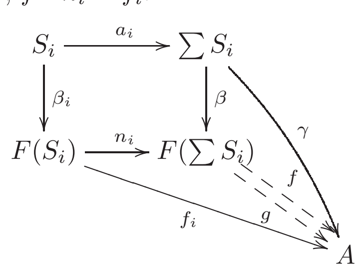
Let $(F(S_i))_{i\in I}$ a family of free Hilbert algebras and the injective functions $a_i : S_i \to \sum S_i$, $\beta_i : S_i \to F(S_i)$ and $\beta : \sum S_i \to F(\sum S_i)$, where by $\sum S_i$ we have note the coproduct of the family of sets $(S_i)_{i\in I}$. For every $i \in I$, since $F(S_i)$ is a free algebra, there will exist an unique morphism $n_i : F(S_i) \to F(\sum S_i)$ such that $n_i|_{S_i} = \beta \circ a_i \Leftrightarrow n_i \circ \beta_i = \beta \circ a_i$.



We must remark that n_i are embeddings, hence injective, and $F(S_i)$ is naturally identified with a subalgebra of $F(\sum S_i)$ generated by $S_i \times \{i\}$.

Proposition 4.1. $(F(\sum S_i), (n_i)_{i\in I})$ *is the coproduct of the family of free Hilbert algebras* $(F(S_i))_{i\in I}$.

Proof. We consider other pair $(A, (f_i)_{i\in I})$, where A is a Hilbert algebra and $f_i : F(S_i) \to A$ morphisms. We prove the existence of an unique morphism $f : F(\sum S_i) \to A$ such that for every $i \in I$, $f \circ n_i = f_i$.



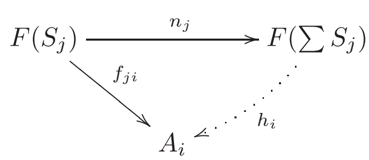
Since $\sum S_i$ is the coproduct of $(S_i)_{i\in I}$, there exists an unique function $\gamma : \sum S_i \to A$ such that $\gamma \circ a_i = f_i \circ \beta_i$, for every $i \in I$. Then by the freeness of $F(\sum S_i)$, there will be an unique morphism $f : F(\sum S_i) \to A$ with the propery that $f|_{\sum S_i} = \gamma \Leftrightarrow f \circ \beta = \gamma$. By the following equalities $f \circ (n_i \circ \beta_i) = f \circ (\beta \circ a_i) = (f \circ \beta) \circ a_i = \gamma \circ a_i = f_i \circ \beta_i$, we obtain $(f \circ n_i) \circ \beta_i = f_i \circ \beta_i$ and this means that both $f \circ n_i, f_i : F(S_i) \to A$ are equal on S_i, $(f \circ n_i)|_{S_i} = f_i|_{S_i}$. So, by Lemma 4.1, $f \circ n_i = f_i$.

To prove the uniqueness of f, let be other morphism $g : F(\sum S_i) \to A$, such that for every $i \in I$, $g \circ n_i = f_i$. Then $f \circ (n_i \circ \beta_i) = g \circ (n_i \circ \beta_i) \Rightarrow f \circ (\beta \circ a_i) = g \circ (\beta \circ a_i)$. So $\gamma \circ a_i = (f \circ \beta) \circ a_i = (g \circ \beta) \circ a_i$. But $\sum S_i$ being the coproduct of the family of sets $(S_i)_{i\in I}$, there will be an unique function $u : \sum S_i \to A$ with $u \circ a_i = \gamma \circ a_i$. We must have $u = f \circ \beta = g \circ \beta$. Combining this with $f \circ \beta = \gamma$, we obtain $\gamma = f \circ \beta = g \circ \beta$, and by the freness of $F(\sum S_i)$, we have f=g. □

Using the above proposition, we will construct now the coproduct of a family of Hilbert algebras $(A_i)_{i\in I}$. We consider a disjoint family of sets $(S_i)_{i\in I}$ such that $|S_i| > |A_i|$. Then, by the freeness of $F(S_i)$, there will exist a family of surjective morphisms $(f_i)_{i\in I}$, $f_i : F(S_i) \to A_i$. We consider the deductive system from $F(S_i)$, $C_i = f_i^{-1}(1)$. Since $F(S_i)$ can be naturally considered as a subalgebra of $F(\sum S_i)$ generated by S_i, we can define the deductive system C generated by $\bigcup_{i\in I} C_i$ in $F(\sum S_i)$.

Lemma 4.2. *$C \cap F(S_i) = C_i$, for every $i \in I$.*

Proof. For every $i, j \in I$, we consider the morphism $f_{ji} : F(S_j) \to A_i$ defined as follows: for every $x \in F(S_j)$, $f_{ji}(x) = \begin{cases} f_i(x), & \text{if} \quad j = i \\ 1, & \text{if} \quad j \neq i\,. \end{cases}$



Since $F(\sum S_j)$ is the coproduct of $(F(S_j))_{j\in I}$, there will exist an unique morphism $h_i : F(\sum S_j) \to A_i$ such that $f_{ji} = h_i \circ n_j$. Using the fact that $F(S_j)$ can be considered as a subalgebra of $F(\sum S_j)$ and n_j as embedding, we have: $(f_{ji})^{-1}(1) = (h_i \circ n_j)^{-1}(1) = h_i^{-1}(1) \cap F(S_j)$. This means that for $i = j$ we have $h_i^{-1} \cap F(S_i) = (f_{ii})^{-1}(1) = f_i^{-1}(1) = C_i$ and for $i \neq j$, $h_i^{-1} \cap F(S_j) = (f_{ji})^{-1}(1) = F(S_j)$.

Next, we prove that for every $i, j \in I$, $C_j \subseteq h_i^{-1}(i)$. Indeed, if $i = j$, then $C_i = F(S_i) \cap h_i^{-1}(i) \subseteq h_i^{-1}(i)$, and if $i \neq j$, from $F(S_i) = F(S_i) \cap h_j^{-1}(i)$, we obtain $C_i = F(S_i) \cap h_i^{-1}(i) = F(S_i) \cap h_i^{-1}(i) \cap h_j^{-1}(i) \subseteq h_j^{-1}(1)$.

Finally, $\bigcup C_j \subseteq \bigcap h_i^{-1}(1)$. Then $C \subset \bigcap h_i^{-1}(1) \Rightarrow C \cap F(S_i) \subseteq \bigcap h_i^{-1}(1) \cap F(S_i) \subseteq C_i$. Conversely $C_i \subseteq F(S_i)$ and $C_i \subseteq \bigcup C_j \subseteq C$, so $C_i \subseteq C \cap F(S_i)$.

□

Let be Θ the congruence generated by the deductive system C.

Lemma 4.3. *Every A_i is isomorphicaly embeddable in $F(\sum S_i)/_\Theta$ such that $m_i \circ f_i = p \circ n_i$, where m_i is the embedding and p is the natural surjection (see the diagram from Theorem 4.1).*

Proof. We note $[F(S_i)]_\Theta = \{[a]_\Theta : a \in F(S_i)\} = (p \circ n_i)(F(S_i))$. From the above lemma, $Ker(f_i) = f_i^{-1}(1) = C_i = C \cap F(S_i)$ and since f_i is surjective, we obtain $F(S_i)/_{Ker(f_i)} \approx Im(f_i) = A_i$.

But $[F(S_i)]_\Theta = (p \circ n_i)(F(S_i)) \approx F(S_i)/_{Ker(p \circ n_i)} = F(S_i)/_{Ker(f_i)}$, because $Ker(p \circ n_i) = \{a \in F(S_i) : a \in C\} = C \cap F(S_i) = Ker(f_i)$.

We deduce that $A \approx [F(S_i)]_\Theta$. So we can define $m_i : A_i \to F(\sum S_i)/_\Theta$, which is an isomorphism from A_i onto $[F(S_i)]_\Theta$ such that $m_i \circ f_i = p \circ n_i$.

□

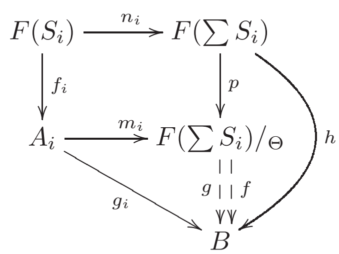
Lemma 4.4. *For every $i \neq j$, $m_i(A_i) \cap m_j(A_j) = \{1\}$.*

Proof. Let $a \in F(S_i)$, $b \in F(S_j)$, with $i \neq j$. From Lemma 4.2, we have $C \subseteq \bigcap h_i^{-1}(1) = \bigcap Ker(h_i)$. Then $\Theta \subseteq \Phi(Ker(h_i))$. So for every $(a, b) \in \Theta$, we have $(a, b) \in \Phi(Ker(h_i))$. This means $h_i(a) = h_i(b) = (h_i \circ n_j)(b) = f_{ji}(b) = 1 \Rightarrow (a, 1) \in \Phi(Ker(h_i))$ and since $h_i \circ n_i = f_i$, we obtain $(a, 1) \in \Phi(Ker(f_i)) \Rightarrow a \in C_i \subseteq C = [1]_\Theta$. By the same way $b \in [1]_\Theta$ and since $m_i(A_i) = [F(S_i)]_\Theta$, $m_j(A_j) = [F(S_j)]_\Theta$, we obtain $m_i(A_i) \cap m_j(A_j) = \{1\}$.

□

Theorem 4.1. *$(F(\sum S_i)/_\Theta, (m_i)_{i \in I})$ is the coproduct of the family of Hilbert algebras $(A_i)_{i \in I}$.*

Proof. We have to prove that for every other pair $(B, (g_i)_{i \in I})$ where $g_i : A_i \to B$ are morphisms of Hilbert algebras, there exists an unique morphism $g : F(\sum S_i)/_\Theta \to B$ such that $g \circ m_i = g_i$, for every $i \in I$.



Since $F(\sum S_i)$ is the coproduct of the family $(F(S_i))_{i \in I}$, then there exists an unique morphism $h : F(\sum S_i) \to B$, such that $h \circ n_i = g_i \circ f_i$ for every $i \in I$. Then $C_i = f_i^{-1}(1) \subseteq (g_i \circ f_i)^{-1}(1) = (h \circ n_1)^{-1}(1) = h^{-1}(1) \cap F(S_i) \subseteq h^{-1}(1)$, so $\bigcup C_i \subseteq h^{-1}(1)$, then $C \subseteq h^{-1}(1)$. This means that $\Phi(Ker(p)) = \Theta \subseteq \Phi(Ker(h))$. We obtain $Ker(p) \subseteq Ker(h)$ and by a homomorphism theorem (see [1], p.10) there exists an unique morphism $g : F(\sum S_i)/_\Theta \to B$ such that $g \circ p = h$. We have the following equalities: $g \circ m_i \circ f_i = g \circ p \circ n_i = h \circ n_i = g_i \circ f_i$, which using the fact that f_i surjective, leeds to $g \circ m_i = g$.

To prove the uniqueness of g, we consider other morphism $k : F(\sum S_i)/_\Theta \to B$ with $k \circ m_i = g_i$. Then $k \circ m_i \circ f_i = g_i \circ f_i \Rightarrow k \circ p \circ n_i = g_i \circ f_i = h \circ n_i$ and since $F(\sum S_i)$ is the coproduct of $(F(S_i))_{i \in I}$, we obtain $k \circ p = h = g \circ p$. p being surjective, we obtain $k = g$. □

5. Conclusions

Since $\mathcal{H}$ has equalizers and products, by a result from [8](Th.9.1), $\mathcal{H}$ will have also limits. In the same time, $\mathcal{H}$ having coequalizers and coproducts, it will have also colimits (see [8],(Th.9.2)), so $\mathcal{H}$ is both complete and cocomplete. This means that in $\mathcal{H}$ we will have some particular cases of limits as fibred products and coproducts.

Although we have fibred products and coproducts and zero objects in $\mathcal{H}$, $\mathcal{H}$ is not an abelian category because by Examples 3.1 and 3.2, not every monomorphism and not every epimorphism is normal (a monomorphism is called *normal* if it is an equalizer and an epimorphism is called *normal* if it is a coequalizer).

We remark that equational subcategories of Hilbert algebras will be both complete and cocomplete (our constructions preserve Hilbert identities, so the constructions remain similar for Hilbert algebras with infimim, Hilbert algebras with supremum).

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