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Solvability of a mixed variational problem

Raluca Ciurcea and Andaluzia Matei

Abstract. We consider a variational problem arising from contact mechanics, that consists into a system of two variational inequalities involving Lagrange multipliers. Using elements of convex analysis, we prove the existence of at least one solution. Moreover, the uniqueness and the stability are discussed.

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1. Introduction

The present paper focuses on the solvability of the following abstract variational problem.

Problem 1. *For given f, $h \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that*

$$a(u, v-u) + \phi(v) - \phi(u) + b(v-u, \lambda) \geq (f, v-u)_X \quad \forall v \in X, \tag{1}$$

$$b(u, \mu - \lambda) \leq b(h, \mu - \lambda) \quad \forall \mu \in \Lambda. \tag{2}$$

Everywhere in this paper $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ denote two Hilbert spaces.

Let us assume that

$$\left.\begin{array}{c} a(\cdot, \cdot) : X \times X \to \mathbb{R}, \text{ is a symmetric bilinear form such that} \\ \text{there exists } M_a > 0 : |a(u,v)| \leq M_a \|u\|_X \|v\|_X, \quad \forall u, v \in X, \\ \text{there exists } m_a > 0 : a(v,v) \geq m_a \|v\|_X^2, \quad \forall v \in X, \end{array}\right\} \tag{3}$$

$$\left.\begin{array}{c} \phi : X \to \mathbb{R}_+ \text{ is a convex lower semicontinuous functional such that} \\ \phi(0_X) = 0, \end{array}\right\} \tag{4}$$

$$\left.\begin{array}{c} b(\cdot, \cdot) : X \times Y \to \mathbb{R} \text{ is a bilinear form such that} \\ \text{there exists } M_b > 0 : |b(v, \mu)| \leq M_b \|v\|_X \|\mu\|_Y, \ \forall v \in X, \ \mu \in Y, \\ \text{there exists } \alpha > 0 : \inf\limits_{\mu \in Y, \mu \neq 0_Y} \sup\limits_{v \in X, v \neq 0_X} \dfrac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha, \end{array}\right\} \tag{5}$$

and

$$\Lambda \text{ is an unbounded, closed, convex subset of } Y \text{ that contains } 0_Y. \tag{6}$$

If $b \equiv 0$, Problem 1 reduces to a variational inequality of the second kind, as follows.

Problem 2. *Find $u \in X$ such that*

$$a(u, v-u) + \phi(v) - \phi(u) \geq (f, v-u)_X \quad \forall v \in X.$$

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The proof of the existence, uniqueness and stability of the solution of Problem 2, can be found in [9]. Since the form a is symmetric, we recall that the unique solution of Problem 2, $u \in X$, verifies

$$\inf_{v \in X} J(v) = J(u),$$

where

$$J : X \to \mathbb{R} \quad J(v) = \frac{1}{2} a(v, v) + \phi(v) - (f, v)_X, \; \forall v \in X.$$

If $\phi \equiv 0$, Problem 1 becomes equivalent with

Problem 3. *For given f, $h \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that*

$$\begin{aligned} a(u, v) + b(v, \lambda) &= (f, v)_X \quad \forall v \in X, \\ b(u, \mu - \lambda) &\leq b(h, \mu - \lambda) \quad \forall \mu \in \Lambda. \end{aligned}$$

Such problems are called in the literature *mixed variational problems*. For their solvability we send the reader to [4]. The interest for this kind of problems arises from contact mechanics; see for example [4, 5, 6, 7], where the weak formulations of the contact models are written via mixed variational problems.

In the present paper, we prove that Problem 1 has at least one solution. We also discuss the uniqueness and the stability of the solution.

2. Preliminaries

For the convenience of the reader, we recall in this section some elements of convex analysis that will be used in this paper. To start, we recall the definition of the saddle point.

Definition 2.1. *Let A and B be two non-empty sets. A pair $(u, \lambda) \in A \times B$ is said to be a saddle point of a functional $\mathcal{L} : A \times B \to \mathbb{R}$ if and only if*

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda), \quad \forall v \in A, \mu \in B.$$

The following existence result will be used in our paper.

Theorem 2.1. *Let $(X, (\cdot, \cdot), \|\cdot\|_X)$, $(Y, (\cdot, \cdot), \|\cdot\|_Y)$ be two Hilbert spaces and let $A \subseteq X$, $B \subseteq Y$ be non-empty, closed, convex subsets. Assume that a real functional $\mathcal{L} : A \times B \to \mathbb{R}$ satisfies the following conditions*

$$v \to \mathcal{L}(v, \mu) \;\; \text{is convex and lower semi-continuous} \;\; \forall \mu \in B, \tag{7}$$

$$\mu \to \mathcal{L}(v, \mu) \;\; \text{is concave and upper semi-continuous} \; \forall v \in A. \tag{8}$$

Moreover,

$$A \text{ is bounded or } \lim_{\|v\|_X \to \infty, v \in A} \mathcal{L}(v, \mu_0) = \infty \text{ for some } \mu_0 \in B \tag{9}$$

and

$$B \text{ is bounded or } \lim_{\|\mu\|_Y \to \infty, \mu \in B} \inf_{v \in A} \mathcal{L}(v, \mu) = -\infty. \tag{10}$$

Then, the functional $\mathcal{L}$ has at least one saddle point.

For more details on the saddle point theory and its applications, we refer to [1, 2, 3, 4].

Another result that will be needed below is the following one.

Proposition 2.1. *Assume that $f : X \to \mathbb{R}$ is a Gâteaux differentiable functional. Then, f is convex if and only if*

$$f(v) \geq f(u) + (\nabla f(u), v-u)_X \quad \forall u, v \in X.$$

The proof of Proposition 2.1 can be found in [8].

3. Well-posedness of Problem 1

Everywhere in this section we assume (3)-(6). Our approach of Problem 1 is based on the saddle point theory, applied to the following functional,

$$\mathcal{L} : X \times \Lambda \to \mathbb{R}, \ \ \mathcal{L}(v,\mu) = \frac{1}{2}a(v,v) + \phi(v) + b(v-h,\mu) - (f,v)_X \qquad \forall v \in X, \mu \in \Lambda. \tag{11}$$

We will prove an auxiliary lemma.

Lemma 3.1. *Assuming that Problem 1 has a solution $(u,\lambda) \in X \times \Lambda$, then this solution is a saddle point of the functional $\mathcal{L}$. Conversely, assuming that the functional $\mathcal{L}$ has a saddle point $(u,\lambda) \in X \times \Lambda$, then, this saddle point verifies Problem 1.*

Proof. To start, by

$$b(u, \mu-\lambda) \leq b(h, \mu-\lambda) \quad \forall \mu \in \Lambda,$$

we obtain immediately

$$\mathcal{L}(u,\mu) \leq \mathcal{L}(u,\lambda) \quad \forall \mu \in \Lambda.$$

Moreover, using the definition of the functional $\mathcal{L}$, we have

$$\begin{aligned}\mathcal{L}(u,\lambda) - \mathcal{L}(v,\lambda) \ &= \ \frac{1}{2}a(u,u) + \phi(u) + b(u,\lambda) - (f,u)_X \\ &- \ \frac{1}{2}a(v,v) - \phi(v) - b(v,\lambda) + (f,v)_X.\end{aligned}$$

Thus, by

$$a(u, v-u) + \phi(v) - \phi(u) + b(v-u,\lambda) \geq (f,\, v-u)_X \quad \forall v \in X,$$

we obtain

$$\mathcal{L}(u,\lambda) - \mathcal{L}(v,\lambda) \leq \frac{1}{2}a(u,u) - \frac{1}{2}a(v,v) + a(u,v-u) = -\frac{1}{2}a(u-v,u-v).$$

Therefore, for all $v \in X$, $\mathcal{L}(u,\lambda) \leq \mathcal{L}(v,\lambda)$.

Conversely, let us assume that $(u,\lambda) \in X \times \Lambda$ is a saddle point of the functional $\mathcal{L}$. It is straightforward to observe that

$$\mathcal{L}(u,\mu) \leq \mathcal{L}(u,\lambda) \quad \forall \mu \in \Lambda$$

implies

$$b(u, \mu-\lambda) \leq b(h, \mu-\lambda) \quad \forall \mu \in \Lambda.$$

Furthermore,

$$\mathcal{L}(u,\lambda) \leq \mathcal{L}(w,\lambda) \quad \forall w \in X$$

yields

$$\frac{1}{2}a(u,u) - \frac{1}{2}a(w,w) + \phi(u) - \phi(w) + b(u-w,\lambda) + (f, w-u)_X \leq 0 \ \forall w \in X.$$

Replacing w by $u+t(v-u)$, with $t>0$, we can write, taking into account the convexity of the functional ϕ,

$$t\,a(u,v-u)+\frac{t^2}{2}a(v-u,v-u)+t\,(\phi(v)-\phi(u))+t\,b(v-u,\lambda)\geq t(f,v-u)_X \quad \forall v\in X.$$

Dividing by t and passing to the limit as $t\to 0$, we are led to

$$a(u,v-u)+\phi(v)-\phi(u)+b(v-u,\lambda)\geq (f,\,v-u)_X \quad \forall v\in X.$$

□

The main result is the following theorem.

Theorem 3.1. *Assume* (3)-(6). *In addition, we assume that the functional ϕ is Lipschitz continuous, more precisely there exists $L_\phi>0$ such that*

$$|\phi(v)-\phi(w)|\leq L_\phi\|v-w\|_X \quad \forall v,w\in X. \tag{12}$$

Then, Problem 1 has at least one solution.

Proof. Let us prove that the functional $\mathcal{L}$ admits at least one saddle point $(u,\lambda)\in X\times\Lambda$. Obviously, the map $v\to\mathcal{L}(v,\,\mu)$ is convex and lower semi-continuous for every $\mu\in\Lambda$. In addition, for every $v\in X$, the map $\mu\to\mathcal{L}(v,\,\mu)$ is concave and upper semi-continuous. On the other hand,

$$\lim_{\|v\|_X\to\infty, v\in X}\mathcal{L}(v,0_Y)=\infty.$$

We next prove that

$$\lim_{\|\mu\|_Y\to\infty,\mu\in\Lambda}\ \inf_{v\in X}\mathcal{L}(v,\,\mu)=-\infty. \tag{13}$$

Indeed, let μ be an arbitrary element in Λ and let $u_\mu\in X$ be the unique solution of the variational inequality of the second kind

$$a(u_\mu,v-u_\mu)+\phi(v)-\phi(u_\mu)\geq (f_\mu,v-u_\mu)_X, \quad \forall v\in X, \tag{14}$$

where $f_\mu\in X$ is defined using Riesz's representation theorem as follows,

$$(f_\mu,v)_X:=(f,v)_X-b(v,\mu) \quad \forall v\in X.$$

Obviously,

$$\inf_{v\in X}\mathcal{L}(v,\mu)=\frac{1}{2}a(u_\mu,u_\mu)+\phi(u_\mu)-(f,u_\mu)_X+b(u_\mu,\mu)-b(h,\mu).$$

Let us put $v=0_X$ in (14). Then, after summing with $\frac{1}{2}a(u_\mu,u_\mu)$, we deduce

$$\frac{1}{2}a(u_\mu,u_\mu)-(f,u_\mu)_X+\phi(u_\mu)+b(u_\mu,\mu)\leq -\frac{m_a}{2}\|u_\mu\|_X^2.$$

Therefore,

$$\inf_{v\in X}\mathcal{L}(v,\mu)\leq -\frac{m_a}{2}\|u_\mu\|_X^2-b(h,\mu).$$

Due to the inf-sup property of the form b, we deduce that there exists $\alpha>0$ such that

$$\alpha\|\mu\|_Y\leq \sup_{v\in X,v\neq 0_X}\frac{b(v,\mu)}{\|v\|_X}$$

and, by (14), we get

$$\alpha\|\mu\|_Y\leq \sup_{v\in X,v\neq 0_X}\frac{(f,v)_X-a(u_\mu,v)+\phi(u_\mu-v)-\phi(u_\mu)}{\|v\|_X}\leq \|f\|_X+M_a\|u_\mu\|_X+L_\phi.$$

Therefore, there exists $c > 0$ such that

$$\|\mu\|_Y^2 \leq c(\|f\|_X^2 + \|u_\mu\|_X^2 + L_\phi^2).$$

Furthermore, there exists $\tilde{c} > 0$ such that

$$\inf_{v\in X} \mathcal{L}(v,\mu) \leq -\tilde{c}(\|\mu\|_Y^2 - \|f\|_X^2 - L_\phi^2) + M_b\|h\|_X\|\mu\|_Y.$$

Since μ was arbitrarily fixed in Λ, by passing to the limit as $\|\mu\|_Y \to \infty$, we obtain (13).

Consequently, based on Theorem 2.1, we deduce that the functional $\mathcal{L}$ has at least one saddle point. Using Lemma 3.1 we deduce that Problem 1 has at least one solution. □

If we assume that

$$\phi \text{ is a Gâteaux differentiable functional,} \tag{15}$$

since $\partial\phi(u) = \{\nabla\phi(u)\}$, Problem 1 can be restated as follows: find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} a(u,v) + (\nabla\phi(u), v)_X + b(v,\lambda) &= (f,v)_X \quad \forall v \in X,\\ b(u,\mu-\lambda) &\leq b(h,\mu-\lambda) \quad \forall \mu \in \Lambda. \end{aligned}$$

Let us investigate the uniqueness of the solution.

Theorem 3.2. *Assume* (3)-(6), (12) *and* (15). *Then, Problem 1 has a solution and only one.*

Proof. Let us consider (u^1, λ^1) and (u^2, λ^2), two solutions of Problem 1. We have

$$a(u^1, v-u^1) + \phi(v) - \phi(u^1) + b(v-u^1, \lambda^1) \geq (f, v-u^1)_X \quad \forall v \in X, \tag{16}$$

and

$$a(u^2, v-u^2) + \phi(v) - \phi(u^2) + b(v-u^2, \lambda^2) \geq (f, v-u^2)_X \quad \forall v \in X. \tag{17}$$

Taking $v = u^2$ in (16) and $v = u^1$ in (17), by summing, we get

$$a(u^1-u^2, u^2-u^1) + b(u^1-u^2, \lambda^2-\lambda^1) \geq 0. \tag{18}$$

Due to the fact that

$$b(u^1-u^2, \lambda^2-\lambda^1) \leq 0,$$

taking into account (3), we deduce from (18) that $u^1 = u^2$.

Furthermore,

$$a(u,v) + (\nabla\phi(u), v)_X + b(v,\lambda) = (f,v)_X \quad \forall v \in X.$$

Thus,

$$b(v, \lambda^1-\lambda^2) = -a(u^1-u^2, v) - (\nabla\phi(u^1) - \nabla\phi(u^2), v)_X \quad \forall v \in X.$$

Since $u^1 = u^2$, by the inf-sup property of the form b, we deduce that

$$\alpha\|\lambda^1-\lambda^2\|_Y \leq 0,$$

and from this, we obtain $\lambda^1 = \lambda^2$. □

Let us establish a stability result.

Theorem 3.3. *Assume* (3)-(6), (12) *and* (15). *In addition, we assume that there exists* $L_{\nabla\phi} > 0$ *such that*

$$\|\nabla\phi(v) - \nabla\phi(w)\|_X \leq L_{\nabla\phi}\|v - w\|_X \quad \forall v, w \in X. \tag{19}$$

Let $(u_1, \lambda_1) \in X \times \Lambda$ *and* $(u_2, \lambda_2) \in X \times \Lambda$ *be the solutions of Problem 1 corresponding respectively the to the data* $(f_1, h_1) \in X \times X$ *and* $(f_2, h_2) \in X \times X$. *Then, there exists* $C = C(\alpha, M_a, m_a, M_b, L_{\nabla\phi}) > 0$, *such that*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \leq C(\|f_1 - f_2\|_X + \|h_1 - h_2\|_X). \tag{20}$$

Proof. Obviously,

$$a(u_1, v - u_1) + \phi(v) - \phi(u_1) + b(v - u_1, \lambda_1) \geq (f_1, v - u_1)_X \quad \forall v \in X, \tag{21}$$

and

$$a(u_2, v - u_2) + \phi(v) - \phi(u_2) + b(v - u_2, \lambda_2) \geq (f_2, v - u_2)_X \quad \forall v \in X. \tag{22}$$

Let us take $v = u_2$ in (21) and $v = u_1$ in (22). By summing, we get

$$a(u_1 - u_2, u_1 - u_2) \quad \leq \quad (f_1 - f_2, u_1 - u_2)_X + b(u_1 - u_2, \lambda_2 - \lambda_1).$$

Notice that

$$b(u_1 - u_2, \lambda_2 - \lambda_1) \quad \leq \quad b(h_1 - h_2, \lambda_2 - \lambda_1).$$

Thus, we can write

$$m_a\|u_1 - u_2\|_X^2 \quad \leq \quad \|f_1 - f_2\|_X\|u_1 - u_2\|_X + M_b\|h_1 - h_2\|_X\|\lambda_1 - \lambda_2\|_Y.$$

On the other hand

$$a(u_1, v) + (\nabla\phi(u_1), v)_X + b(v, \lambda_1) = (f_1, v)_X \quad \forall v \in X$$

and

$$a(u_2, v) + (\nabla\phi(u_2), v)_X + b(v, \lambda_2) = (f_2, v)_X \quad \forall v \in X.$$

Therefore,

$$b(v, \lambda_1 - \lambda_2) = -a(u_1 - u_2, v) + (f_1 - f_2, v)_X - (\nabla\phi(u_1) - \nabla\phi(u_2), v) \quad \forall v \in X.$$

Consequently, for every $v \in X$, $v \neq 0_X$, taking into account (19),

$$\frac{b(v, \lambda_1 - \lambda_2)}{\|v\|_X} \leq M_a\|u_1 - u_2\|_X + \|f_1 - f_2\|_X + L_{\nabla\phi}\|u_1 - u_2\|_X.$$

By the inf-sup property of the form b, we deduce

$$\alpha\|\lambda_1 - \lambda_2\|_Y \leq \|f_1 - f_2\|_X + (M_a + L_{\nabla\phi})\|u_1 - u_2\|_X.$$

Now, we can write the following inequalities,

$$\begin{aligned} m_a\|u_1 - u_2\|_X^2 \quad &\leq \quad \frac{\|f_1 - f_2\|_X^2}{2k_1} + \frac{k_1\|u_1 - u_2\|_X^2}{2} \\ &+ \quad \frac{M_b^2\|h_1 - h_2\|_Y^2}{2k_2} + \frac{k_2\|\lambda_1 - \lambda_2\|_Y^2}{2}, \\ \|\lambda_1 - \lambda_2\|_Y^2 \quad &\leq \quad \frac{2}{\alpha^2}(\|f_1 - f_2\|_X^2 + (M_a + L_{\nabla\phi})^2\|u_1 - u_2\|_X^2), \end{aligned} \tag{23}$$

where k_1, k_2 are strictly positive real constants. Let us choose k_1 and k_2 such that

$$m_a - \frac{k_1}{2} - \frac{k_2(M_a + L_{\nabla\phi})^2}{\alpha^2} > 0.$$

Consequently,

$$\|u_1 - u_2\|_X^2 \le \frac{1}{m_a - \frac{k_1}{2} - \frac{k_2(M_a + L_{\nabla\phi})^2}{\alpha^2}} \Big(\frac{\|f_1 - f_2\|_X^2}{2k_1} + \frac{M_b^2 \|h_1 - h_2\|_Y^2}{2k_2} \Big)$$
$$+ \frac{2k_2}{2m_a\alpha^2 - k_1\alpha^2 - 2k_2(M_a + L_{\nabla\phi})^2} \|f_1 - f_2\|_X^2. \qquad (24)$$

By (23) and (24), we deduce that there exists $C_1 > 0$ such that

$$\|\lambda_1 - \lambda_2\|_Y^2 \le C_1(\|f_1 - f_2\|_X^2 + \|h_1 - h_2\|_X^2). \qquad (25)$$

Based on (24) and (25), we deduce (20).

□

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(R. Ciurcea and A. Matei) Department of Mathematics, University of Craiova, Al.I. Cuza Street, No. 13, Craiova RO-200585, Romania, Tel. & Fax: 40-251412673
E-mail address: ciurcearaluca@yahoo.com, andaluziamatei2000@yahoo.com