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On the series of Kempner-Irwin type

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Abstract. In 1914, Kempner proved that the series consisting of the inverses of natural numbers which are free of the digit 9 is convergent. In 1916, Irwin considered the convergence problem of the series containing the inverses of all numbers that contain a group of digits a number of times. These types of series are still under the attention of many mathematicians such as R. Baillie, T. Schmelzer, H. Behforooz, B. Farhi, etc. In this paper we will deal with the problem of computing the sum of series of Kempner-Irwin type.

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1. Introduction

Let X_m be a string of $m \geq 0$ digits. We denote by S^- the set of all the positive integers that *do not contain* X_m in their decimal representation, by S^+ we denote the set of all positive integers that *contain* X_m, and by $S^{(p)}$ the set of all positive integers that *contain* X_m *exactly* p times. We will also make use of the set $S^{(\leq p)}$, of all positive integers that *contain* X_m *no more* than p times, and of the set $S^{(\geq p)}$, of all positive integers that *contain* X_m *at least* p times.

The *Kempner type* series are the series of one of the following form:

$$\Psi^-_{k;X_m} = \sum_{s \in S^-} \frac{1}{s^k}, \tag{1}$$

$$\Psi^+_{k;X_m} = \sum_{s \in S^+} \frac{1}{s^k}, \tag{2}$$

$$\Psi^{(p)}_{k;X_m} = \sum_{s \in S^{(p)}} \frac{1}{s^k}, \tag{3}$$

$$\Psi^{(\leq p)}_{k;X_m} = \sum_{s \in S^{(\leq p)}} \frac{1}{s^k}, \tag{4}$$

$$\Psi^{(\geq p)}_{k;X_m} = \sum_{s \in S^{(\geq p)}} \frac{1}{s^k}, \qquad \text{for } k \in \mathbb{N}^*, \tag{5}$$

See [10], who considered the case where $X_m = \{9\}$.

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Notice that

$$\Psi_{k;X_m}^{-} = \Psi_{k;X_m}^{(\leq 0)} = \Psi_{k;X_m}^{(0)};$$
$$\Psi_{k;X_m}^{(\leq p)} = \sum_{i=1}^{p} \Psi_{k;X_m}^{(i)};$$
$$\Psi_{k;X_m}^{+} = \sum_{i=1}^{\infty} \Psi_{k;X_m}^{(i)}.$$

We may extend the notion of *Kempner type* series by considering a set X, of strings of digits, and a set S, of numbers that meet different conditions, expressed in one of the five forms above. The series

$$\Psi_{k;X} = \sum_{s \in S} \frac{1}{s^k}, \qquad \text{for } k \in \mathbb{N}^*, \tag{6}$$

are said to be of *Irwin type*.

The definitions above make sense for any numerical base b.

If $m = 0$ and $k = 1$, the series

$$\Psi_{1;X_0}^{-} = \sum_{s=1}^{\infty} \frac{1}{s} \tag{7}$$

is precisely the *harmonic*. This series is divergent because

$$\begin{aligned}\sum_{s=1}^{\infty} \frac{1}{s} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty.\end{aligned}$$

Euler noticed that the partial sums H_n of this series satisfy the relation

$$\gamma = \lim_{n \to \infty} \left(H_n - \ln n\right), \tag{8}$$

where γ is the *Euler-Mascheroni constant*.

If $m = 0$ and $k \geq 2$, the series

$$\Psi_{k;X_0}^{-} = \sum_{s=1}^{\infty} \frac{1}{s^k} \tag{9}$$

is precisely the *generalized harmonic series*. It is absolutely convergent and the sum of the series is the value $\zeta(k)$, of Riemann's function.

Since for $k \geq 2$ the series (1)-(5) are sub-series of (9), they are also convergent. It remains to consider the case $k = 1$.

In 1914, A. J. Kempner [10] showed that the series $\Psi_{1;X_1}^{-} = \sum_{s \in S^-} 1/s$ and $\Psi_{1;X_1}^{+} = \sum_{s \in S^+} 1/s$ are convergent. F. Irwin [8] has generalized in 1916 this result as follows:

Theorem 1.1. *The series*

$$\sum_{s \in S} \frac{1}{s}, \tag{10}$$

where S represents the set of all natural numbers that have in their decimal representation at most a_9-times the digit 9*, at most a_8-times the digit* 8 ... *and at most a_0-times the digit* 0*, is convergent.*

Later, H. Behforooz [5] proved the following result regarding *the density of the terms* in harmonic subseries which generates convergence:

Theorem 1.2. *Suppose that S is an infinite set of positive integers for which the series $\sum_{s\in S} 1/s$ is convergent. If for each positive integer k we denote by N_k the number of all elements of S which do not exceed 10^k and $M_k = 10^k - N_k$, then*

$$\lim_{k\to\infty} \frac{N_k}{M_k} = 0.$$

The convergence of the Kempner-Irwin series is *very slow*. For this reason various algorithms were built in order to compute their sums accurately (see [2], [3], [11]). The aim of this paper is to discuss the problem of computing the sum of a series of Kempner-Irwin type in various numerical bases.

2. The main results

B. Farhi [4] has studied the convergence of the sequences $\left(\Psi^{(r)}_{1;"d"}\right)_{r\in\mathbb{N}}$, for $d \in \{0, 1, ..., 9\}$:

Lemma 2.1. (B. Farhi) *For all $d \in \{0, 1, ..., 9\}$, the sequence $\left(\Psi^{(r)}_{1;"d"}\right)_{r\in\mathbb{N}^*}$ is convergent decreasingly to* $10 \log 10$.

We will consider now the convergence problem of the sequence $\left(\Psi^{(r)}_{1;"89"}\right)_{r\in\mathbb{N}}$, defined by

$$\Psi^{(r)}_{1;"89"} = \sum_{s\in S^{(r)}} \frac{1}{s}, \tag{11}$$

where $S^{(r)}$ is the set of all positive integers whose decimal representations contains the string "89" exactly r times. Using the technique of Baillie [3], we split the set $S^{(r)}$ into two sets: $S^{1(r)}$, the set of all numbers of $S^{(r)}$ that have the last digit 8, and $S^{2(r)} = S^{(r)} \backslash S^{1(r)}$.

We note

$$S_i^{j(r)} = S^{j(r)} \cap \left[10^i, 10^{i+1}\right), \quad S_i^{(r)} = S^{(r)} \cap \left[10^i, 10^{i+1}\right),$$

where $r, i \in \mathbb{N},\ j \in \{1, 2\}$ and

$$\begin{aligned}
&S_0^{j(r)} = \emptyset,\ r \geq 1,\\
&S_0^{1(0)} = \{8\},\ S_0^{2(0)} = \{1, 2, 3, 4, 5, 6, 7, 9\},\\
&S_1^{1(0)} = \{18, .., 98\},\ S_1^{2(1)} = \{89\},\\
&S_1^{2(0)} = \left\{\overline{ab} : a = 1, ..., 9;\ b = 0, ..., 7, 9\right\} \setminus \{89\},\\
&S_2^{1(0)} = \left\{\overline{ab8} : a = 1, ..., 9;\ b = 0, ..., 9\right\} \setminus \{898\},\\
&S_2^{1(1)} = \{898\},\ S_2^{2(1)} = \{189, ..., 989\} \cup \{890, ..., 897, 899\},\\
&S_2^{2(0)} = \left(\left[10^2, 10^3\right) \cap \mathbb{N}\right) \setminus \left(S_2^{1(0)} \cup S_2^{1(1)} \cup S_2^{2(1)}\right), ...
\end{aligned}$$

Put:

$$t_{i,j;r} = \sum_{s\in S_i^{j(r)}} \frac{1}{s}, \quad t_{i;r} = \sum_{s\in S_i^{(r)}} \frac{1}{s}, \quad \Psi'^{(r)}_{1;"89"} = \sum_{s\in S^{1(r)}} \frac{1}{s}. \tag{12}$$

We have

$$t_{0,1;0} = \frac{1}{8}, \quad t_{0,1;\,r} = 0,\ r\geq 1$$

$$t_{1,1;0} = \sum_{u=1}^{9} \frac{1}{10u+8}, \quad t_{1,1;\,r} = 0,\ r\geq 1,$$

$$t_{i,1;r} = \sum_{s\in S_i^{1(r)}} \frac{1}{s} = \sum_{u\in S_{i-1}^{(r)}} \frac{1}{10u+8},\ i\geq 1,\ r\geq 0,$$

and

$$t_{0,2;0} = \sum_{l=1}^{7} \frac{1}{l} + \frac{1}{9},\ t_{0,2;r} = 0,\ r\geq 1,$$

$$t_{1,2;0} = \sum_{l=0}^{7}\sum_{u=1}^{9} \frac{1}{10u+l} + \sum_{u=1}^{7} \frac{1}{10u+9} + \frac{1}{99},\ t_{1,2;1} = \frac{1}{89},\ t_{1,2;r} = 0,\ r\geq 2,$$

$$t_{i,2;0} = \sum_{s\in S_i^{2(0)}} \frac{1}{s} = \sum_{l=0}^{7}\sum_{u\in S_{i-1}^{(0)}} \frac{1}{10u+l} + \sum_{u\in S_{i-1}^{2(0)}} \frac{1}{10u+9},\ i\geq 1,$$

$$t_{i,2;r} = \sum_{s\in S_i^{2(r)}} \frac{1}{s} = \sum_{l=0}^{7}\sum_{u\in S_{i-1}^{(r)}} \frac{1}{10u+l} + \sum_{u\in S_{i-1}^{2(r)}} \frac{1}{10u+9} + \sum_{u\in S_{i-1}^{1(r-1)}} \frac{1}{10u+9},\ i\geq 1,\ r\geq 1.$$

Also,

$$t_{0;0} = \sum_{l=1}^{9} \frac{1}{l},\ t_{0;r} = 0,\ r\geq 1,$$

$$t_{1;0} = \sum_{l=0}^{8}\sum_{u=1}^{9} \frac{1}{10u+l} + \sum_{u=1}^{7} \frac{1}{10u+9} + \frac{1}{99},$$

$$t_{1;1} = \frac{1}{89},\ t_{1;r} = 0,\ r\geq 2,$$

$$t_{i;0} = \sum_{l=0}^{8}\sum_{u\in S_{i-1}^{(0)}} \frac{1}{10u+l} + \sum_{u\in S_{i-1}^{2(0)}} \frac{1}{10u+9},\ i\geq 2,$$

$$t_{i;r} = \sum_{l=0}^{8}\sum_{u\in S_{i-1}^{(r)}} \frac{1}{10u+l} + \sum_{u\in S_{i-1}^{2(r)}} \frac{1}{10u+9} + \sum_{u\in S_{i-1}^{1(r-1)}} \frac{1}{10u+9},\ i\geq 1,\ r\geq 1.$$

We use for approximating $t_{i;r}$ (with $i, r\in\mathbb{N}^*$) the formulas

$$T_{i;r} = \sum_{l=0}^{8}\sum_{u\in S_{i-1}^{(r)}} \frac{1}{10u} + \sum_{u\in S_{i-1}^{2(r)}} \frac{1}{10u} + \sum_{u\in S_{i-1}^{1(r-1)}} \frac{1}{10u} = \frac{9}{10}t_{i-1;r} + \frac{1}{10}t_{i-1,2;r} + \frac{1}{10}t_{i-1,1;r-1},$$

$$T_{1;r} = \frac{1}{10}t_{0,1;r-1}, \quad T_{1;1} = \frac{1}{80}$$

The approximation error is given by

$$\begin{aligned}
&C_{i;r} = T_{i;r} - t_{i;r}, \quad r, i \in \mathbb{N},\\
&C_{1;1} = \frac{1}{80} - \frac{1}{89} = \frac{9}{7120},\\
&C_{1;r} = \frac{1}{10} t_{0,1;r-1} - t_{1;r} = 0, \quad r \geq 2.
\end{aligned}$$

Therefore

$$t_{i;r} = \frac{9}{10} t_{i-1;r} + \frac{1}{10} t_{i-1,2;r} + \frac{1}{10} t_{i-1,1;r-1} - C_{i;r}. \tag{13}$$

Lemma 2.2. *The following formula holds true:*

$$0 \leq \sum_{r=1}^{\infty} \sum_{i=1}^{\infty} C_{i;r} < \infty.$$

Proof. In fact, for $r, i \in \mathbb{N}^*$ we have

$$C_{i;r} = \sum_{l=0}^{8} \sum_{u \in S_{i-1}^{(r)}} \frac{l}{10u\,(10u + l)} + \sum_{u \in S_{i-1}^{2(r)}} \frac{9}{10u\,(10u + 9)} + \sum_{u \in S_{i-1}^{1(r-1)}} \frac{9}{10u\,(10u + 9)}.$$

From this identity it results that $C_{i;r} \geq 0$. On the other hand,

$$C_{i;r} \leq \frac{9}{25} \sum_{u \in S_{i-1}^{(r)}} \frac{1}{u^2} + \frac{9}{100} \sum_{u \in S_{i-1}^{2(r)}} \frac{1}{u^2} + \frac{9}{100} \sum_{u \in S_{i-1}^{1(r-1)}} \frac{1}{u^2}.$$

Because the sets $S_i^{(r)} = S_i^{1(r)} \cup S_i^{2(r)}$ $(r, i \in \mathbb{N})$ form a partition of $\mathbb{N}^*$,

$$\sum_{r=1}^{\infty} \sum_{i=1}^{\infty} C_{i;r} \leq \left(\frac{9}{25} + \frac{9}{50} \right) \sum_{u=1}^{\infty} \frac{1}{u^2} = \frac{27}{50} \cdot \frac{\pi^2}{6} < \infty,$$

and the proof is done □

We note:

$$C_r = \sum_{i=1}^{\infty} C_{i;r}, \quad \text{for } r \in \mathbb{N}^*.$$

Lemma 2.3. *For every $r \in \mathbb{N}^*$ the series $\Psi_{1;\text{"}89\text{"}}^{(r)}$ and $\Psi'^{(r)}_{1;\text{"}89\text{"}}$ are convergent and*

$$\Psi'^{(1)}_{1;\text{"}89\text{"}} = \Psi'^{(0)}_{1;\text{"}89\text{"}} - 10C_1 + \frac{1}{8}, \tag{14}$$

$$\Psi'^{(r)}_{1;\text{"}89\text{"}} = \Psi'^{(r-1)}_{1;\text{"}89\text{"}} - 10C_r, \ r \geq 2. \tag{15}$$

Proof. The fact that the series $\Psi^{(r)}_{1;"89"}$ and $\Psi'^{(r)}_{1;"89"}$ ($r \in \mathbb{N}^*$) are convergent is obvious . Let's demonstrate the relations *(14)* and *(15)*. Using *(13)*, we have for all $r \in \mathbb{N}$:

$$\begin{aligned}
\Psi^{(r)}_{1;"89"} &= \sum_{i=2}^{\infty} t_{i;r} + t_{1;r} \\
&= \sum_{i=2}^{\infty} \left(\frac{9}{10} t_{i-1;r} + \frac{1}{10} t_{i-1,2;r} + \frac{1}{10} t_{i-1,1;r-1} - C_{i;r} \right) + t_{1;r} \\
&= \frac{9}{10} \sum_{i=1}^{\infty} t_{i;r} + \frac{1}{10} \sum_{i=1}^{\infty} t_{i,2;r} + \frac{1}{10} \sum_{i=1}^{\infty} t_{i,1;r-1} - \sum_{i=1}^{\infty} C_{i;r} + C_{1;r} + t_{1;r} \\
&= \frac{9}{10} \Psi^{(r)}_{1;"89"} + \frac{1}{10} \left(\Psi^{(r)}_{1;"89"} - \Psi'^{(r)}_{1;"89"} \right) + \frac{1}{10} \Psi'^{(r-1)}_{1;"89"} - C_r + C_{1;r} + t_{1;r} \\
&= \Psi^{(r)}_{1;"89"} - \frac{1}{10} \Psi'^{(r)}_{1;"89"} + \frac{1}{10} \Psi'^{(r-1)}_{1;"89"} - C_r + C_{1;r} + t_{1;r}.
\end{aligned}$$

Thus

$$\Psi'^{(r)}_{1;"89"} = \Psi'^{(r-1)}_{1;"89"} - 10C_r + 10C_{1;r} + 10t_{1;r} = \begin{cases} \Psi'^{(r-1)}_{1;"89"} - 10C_r + \frac{1}{8} & \text{if} \quad r = 1 \\ \Psi'^{(r-1)}_{1;"89"} - 10C_r & \text{if} \quad r \geq 2. \end{cases}$$

and the demonstration is closed . □

Theorem 2.1. *The sequence* $\left(\Psi'^{(r)}_{1;"89"} \right)_{r \in \mathbb{N}^*}$ *decreases and*

$$\lim_{r \to \infty} \Psi'^{(r)}_{1;"89"} = 22.217649... \,. \tag{16}$$

Proof. Because $C_r > 0$, the formula *(15)* shows that $\left(\Psi'^{(r)}_{1;"89"} \right)_{r \in \mathbb{N}^*}$ is decreasing. Because this sequence is positive, it is necessarily convergent. It remains to compute the limit. For every integer $R \geq 2$,

$$\Psi'^{(R)}_{1;"89"} = \sum_{r=2}^{R} \left(\Psi'^{(r)}_{1;"89"} - \Psi'^{(r-1)}_{1;"89"} \right) + \Psi'^{(1)}_{1;"89"} = -10 \sum_{r=1}^{R} C_r + \Psi'^{(0)}_{1;"89"} + \frac{1}{8},$$

which yields

$$\lim_{R \to \infty} \Psi'^{(R)}_{1;"9"} = -10 \sum_{r=1}^{\infty} C_r + \Psi'^{(0)}_{1;"89"} + \frac{1}{8}. \tag{17}$$

We have

$$\sum\nolimits_{r=1}^{\infty} C_r = \sum\nolimits_{r=1}^{\infty} \sum\nolimits_{i=1}^{\infty} C_{i;r},$$

where

$$C_{i;r} = T_{i;r} - t_{i;r} = \sum_{l=0}^{8} \sum_{u \in S^{(r)}_{i-1}} \frac{l}{10u\,(10u + l)} + \sum_{u \in S^{2(r)}_{i-1}} \frac{9}{10u\,(10u + 9)} + \sum_{u \in S^{1(r-1)}_{i-1}} \frac{9}{10u\,(10u + 9)}.$$

Because the sets $S_{i-1}^{(r)}$ $(i, r \in \mathbb{N}^*)$ form a partition of $\mathbb{N}^* \backslash S^{(0)}$ and the sets $S_{i-1}^{(r-1)}$ $(i, r \in \mathbb{N}^*)$ form a partition of $\mathbb{N}^*$, we infer that

$$\sum_{r=1}^{\infty}\sum_{i=1}^{\infty} C_{i;r} = \sum_{l=0}^{8}\sum_{u\in\mathbb{N}^*\backslash S^{(0)}}\left(\frac{1}{10u}-\frac{1}{10u+l}\right)+\sum_{u\in\mathbb{N}^*}\left(\frac{1}{10u}-\frac{1}{10u+9}\right)$$

$$+\sum_{u\in\mathbb{S}^{1(0)}}\left(\frac{1}{10u}-\frac{1}{10u+9}\right)=\sum_{l=0}^{8}\left\{\sum_{u=1}^{\infty}\left(\frac{1}{10u}-\frac{1}{10u+l}\right)-\sum_{u\in S^{(0)}}\left(\frac{1}{10u}-\frac{1}{10u+l}\right)\right\}$$

$$+\sum_{u=1}^{\infty}\left(\frac{1}{10u}-\frac{1}{10u+9}\right)+\sum_{u\in\mathbb{S}^{1(0)}}\left(\frac{1}{10u}-\frac{1}{10u+9}\right)$$

$$=\sum_{l=0}^{9}\sum_{u=1}^{\infty}\left(\frac{1}{10u}-\frac{1}{10u+l}\right)-\sum_{l=0}^{8}\sum_{u\in S^{(0)}}\left(\frac{1}{10u}-\frac{1}{10u+l}\right)+\sum_{u\in\mathbb{S}^{1(0)}}\left(\frac{1}{10u}-\frac{1}{10u+9}\right)$$

$$=\sum_{l=0}^{9}\sum_{u=1}^{\infty}\left(\frac{1}{10u}-\frac{1}{10u+l}\right)-\frac{9}{10}\sum_{t\in S^{(0)}}\frac{1}{u}+\sum_{l=0}^{8}\sum_{u\in S^{(0)}}\frac{1}{10u+l}+\frac{1}{10}\sum_{u\in\mathbb{S}^{1(0)}}\frac{1}{u}-\sum_{u\in\mathbb{S}^{1(0)}}\frac{1}{10u+9}.$$

From

$$\sum_{u\in S^{(0)}}\frac{1}{u}=\Psi_{1;\text{"}89\text{"}}^{(0)}$$

it results

$$\sum_{u\in S^{(0)}}\frac{1}{u}=\sum_{u=1}^{9}\frac{1}{u}+\sum_{u\in S^{(0)}\cap[10,\infty)}\frac{1}{u}=\sum_{u=1}^{9}\frac{1}{u}+\sum_{l=0}^{8}\sum_{u\in S^{(0)}}\frac{1}{10u+l}+\sum_{u\in S^{2(0)}}\frac{1}{10u+9}.$$

Or,

$$\sum_{u\in S^{1(0)}}\frac{1}{u}=\Psi_{1;\text{"}89\text{"}}^{\prime(0)}$$

so that

$$\sum_{r=1}^{\infty}\sum_{i=1}^{\infty} C_{i;r}=\Delta+\frac{1}{10}\Psi_{1;\text{"}89\text{"}}^{(0)}+\frac{1}{10}\Psi_{1;\text{"}89\text{"}}^{\prime(0)}-\sum_{u=1}^{9}\frac{1}{u}-\sum_{u\in S^{(0)}}\frac{1}{10u+9}, \qquad (18)$$

where

$$\Delta=\sum_{l=1}^{9}\sum_{u=1}^{\infty}\left(\frac{1}{10u}-\frac{1}{10u+l}\right)=\sum_{u=1}^{\infty}\left\{\frac{1}{u}-\sum_{10u\leq s<10(u+1)}\frac{1}{s}\right\}.$$

We pass now to the computation of Δ. For every N sufficiently big, by applying *(8)* and using *the symbols of Landau and Hardy* (see [7], p. 7), we have:

$$\sum_{u=1}^{N}\left\{\frac{1}{u}-\sum_{10u\leq s<10(u+1)}\frac{1}{s}\right\}=\sum_{t=1}^{N}\frac{1}{t}-\sum_{s=10}^{10N+9}\frac{1}{s}$$

$$=(\log N+\gamma)-\left\{\log(10N+9)+\gamma-\sum_{s=1}^{9}\frac{1}{s}\right\}+o_N(1)$$

$$=\sum_{s=1}^{9}\frac{1}{s}-\log 10+o_N(1),$$

where γ is the constant of Euler. Passing to the limit we have:

$$\Delta = \sum_{s=1}^{9} \frac{1}{s} - \log 10. \tag{19}$$

According to *(18)*, we obtain:

$$\sum_{r=1}^{\infty} C_r = \sum_{r=1}^{\infty}\sum_{i=1}^{\infty} C_{i;r} = \frac{1}{10}\Psi^{(0)}_{1;"89"} + \frac{1}{10}\Psi'^{(0)}_{1;"89"} - \log 10 - \sum_{u\in S^{(0)}} \frac{1}{10u+9}.$$

Finally, by replacing the value of $\sum_{r=1}^{\infty} C_r$ in *(17)*, it results

$$\lim_{R\to\infty} \Psi'^{(R)}_{1;"89"} = -10\sum_{r=1}^{\infty} C_r + \Psi'^{(0)}_{1;"89"} + \frac{1}{8} = \frac{1}{8} - \Psi^{(0)}_{1;"89"} + 10\log 10 + 10\sum_{u\in S^{(0)}} \frac{1}{10u+9}$$
$$= \frac{1}{8} + 10\log 10 - 9\sum_{u\in S^{(0)}} \frac{1}{u\,(10u+9)}.$$

Using *Maple 7*,

$$\sum_{u\in S^{(0)}} \frac{1}{u\,(10u+9)} \simeq .103689...$$

and thus

$$\lim_{R\to\infty} \Psi'^{(R)}_{1;"89"} \simeq 22.217649... .$$

The proof is done. □

Similarly one can study sequences of the form $\left(\Psi^{(r)}_{1;"d_1 d_2"}\right)_{r\in\mathbb{N}}$. It would be interesting to find the exact value of their terms and their limit, but this problem remains open.

3. Kempner-Irwin series in binary base

In other numerical bases (e.g., base 2) there are Kempner-Irwin series with faster convergence.

We will consider several examples.

1) Of first example is the series summation,

$$_{(2)}\Phi^{-}_{1;"0"} = \sum_{s\in S^{-}} \frac{1}{s_{(2)}}, \tag{20}$$

where S^{-} is the set of all the positive integers that *do not contain* "0", in the base 2. Thus,

$$_{(2)}\Phi^{-}_{1;"0"} = \frac{1}{1_{(2)}} + \frac{1}{11_{(2)}} + \frac{1}{111_{(2)}} + \frac{1}{1111_{(2)}} + ...$$

Transformed in the base 10, the series becomes:

$$_{(10)}\Phi^{-}_{1;"0"} = \frac{1}{1_{(10)}} + \frac{1}{3_{(10)}} + \frac{1}{7_{(10)}} + \frac{1}{15_{(10)}} + ... = \sum_{n=1}^{\infty} \frac{1}{(2^n-1)_{(10)}},$$

We evaluate this series using *Maple 7*,

```
>evalf[60](sum('1/(2^n-1)', 'n'=1..infinity));
```

concluding that it *rapidly converges* to

$$1.6066951524152917637833015231909245804805796715057564357780 81..._{(10)}.$$

Converting in binary basis

```
>convert(1.60669515241529176378330152319092458048057967150,binary,60);
```

we find

$$_{(2)}\Phi^{-}_{1;"0"} \approx 1.1001101101010000010111111001111001000011111110010001001000 01..._{(2)}.$$

2) Another example is offered by the series

$$_{(2)}\Phi^{(1)}_{1;"1"} = \sum_{s\in S^{(1)}} \frac{1}{s_{(2)}} = \frac{1}{1_{(2)}} + \frac{1}{10_{(2)}} + \frac{1}{100_{(2)}} + \frac{1}{1000_{(2)}} + \cdots, \tag{21}$$

where $S^{(1)}$ is the set of all the positive integers that *contain* "1" *exactly* once, in base 2. In base 10 we have

$$\begin{aligned} _{(10)}\Phi^{(1)}_{1;"1"} &= \frac{1}{1_{(10)}} + \frac{1}{2_{(10)}} + \frac{1}{2^2_{(10)}} + \frac{1}{2^3_{(10)}} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{2^n_{(10)}} = 2_{(10)} = 10_{(2)}. \end{aligned}$$

3) A third example,

$$_{(2)}\Phi^{(1)}_{1;"0"} = \sum_{s\in S^{(1)}} \frac{1}{s_{(2)}}, \tag{22}$$

where $S^{(1)}$ is the set of all the positive integers that *contain* "0" *exactly* once, in base 2. We have

$$\begin{aligned} _{(2)}\Phi^{(1)}_{1;"0"} &= \frac{1}{10_{(2)}} + \frac{1}{101_{(2)}} + \frac{1}{110_{(2)}} + \frac{1}{1011_{(2)}} + \frac{1}{1101_{(2)}} + \frac{1}{1110_{(2)}} + \cdots \\ &= \frac{1}{10_{(2)}} + \left(\frac{1}{111_{(2)} - 10_{(2)}} + \frac{1}{111_{(2)} - 1_{(2)}}\right) \\ &\quad + \left(\frac{1}{1111_{(2)} - 100_{(2)}} + \frac{1}{1111_{(2)} - 10_{(2)}} + \frac{1}{1111_{(2)} - 1_{(2)}}\right) + \cdots \end{aligned}$$

Transformed in base 10, this series becomes

$$\begin{aligned} _{(10)}\Phi^{(1)}_{1;"0"} &= \frac{1}{(2^2-1-1)_{(10)}} + \left(\frac{1}{(2^3-1-2)_{(10)}} + \frac{1}{(2^3-1-1)_{(10)}}\right) \\ &\quad + \left(\frac{1}{(2^4-1-2^2)_{(10)}} + \frac{1}{(2^4-1-2)_{(10)}} + \frac{1}{(2^4-1-1)_{(10)}}\right) + \cdots \\ &= \frac{1}{(2^2-1-1)_{(10)}} + \sum_{k=0}^{1} \frac{1}{(2^3-1-2^k)_{(10)}} + \sum_{k=0}^{2} \frac{1}{(2^4-1-2^k)_{(10)}} + \cdots \\ &= \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \frac{1}{(2^n-1-2^k)_{(10)}}. \end{aligned}$$

With Maple 7,

```
>evalf[60](sum(sum('1/(2^n-1-2^k)','k'=0..n-2),'n'=2..infinity));
```

we obtain

$$_{(10)}\Phi^{(1)}_{1;"0"} \approx 1.462590735044364699546145446720534621074744864748821109364 20..._{(10)}$$

and converted in base 2:

```
>convert(1.46259073504436469954614544672053462107474486474,binary,60);
```

we conclude that

$$_{(2)}\Phi^{(1)}_{1;"0"} \approx 1.0111011001101100010110001010111001110010101110011100101 1010..._{(2)}.$$

More examples can be easily exhibited by using the beautiful result of Thomas Schmelzer and Robert Baillie [11] on the asymptotic behavior of Kempner-Irwin series in base 10:

Theorem 3.1. a) *Let X_m be a string with m digits having period p, i.e.,*

$$X_m = "\underbrace{d_1 d_2 ... d_p d_1 d_2 ... d_p ... d_1 d_2 ... d_p}_{m=kp \ digits}". \tag{23}$$

Let $\Psi^-_{X_m}$ be the sum of all numbers $1/s$, where s does not contain the substring X_m. Then

$$\lim_{m\to\infty} \frac{\Psi^-_{1;X_m}}{10^m} = \frac{10^p}{10^p - 1} \log 10. \tag{24}$$

b) *Let $X_m = "d_1 d_2 ... d_m"$ be a string with m digits non-periodical and let $\Psi^-_{X_m}$ be the sum of all numbers $1/s$, where s does not contain the substring X_m. Then*

$$\lim_{m\to\infty} \frac{\Psi^-_{1;X_m}}{10^m} = \log 10. \tag{25}$$

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